Formulation en flux des schémas Lagrange-projection et variantes

Introduction à la table ronde

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Outline

Outline **Lagrange-Remap schemes** Toward Lagrange-flux schemes What about multimaterial in Lagrange-Flux ?

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Construction

Lagrange-remap (LR) schemes – Construction

- Define a piecewise $\mathscr{C}^1$ Lagrange transformation operator
  $\mathscr{L}(\boldsymbol{x}, t; t_0, \boldsymbol{x}_0) : d\boldsymbol{x}/dt = \boldsymbol{u}(\boldsymbol{x})$, $\boldsymbol{x}(t = 0) = \boldsymbol{x}_0$.

- Use the Reynolds transport theorem:

$$\int_{\Omega_t} q(\boldsymbol{x}, t) \, d\boldsymbol{x} = \int_{\Omega_t} \{\partial_t q + \nabla \cdot (q\boldsymbol{u})\} \, d\boldsymbol{x}$$

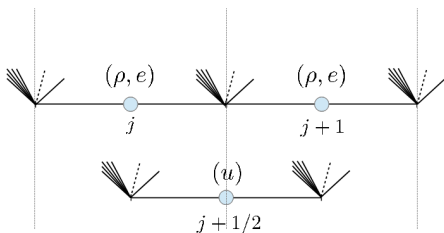 for any moving domain Ω_t, quantity q.

- Finite volume approximation

- Move the mesh according to a Lagrange operator $\mathscr{L}$ over
  $[t^n, t^{n+1})$

- Project the quantities on the initial mesh.

Construction

Attributes of a LR scheme

- Definition of the Lagrange operator $\mathscr{L}$
- Pseudo-viscosity:
    - Linear part: for linear stability;
    - Quadratic part: for $[\![U]\!]^3$ entropy production (nonlinear entropy stability).
- Time integration of the Lagrange step
- Order of accuracy of the projection: requires an interpolator operator $\mathscr{I}$ (Llor, ETSN Roscoff 2012)

Outline   **Lagrange-Remap schemes**   Toward Lagrange-flux schemes   What about multimaterial in Lagrange-Flux ?

Conservative reformulation (1D case)

Conservative reformulation
(staggered variables, 1D case)

Geometric conservation law (GCL):

$$h_j^{n+1,L} = h + \Delta t (u_{j+1/2}^{n+1/2,L} - u_{j-1/2}^{n+1/2,L}).$$

Mass conservation law:

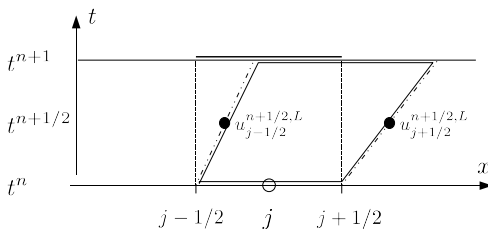$$\rho_j^{n+1,L} h_j^{n+1,L} = h \rho_j^n.$$

Outline **Lagrange-Remap schemes** Toward Lagrange-flux schemes What about multimaterial in Lagrange-Flux ?
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Conservative reformulation (1D case)

Mass conservation equation

i.e.

$$\rho_j^{n+1,L} = \frac{\rho_j^n}{1 + \frac{\Delta t}{h}(\Delta u)_j^{n+1/2,L}}, \quad (\Delta u)_j^{n+1/2,L} := u_{j+1/2}^{n+1/2,L} - u_{j-1/2}^{n+1/2,L}.$$

Projection step:

$$\rho_j^{n+1} = \frac{1}{h} \int_{I_j} \mathscr{I} \rho^{n+1,L}(x)\, dx = \frac{1}{h} \int_{I_j^{n+1,L}} ... - ... + ... \,.$$

Outline   **Lagrange-Remap schemes**   Toward Lagrange-flux schemes   What about multimaterial in Lagrange-Flux ?
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Conservative reformulation (1D case)

Mass equation (cont.)

