Inconsistency and stability of the compatible staggered Lagrangian scheme

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Plan

- Introduction and motivation
- 2D Lagrangian Staggered Hydrodynamics scheme
  - Subcell formalism
  - Specifics : Artificial viscosity, subpressure forces
  - Properties
- Deaper studies
  - Internal (and volume) consistency ?
  - Stability ?
- Conclusions and perspectives

Introduction and motivation

Why do we still analyse a staggered Lagrangian scheme from the 50's ?

2D Staggered Lagrangian scheme for hydrodynamics

- dates back to von Neumann, Richtmyer J. Appl. Phy. 1950)], Schultz, Wilkins [Green book (1964)] era.
- later improved by many authors in national labs or academy
- important subcell based compatible discretization of div/grad [Favorskii, Burton, Caramana]
  ▷ improved artificial viscosity, hourglass filters, accuracy time/space, axisymetric geo.
  ▷ coupling with slide line, materials, diffusion, elastoplasticity, etc.
  ▷ "engine" of many ALE codes
- most of all this scheme <u>has been and still is</u> routinely used !

$\Longrightarrow$ Need to deeply understand its behaviors !

- to explain already known features
- to chose between different "versions"
- to measure the relative importance of "improvements"
- to fight back, justify or simply understand urban legends

2D Lagrangian Staggered Hydro scheme

Governing equations

2D gas dynamics equations

$$\rho \frac{d}{dt}\left(\frac{1}{\rho}\right) - \nabla \cdot \boldsymbol{U} = 0 \qquad \rho \frac{d}{dt}\boldsymbol{U} + \boldsymbol{\nabla} P = \boldsymbol{0} \qquad \rho \frac{d}{dt}\varepsilon + P\nabla \cdot \boldsymbol{U} = 0$$

Equation of state EOS $P = P(\rho, \varepsilon)$, where $\varepsilon = E - \frac{\boldsymbol{U}^2}{2}$.
Internal energy equation can be viewed as an entropy evolution equation (Gibbs relation $TdS = d\varepsilon + Pd\left(\frac{1}{\rho}\right) \geq 0$)

$$\rho \frac{d}{dt}\varepsilon + P\nabla \cdot \boldsymbol{U} = \rho \left(\frac{d}{dt}\varepsilon + P\frac{d}{dt}\left(\frac{1}{\rho}\right)\right) \geq 0$$

Trajectory equations

$$\frac{d\boldsymbol{X}}{dt} = \boldsymbol{U}(\boldsymbol{X}(t), t), \qquad \boldsymbol{X}(0) = \boldsymbol{x},$$

Lagrangian motion of any point initially located at position $\boldsymbol{x}$.

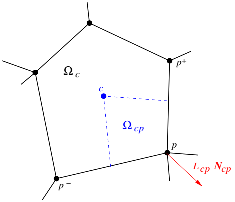
Staggered placement of variables

Point velocity $\boldsymbol{U}_p$, cell-centered density ρ_c and internal energy ε_c

Subcells are Lagrangian volumes

Subcell mass m_{cp} is constant in time so are cell/point masses

$$m_c = \sum_{p \in \mathcal{P}(c)} m_{cp}, \qquad m_p = \sum_{c \in \mathcal{C}(p)} m_{cp},$$



Compatible discretization

Given total energy definition and momentum discretization (Newton's 2nd law) imply energy discretization as sufficient condition
Cornerstone : subcell force $\boldsymbol{F}_{cp}$ that acts from subcell Ω_{cp} on p.
$\triangleright$ compile pressure gradient $\boldsymbol{F}_{cp} = -P_c L_{cp} \boldsymbol{N}_{cp}$, artificial visco, anti-hourglass, elasto forces.
Galilean invariance and/or momentum conservation implies $\displaystyle\sum_{p \in \mathcal{P}(c)} \boldsymbol{F}_{cp} = \boldsymbol{0}$

2D Lagrangian Staggered Hydro scheme

Discretization

Time discretization : $t^n \longrightarrow t^{n+1}$

- Originaly staggered placement of variable in time $\boldsymbol{U}^{n+1/2}$ and ρ^n, ε^n.
- Improvement gained by same time location $\boldsymbol{U}^n, \rho^n, \varepsilon^n$. Side effect : This helped total energy conservation.

