Investigation of exact solutions of a coupled Kerr-SBS system

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The coupled Kerr-stimulated Brillouin scattering system

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After a reductive perturbation, three complex equations in three complex amplitudes U_1, U_2, Q depending on four independent variables x, y, t, z

$$
\begin{aligned}
& i(U_{1,z} + v_g U_{1,t}) + \frac{U_{1,xx} + U_{1,yy}}{2k_0} + b\left(|U_1|^2 + 2|U_2|^2\right) U_1 + i\frac{g}{2} Q U_2 = 0, \\
& -i(U_{2,z} - v_g U_{2,t}) + \frac{U_{2,xx} + U_{2,yy}}{2k_0} + b\left(|U_2|^2 + 2|U_1|^2\right) U_2 - i\frac{g}{2} \overline{Q} U_1 = 0, \\
& \tau Q_t + Q - U_1 \overline{U}_2 = 0,
\end{aligned}
$$

in which v_g, k_0, b, g, τ are real constants,
t is time, z is the longitudinal coordinate.
We restrict here to the generic case $bg\tau\partial_t \neq 0$.
Our goal: find closed-form particular solutions of physical interest.

Search for particular solutions (closed form)

Multivalued particular solutions: there exists no method.
Singlevalued particular solutions: there exists a method.

This method (Kowalevski, school of Painlevé) takes advantage of the singularities which depend on initial conditions ("movable"), in **two steps**:
1. (Local) Near every movable singularity, require the solution to be singlevalued.
2. (Global) Assume a closed form expression matching the singularity structure.

Example. Travelling waves of modified KdV

Methods in *The Painlevé handbook*, RC and Musette, Springer 2008

$$u_t + u_{xxx} - 6u^2 u_x = 0,$$

Traveling wave reduction:

$$u = U(\xi),\ \xi = x - ct,\ U''' - 6U^2 U' - cU' = 0.$$

1. (Local) $\exists$ two Laurent series (simple pole)

$$U = \sum_{j=0}^{+\infty} U_j (\xi - \xi_0)^{j-1},\ U_0 = \pm 1,\ \text{with } \xi_0, U_3 \text{ and } U_4 \text{ arbitrary.}$$

2. (Global)

2a. Assume U to have only one (not two) simple pole

$U = \pm \partial_\xi \log \psi$, $\psi =$ entire f., e.g. $\psi'' - k^2 \psi = 0$.

Output is **two fronts** $U = \pm k \tanh k(\xi - \xi_0),\ c = -2k^2$.

2b. Assume U to have two simple poles

$U = \partial_\xi \log \psi_1 - \partial_\xi \log \psi_2,\ \psi_j$ entire, e.g. $\psi_j'' - k^2 \psi_j = 0$.

Output is **one pulse** $U = k \,\mathrm{sech}\, k(\xi - \xi_0),\ c = k^2$.

Kerr-SBS, local analysis

$\exists$ a movable singularity $\varphi(x,y,z,t)=0$ where U_1, U_2, Q all have simple poles,

$$U_1 = Me^{ia_1}\varphi^{-1}+\cdots\ ,\ U_2 = Me^{ia_2}\varphi^{-1}+\cdots\ ,\ Q = Ne^{ia_1-ia_2}\varphi^{-1}+\cdots\ ,$$
$$M=\pm\sqrt{-N\tau\varphi_t},\ N=\frac{\varphi_x^2+\varphi_y^2}{3k_0 b\tau\varphi_t},$$

$a_1, a_2 =$ arbitrary real functions.
Same situation as mKdV (two opposite values of M).
The 10 arbitrary coefficients occur at Fuchs indices

$$-1,0,0,1,1,3,3,4,\frac{3}{2}+\frac{\sqrt{11}}{2\sqrt{3}},\frac{3}{2}-\frac{\sqrt{11}}{2\sqrt{3}}, \qquad (1)$$

and 5 constraints arise from the positive integer indices $1,1,3,3,4$.

Bad news: nonintegrable (some Fuchs indices are irrational, 5 constraints).
Good news: Laurent series do exist which depend on $10-5$ arb functions. Singlevalued closed form solutions **may** exist.

Kerr-SBS. Global assumption with one family

Assume the closed form

$$U_k = e^{ia_k}(\varphi^{-1}M + U_{k,1}),\ k=1,2,\ Q = e^{ia_1 - ia_2}(\varphi^{-1}N + Q_1),$$

and identify the equations (finite Laurent series in φ) to 0.
16 real equations, 9 real unknown functions $(\varphi, a_1, a_2, U_{k,1}, Q_1)$: not too bad.
The constraint from Fuchs index 1

$$3\left(\varphi_x^2+\varphi_y^2\right)^2(\tau^{-1}\varphi_t - \varphi_{tt}) + 6\left(\varphi_x^2+\varphi_y^2\right)\varphi_t(\varphi_x\varphi_{xt}+\varphi_y\varphi_{yt}) + \varphi_t^2(\varphi_x^2(3\varphi_{xx}+\varphi_{yy}) + \varphi_y^2(3\varphi_{yy}+\varphi_{xx}) + 4\varphi_x\varphi_y\varphi_{xy}) = 0,$$

proves that **no travelling wave solution exists** $\varphi = \Phi(\xi)$ with $\xi = k_x x + k_y y + k_z z + k_t t$.
In the coordinates (R, θ, z, T),

$$x = \rho\cos\theta,\ y = \rho\sin\theta,\ t = \tau\log T,\ R = \rho^{2/3},$$

we could only find a solution in the radial case

$$\partial_\theta = 0:\ \ T = G_1(\varphi, z)(R + G_0(\varphi, z)),\ G_0, G_1 = \text{ arb f.}$$

Kerr-SBS. Global assumption with one family
(continued)

Next equations then yield the two “arbitrary” functions a_1, a_2,

$$a_1 + a_2 = R\ g_1(z, T) + g_0(z, T),$$
$$a_1 - a_2 = -\frac{g}{3b}\left[\log T - G_2(\varphi, z)\right],$$

in which $g_0(z, T), g_1(z, T), G_2(\varphi, z)$ are arbitrary functions.
Next equation

$$(g_1(z, T))^2 = \text{rational}(T; G_0(\varphi, z), G_1(\varphi, z)),$$

requires the rhs to be independent of φ, and this admits [To be checked] **no solution** (at least in the radial case $\partial_\theta = 0$).

Kerr-SBS. Global assumption with two families

Assume the closed form

$$U_k = e^{ia_k} \left(M\chi^{-1} + U_{k,1} - M\chi \right), \; k = 1, 2,$$
$$Q = e^{ia_1 - ia_2} \left(N\chi^{-1} + Q_1 + N\chi \right),$$

and identify the equations (finite Laurent series in χ) to 0,
with $\chi =$ some precise homographic function of $\varphi(x, y, z, t)$.
19 real equations, 9 real unknown functions $(\varphi, a_1, a_2, U_{k,1}, Q_1)$.
If such a solution exists, it will look like
$U_1 = \mathrm{sech}\, X, \; U_2 = \mathrm{sech}\, X, \; Q = \tanh X$,
with X some function of (x, y, z, t).
In progress. Should succeed according to the numerical simulations of Mauger *et al.* (2011).