Mass balance rewritting : under some convenient CFL condition, we have

$$
\begin{aligned}
h\rho_j^{n+1} &= h_j^{n+1,L}\rho_j^{n+1,L} - \Delta t \rho_{j+1/2}^{upw,n+1} u_{j+1/2}^{n+1/2,L} + \Delta t \rho_{j+1/2}^{upw,n+1} u_{j-1/2}^{n+1/2,L} \\
&= h\rho_j^{n} - \Delta t \rho_{j+1/2}^{upw,n+1} u_{j+1/2}^{n+1/2,L} + \Delta t \rho_{j+1/2}^{upw,n+1} u_{j-1/2}^{n+1/2,L}
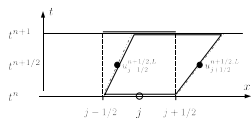\end{aligned}
$$

in the form

$$
\rho_j^{n+1} = \rho_j^n - \frac{\Delta t}{h}\left(\Phi_{m,j+1/2}^{n+1/2,n+1} - \Phi_{m,j-1/2}^{n+1/2,n+1}\right),
$$

$$
\Phi_{m,j+1/2}^{n+1/2,n+1} = \rho_{j+1/2}^{upw,n+1} u_{j+1/2}^{n+1/2,L}.
$$

Outline   **Lagrange-Remap schemes**   Toward Lagrange-flux schemes   What about multimaterial in Lagrange-Flux ?
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Conservative reformulation (1D case)

Flux accuracy

$$\Phi^{n+1/2,n+1}_{m,j+1/2} = \rho^{upw,n+1}_{j+1/2} u^{n+1/2,L}_{j+1/2}.$$

Let $\delta x = \dfrac{\Delta t}{2} u(x, t + \Delta t/2)$.

$$\rho(x + \delta x, t + \Delta t) u(x + \delta x, t + \frac{\Delta t}{2}) =$$

$$= \left(\rho(x + \delta x, t + \Delta t/2) + \frac{\Delta t}{2} \partial_t \rho + o(\Delta t) \right) u(x + \delta x, t + \Delta t/2)$$

$$= (\rho u)(x, t + \Delta t/2) + \delta x \partial_x (\rho u)(x, t + \Delta t/2)) + \frac{\Delta t}{2} (u \partial_t \rho)(x, t + \Delta t/2) + o(\Delta t)$$

$$= (\rho u)(x, t + \Delta t/2) + \frac{\Delta t}{2} u (\partial_t \rho + \partial_x (\rho u))(x, t + \Delta t/2) + o(\Delta t)$$

$$= (\rho u)(x, t + \Delta t/2) + o(\Delta t).$$

Outline   **Lagrange-Remap schemes**   Toward Lagrange-flux schemes   What about multimaterial in Lagrange-Flux ?

Conservative reformulation (1D case)

Momentum equation (staggered)

$$\frac{\rho_j^{n+1} + \rho_{j+1}^{n+1}}{2} u_{j+1/2}^{n+1} = \frac{\rho_j^n + \rho_{j+1}^n}{2} u_{j+1/2}^n - \frac{\Delta t}{h} \left(\Phi_{mu,j+1}^{n,n+1} - \Phi_{mu,j}^{n,n+1} \right) \quad (1)$$

with

$$\Phi_{mu,j}^{n,n+1} = \overline{u}_j^{n+1,L} \frac{(\overline{\rho}_{j-1/2}^{n+1,L} u_{j-1/2}^{n+1/2,L} + \overline{\rho}_{j+1/2}^{n+1,L} u_{j+1/2}^{n+1/2,L})}{2} + (p+q)_j^{n+1/2,L}. \quad (2)$$

Outline   **Lagrange-Remap schemes**   Toward Lagrange-flux schemes   What about multimaterial in Lagrange-Flux ?
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Conservative reformulation (1D case)