▷ Predictor-Corrector P/C type of scheme is very often considered.
Predictor step is often used as to time center the pressure for correction step.
▷ Very seldom : GRP, ADER to reduce the cost of a two-step P/C process

Space discretization : Ω_p, Ω_c

$$\frac{d}{dt} V_c - \sum_{p \in \mathcal{P}(c)} L_{cp} \boldsymbol{N}_{cp} \cdot \boldsymbol{U}_p = 0 \quad \text{or} \quad \frac{d}{dt} \boldsymbol{X}_p = \boldsymbol{U}_p, \quad \boldsymbol{X}_p(0) = \boldsymbol{x}_p$$

$$m_p \frac{d}{dt} \boldsymbol{U}_p + \sum_{c \in \mathcal{C}(p)} \boldsymbol{F}_{cp} = \boldsymbol{0}$$

$$m_c \frac{d}{dt} \varepsilon_c - \sum_{p \in \mathcal{P}(c)} \boldsymbol{F}_{cp} \cdot \boldsymbol{U}_p = 0$$

2D Lagrangian Staggered Hydro scheme

Properties

- General grid formulation
- GCL
- First order accurate scheme in space on non-regular grid,
- Conservation of mass, momentum, total energy

Expected properties

- Expected (internal) consistency
- Expected second-order accuracy in time
- Expected stability under classical CFL condition

Biblio

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Internal consistency

General remark

The equations are essentially created in discrete form, as opposed to being the discretization of a system of PDE's. As such, one may or may not be able to rigorously take the continuum limit to obtain the latter ; this depends on the kinds of forces that are employed to resolve shocks and to counteract spurious grid motions.

Ambiguity of cell volume definition

Results from requiring both total energy conservation and the modeling of the internal energy advance from the differential equation $\frac{d}{dt}\varepsilon + p\frac{d}{dt}(1/\rho) = 0$ under assumptions

- V_c can be computed from $\boldsymbol{X}_p$ for all $p \in \mathcal{P}(c)$
- $\boldsymbol{U}_p$ is constant for all $t \in [t^n; t^{n+1}]$, so that $\boldsymbol{X}_p(t) = \boldsymbol{X}_p^n + \boldsymbol{U}_p(t - t^n)$

There exist a coordinate and a compatible cell volume which may be different !

Internal consistency

Ambiguity of cell volume definition

Implied coordinate cell volume

$$V_c^{n+1} - V_c^n = \int_{t^n}^{t^{n+1}} \frac{dV_c}{dt} dt = \sum_{p \in \mathcal{P}(c)} u_p \int_{t^n}^{t^{n+1}} \frac{\partial V_c}{\partial x_p} dt + v_p \int_{t^n}^{t^{n+1}} \frac{\partial V_c}{\partial y_p} dt$$

$$= \sum_{p \in \mathcal{P}(c)} u_p \mathbf{A}_{cp} + v_p \mathbf{B}_{cp}$$

with ***A***, ***B*** are rectangular sparce matrices.

Remark

Not simple average of integrands unless for Cartesian geometry.

Internal consistency

Ambiguity of cell volume definition

Implied coordinate cell volume

$$V_c^{n+1} - V_c^n = \sum_{p\in\mathcal{P}(c)} u_p \mathbf{A}_{cp} + v_p \mathbf{B}_{cp}$$

Implied compatible cell volume

Discrete momentum + total energy conservation implicitely defines

$$m_p(u_p^{n+1} - u_p^n) - \sum_{c\in\mathcal{C}(p)} P_c \mathbf{a}_{cp} = 0, \qquad m_p(v_p^{n+1} - v_p^n) - \sum_{c\in\mathcal{C}(p)} P_c \mathbf{b}_{cp} = 0$$

$$m_c(\varepsilon_c^{n+1} - \varepsilon_c^n) + P_c \sum_{p\in\mathcal{P}(c)} u_p \mathbf{a}_{cp} + v_p \mathbf{b}_{cp} = 0$$

with $(\mathbf{a}_{cp}, \mathbf{b}_{cp}) = \Delta t\, L_{cp} \boldsymbol{N}_{cp}$. For adiabatic flows the entropy S satisfies $T\frac{dS}{dt} = \frac{d\varepsilon}{dt} + P\frac{dV}{dt} = 0$.