Energy conservation (staggered)

$$\rho_j^{n+1} \widetilde{E}_j^{n+1} = \rho_j^n \widetilde{E}_j^n - \frac{\Delta t}{h} \left(\Phi_{mE,j+1/2}^{n,n+1} - \Phi_{mE,j-1/2}^{n,n+1} \right) \tag{3}$$

with

$$
\begin{aligned}
\Phi_{mE,j+1/2}^{n,n+1} &= \overline{e}_{j+1/2}^{n+1,L} u_{j+1/2}^{n+1/2,L} \\
&+ \frac{1}{2} \Bigg(\frac{1}{2} \overline{(u^2)}_j^{n+1,L} \frac{(\overline{\rho}_{j-1/2}^{n+1,L} u_{j-1/2}^{n+1/2,L} + \overline{\rho}_{j+1/2}^{n+1,L} u_{j+1/2}^{n+1/2,L})}{2} \\
&+ \frac{1}{2} \overline{(u^2)}_{j+1}^{n+1,L} \frac{(\overline{\rho}_{j+1/2}^{n+1,L} u_{j+1/2}^{n+1/2,L} + \overline{\rho}_{j+3/2}^{n+1,L} u_{j+3/2}^{n+1/2,L})}{2} \Bigg) + \Phi_{pu,j+1/2}^{n+1/2,L} \tag{4}
\end{aligned}
$$

Conservative reformulation (1D case)

Paper submitted

Lagrange-Remap schemes in conservative form

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Abstract

Staggered Lagrange-Remap schemes are reformulated as Eulerian finite volume schemes in conservative form, with
particular focus on the numerical fluxes. The numerical fluxes naturally involve the states after the Lagrange step.
To cite this article: F. De Vuyst, C. Fochesato et al., C. R. Mécanique 333 (2012).

Résumé

Ecriture conservative des schémas Lagrange-projection Dans cette Note, on réécrit un schéma Lagrange-
projection à variables décalées sous la forme d'un schéma eulérien conservatif classique. Les flux numériques sont
naturellement définis à partir des états issus de l'étape Lagrange. *Pour citer cet article : F. De Vuyst, C. Fochesato
et al., C. R. Mécanique 333 (2012).*

[De Vuyst, Fochesato, Loubère, Rouzier, Saas, Motte, Ghidaglia, CRAS Mécanique,
submitted 2012]

Outline **Lagrange-Remap schemes** Toward Lagrange-flux schemes What about multimaterial in Lagrange-Flux ?
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Conservative reformulation (1D case)

Some remarks

Consider $u_{j+1/2}^{n+1/2,L} \geq 0$, then $\rho_{j+1/2}^{upw,n+1} = \rho_j^{n+1,L}$ and

$$
\begin{aligned}
\Phi_{m,j+1/2}^{n+1/2,n+1} &= \rho_{j+1/2}^{upw,n+1} u_{j+1/2}^{n+1/2,L} \\
&= \frac{\rho_j^n}{1 + \frac{\Delta t}{h}(\Delta u)_j^{n+1/2,L}} u_{j+1/2}^{n+1/2,L}.
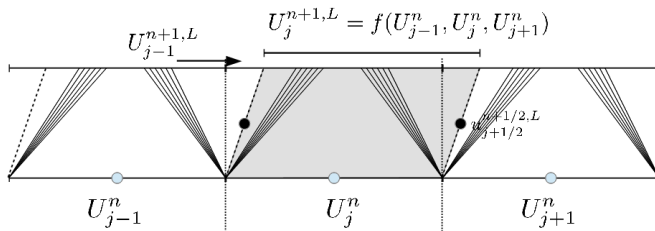\end{aligned}
$$

For $t \in (0, \Delta t)$,

$$
\Phi_{m,j+1/2}^{n+1/2}(t) = \frac{\rho_j^n}{1 + \frac{t}{h}(\Delta u)_j^{n+1/2,L}} u_{j+1/2}^{n+1/2,L}
$$

depends on t, whereas a flux from an approximate Riemann solver does not depend on t (autosimilar solutions of Riemann pbs).

Outline   **Lagrange-Remap schemes**   Toward Lagrange-flux schemes   What about multimaterial in Lagrange-Flux ?