$$m_c\left(\varepsilon_c^{n+1} - \varepsilon_c^n\right) + P_c\left(V_c^{n+1} - V_c^n\right) = 0$$

Internal consistency

Ambiguity of cell volume definition

Implied coordinate cell volume

$$V_c^{n+1} - V_c^n = \sum_{p \in \mathcal{P}(c)} u_p \mathbf{A}_{cp} + v_p \mathbf{B}_{cp}$$

Implied compatible cell volume

Discrete momentum + total energy conservation implicitely defines

$$m_p(u_p^{n+1} - u_p^n) - \sum_{c \in \mathcal{C}(p)} P_c \mathbf{a}_{cp} = 0, \qquad m_p(v_p^{n+1} - v_p^n) - \sum_{c \in \mathcal{C}(p)} P_c \mathbf{b}_{cp} = 0$$

$$m_c(\varepsilon_c^{n+1} - \varepsilon_c^n) + P_c \sum_{p \in \mathcal{P}(c)} u_p \mathbf{a}_{cp} + v_p \mathbf{b}_{cp} = 0$$

with $(\mathbf{a}_{cp}, \mathbf{b}_{cp}) = \Delta t\, L_{cp} \boldsymbol{N}_{cp}$, for adiabatic flows the entropy S satisfies $T\frac{dS}{dt} = \frac{d\varepsilon}{dt} + P\frac{dV}{dt} = 0$.

$$m_c\left(\varepsilon_c^{n+1} - \varepsilon_c^n\right) + P_c \sum_{p \in \mathcal{P}(c)} u_p \mathbf{A}_{cp} + v_p \mathbf{B}_{cp} = 0$$

Internal consistency

Ambiguity of cell volume definition

Condition for uniqueness of cell volume definition

Same volume definition if

$$\mathbf{A}_{cp} = \mathbf{a}_{cp}, \quad \text{and} \quad \mathbf{B}_{cp} = \mathbf{b}_{cp} \quad \forall c, p$$

along with total energy conservation and *PdV* work.
But **a**, **b** correspond to your prefered discrete gradient and **A**, **B** are given by the geometry !

Do the matrices match for different geometry and classical discrete gradient ?

- 1D Cartesian - Yes
- 1D cylindrical - No unless (time centering grid vectors + force=0)
- 1D spherical - No unless (time centering + 1D vector manipulation)
- 2D Cartesian - No unless (time centering + force=0).
- 2D cylindrical $r - z$ - No

Remark : 2D Cartesian analysis shows that the difference is small ($\mathcal{O}(\Delta t^3)$ for one time step)

Internal consistency

Wendroff's idea [(JCP, 227, 2010)]

Derive $\mathbf{A}, \mathbf{B}$ for different geometries and deduce appropriate discrete gradient.

1D spherical : cell half-index $i+\frac{1}{2}$, vertices r_i, r_{i+1}, cell volume $V_{i+\frac{1}{2}} = \frac{1}{3}\left(r_{i+1}^3 - r_i^3\right)$.

$$V_{i+\frac{1}{2}}^{n+1} - V_{i+\frac{1}{2}}^{n} = u_{i+1} \overbrace{\int_{t^n}^{t^{n+1}} \left(r_{i+1}^n + u_{i+1}(t-t^n)\right)^2 dt}^{\mathbf{A}_{i+\frac{1}{2},i+1}} - u_i \overbrace{\int_{t^n}^{t^{n+1}} \left(r_i^n + u_i(t-t^n)\right)^2 dt}^{\mathbf{A}_{i+\frac{1}{2},i}}$$

Matrix $\mathbf{A}$ is given by

$$\mathbf{A}_{i+\frac{1}{2},k} = \begin{cases} -\frac{\Delta t}{3}\left((r_i^n)^2 + \left(r_i^{n+1}\right)^2 + r_i^n r_i^{n+1}\right) & \text{if} \quad k=i \\ \frac{\Delta t}{3}\left(\left(r_{i+1}^n\right)^2 + \left(r_{i+1}^{n+1}\right)^2 + r_{i+1}^n r_{i+1}^{n+1}\right) & \text{if} \quad k=i+1 \\ 0 & \text{if} \quad k\neq i, \ \ k\neq i+1 \end{cases}$$

Imposing $\mathbf{a}_{i\pm\frac{1}{2},i} \equiv \mathbf{A}_{i\pm\frac{1}{2},i}$ leads to

$$m_i(u_i^{n+1} - u_i^n) = \mathbf{A}_{i+\frac{1}{2},i}\, p_{i+\frac{1}{2}} + \mathbf{A}_{i-\frac{1}{2},i}\, p_{i-\frac{1}{2}} = -\Delta t \frac{(r_i^n)^2 + \left(r_i^{n+1}\right)^2 + r_i^n r_i^{n+1}}{3}\left(P_{i+\frac{1}{2}} - P_{i-\frac{1}{2}}\right)$$

$\longrightarrow$ This is the good discrete gradient.

Internal consistency

2D cylindrical $r-z$: quad. cell V_j, nodes (r_i, z_i), $i = 1, \ldots, 4$. Define

$$\begin{aligned} R_{i\to j} &= \left(2r_i^n + r_j^n\right)\left(z_j^{n+1} - z_i^{n+1}\right) + \left(2r_i^{n+1} + r_j^{n+1}\right)\left(z_j^n - z_i^n\right) \\ &\quad +2\left\{\left(2r_i^n + r_j^n\right)\left(z_j^n - z_i^n\right) + \left(2r_i^{n+1} + r_j^{n+1}\right)\left(z_j^{n+1} - z_i^{n+1}\right)\right\}, \\ Z_{i\to j} &= \left(2r_i^n + r_j^n\right)\left(r_j^{n+1} - r_i^{n+1}\right) + \left(2r_i^{n+1} + r_j^{n+1}\right)\left(r_j^n - r_i^n\right) \\ &\quad +2\left\{\left(2r_i^n + r_j^n\right)\left(r_j^n - r_i^n\right) + \left(2r_i^{n+1} + r_j^{n+1}\right)\left(r_j^{n+1} - r_i^{n+1}\right)\right\}, \end{aligned}$$

$$\begin{aligned} V_j^{n+1} - V_j^n &= \frac{\Delta t}{36}\Big\{ \\ &\left(u_1\left[R_{1\to4} - R_{1\to2}\right] + u_2\left[R_{2\to3} - R_{2\to1}\right] + u_3\left[R_{3\to4} - R_{3\to2}\right] + u_4\left[R_{4\to3} - R_{4\to1}\right]\right) \\ + \quad &\left(v_1\left[Z_{1\to4} - Z_{1\to2}\right] + v_2\left[Z_{2\to3} - Z_{2\to1}\right] + v_3\left[Z_{3\to4} - Z_{3\to2}\right] + v_4\left[Z_{4\to3} - Z_{4\to1}\right]\right)\Big\}, \end{aligned}$$

$[R_{1\to4} - R_{1\to2}]$ defines $\mathbf{A}_{jp}$ for p global index of vertex 1, $[Z_{1\to4} - Z_{1\to2}]$ defines $\mathbf{B}_{jp}$

$\mathbf{A}, \mathbf{B}$ being defined, it uniquely implies the discretizations of discrete gradient with $\mathbf{a} = \mathbf{A}, \mathbf{b} = \mathbf{B}$.

$\longrightarrow$ This is the good discrete gradient.

Internal consistency

Numerical scheme

Initialization : $P_c = P_c^n$, $\mathbf{a}_{cp}^{n+1} = \mathbf{a}_{cp}^n$, $\mathbf{b}_{cp}^{n+1} = \mathbf{b}_{cp}^n$

0- Outer iterations :

- 0- Inner consistency iterations :
  Pressure P_c fixed solve the implicit system 1-2
- 1- Velocity

$$m_p(u_p^{n+1} - u_p^n) - \sum_{c \in \mathcal{C}(p)} P_c\, \mathbf{a}_{cp}^{n+1} = 0, \qquad m_p(v_p^{n+1} - v_p^n) - \sum_{c \in \mathcal{C}(p)} P_c\, \mathbf{b}_{cp}^{n+1} = 0$$