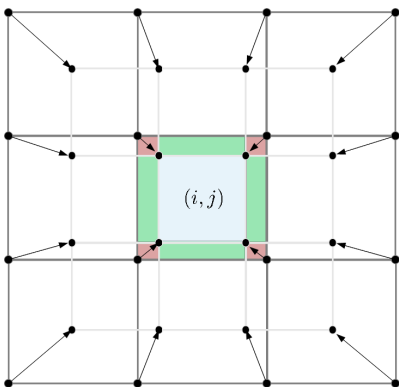Conservative reformulation (1D case)

Some remarks (contd)

First-order (1st-order remap) LR schemes are actually 5-point schemes !

$$U_j^{n+1} = U_j^n - \frac{\Delta t}{h}\left(\Phi_{j+1/2}^{n,n+1} - \Phi_{j-1/2}^{n,n+1}\right),$$

$$\Phi_{j+1/2}^{n,n+1} = \Phi\left(U_{j-1}^n, U_j^n, U_{j+1}^n, U_{j+2}^n, \Delta t\right).$$

Outline   **Lagrange-Remap schemes**   Toward Lagrange-flux schemes   What about multimaterial in Lagrange-Flux ?

Conservative reformulation (2D case)

Conservative reformulation (2D case)

$\Rightarrow$ 1st order LR is a 25-point scheme !

Multidimensional corner effects

Conservation reformulation : 2D case

Remap step by Alternating Directions (AD)

See C. Fochesato's talk (this workshop).

Outline | **Lagrange-Remap schemes** | Toward Lagrange-flux schemes | What about multimaterial in Lagrange-Flux ?
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Partial concluding remarks

Concluding remarks: properties on LR

- Lagrange step + projection: no numerical flux required (except pressure)
- Discrete entropy inequality holds for entropy-satisfying Lagrange steps (under convenient CFL)
- 1D, 1st-order remap: LR is a particular 5-points FV scheme.
- 2D, 1st-order remap : LR is a 25-point scheme (corners) !
- Lagrange step: allows multi-dimensional treatment of linear (contact) waves.
- Projection step can be interpreted as a conservative (flux balance) scheme.
- The underlying numerical flux should not be understood as a flux from a Riemann problem (time-dependent, hidden averages behind projection of the nonlinear waves).

Outline   **Lagrange-Remap schemes**   Toward Lagrange-flux schemes   What about multimaterial in Lagrange-Flux ?
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Partial concluding remarks

Related works

- Després, Lagoutière, J. Sci. Comp., 2002
- Del Pino, Jourdren, CRAS Série I, 2006
- Kokh-Lagoutière, JCP, 2010

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Partial concluding remarks

Benefits of a reformulation by flux

- Conservation ensured (up to FP precision)
- Can be integrated in a general FV code
- Parallelism made simple
- Numerical analysis (accuracy, ...)
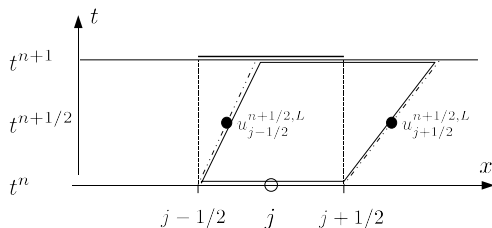- Use of FV tools (flux limiters, antidiffusive, ...)

Issue

- Multimaterial case

Outline   Lagrange-Remap schemes   **Toward Lagrange-flux schemes**   What about multimaterial in Lagrange-Flux ?
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Requirements

Toward Lagrange-flux schemes

Design a more general framework of conservative FV eulerian schemes with two ingredients:

- Lagrange solvers (Lagrange step)
- Computation of numerical fluxes

- Optional: (reduced stencil) 3-point 1st-order schemes ?

What about multimaterial in Lagrange-Flux ?

- The question is open;
- There are connections between interface reconstruction, interpolation operator (remap), and antidiffusive fluxes (reservoir [Le Coq, Alouges, De Vuyst, Lorin], [Lagoutière et al.]);
- Flux determination is subject to some requirements: nonlinear diffusion for compactness, second order accuracy, smallest stencil (ref [Llor Roscoff 2012]).
- The secret is certainly behind the choice of the interpolation operator $\mathscr{I}$.
- See also B. Rebourcet's talk (this workshop)