- 2- Position and $\mathbf{a}_{cp}, \mathbf{b}_{cp}$

$$x_p^{n+1} = x_p^n + \Delta t \frac{u_p^n + u_p^{n+1}}{2} = 0, \qquad y_p^{n+1} = y_p^n + \Delta t \frac{v_p^n + v_p^{n+1}}{2} = 0$$

- 3- Exit when convergence is reached for x_p, y_p, u_p, v_p

1- Compute new cell volume V_c^{n+1} and deduce internal energy

$$m_c(\varepsilon_c^{n+1} - \varepsilon_c^n) + P_c(V_c^{n+1} - V_c^n) = 0$$

2- Deduce new pressure P_c^{n+1} and $P_c = \frac{1}{2}(P_c^{n+1} + P_c^n)$

3- Exit when convergence is reached for ε_c^{n+1}

Internal consistency

Numerical scheme

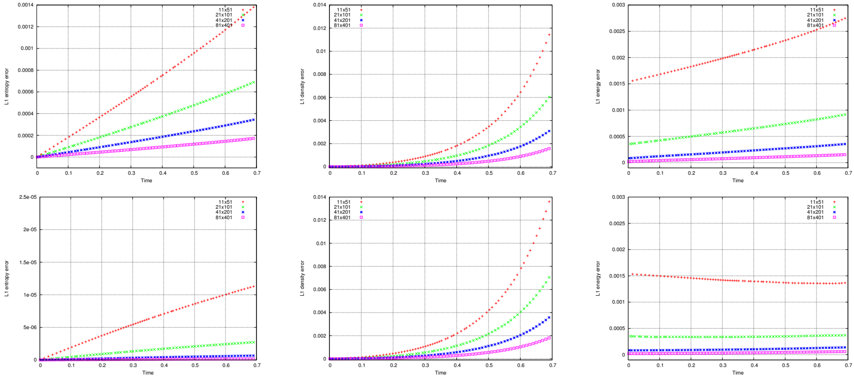
Remarks

These schemes are indexed by (#outer, #inner)

- Classical P/C staggered compatible scheme is a $(2, 1)$ scheme. For 2D axisymetric problem the Cartesian geometrical vectors are modified but this can not fulfill volume consistency and total energy conservation.
- Conversely our proposed scheme is a $(2, \infty)$ scheme which enjoys these properties.

Numerical results : Coggeshall adabatic compression in 2D $r - z$ geometry - No artificial visco - Exact solution exists

L_1 Entropy error **L_1 Density err** **L_1 Energy err**



Stability

Which stability ?

If the continuum system has no growing solutions, the discretized form should also contains no growing solutions.

Predictor-corrector scheme

In general the prediction step only serves as to predict a time advanced pressure $P^* = \alpha P^{predicted} + (1-\alpha)P^n$ with $t^* \in [t^n; t^{n+1}]$.

Scheme #1

Prediction

1- Predict $\boldsymbol{U}_p^{n+*} = \boldsymbol{U}_p^n + \Delta t\, f(P_c^n)$, and $\boldsymbol{U}^{n+1/2} = \frac{1}{2}(\boldsymbol{U}^n + \boldsymbol{U}^{n+*})$

2- Predict $\boldsymbol{X}_p^{n+*} = X_p^n + \Delta t \boldsymbol{U}_p^{n+1/2}$

3- Compute V_c^{n+*}, ρ_c^{n+*}

4- Predict $\varepsilon_c^{n+*} = \varepsilon_c^n + \Delta t\, f(\boldsymbol{U}_p^{n+1/2}, P_c^n)$

5- Predict $P_c^* \equiv \alpha P_c^{n+*} + (1-\alpha) P_c^n$

Correction

1- Compute $\boldsymbol{U}_p^{n+1} = \boldsymbol{U}_p^n + \Delta t\, f(P_c^*)$, and $\boldsymbol{U}^{n+1/2} = \frac{1}{2}(\boldsymbol{U}^n + \boldsymbol{U}^{n+1})$

2- compute $\boldsymbol{X}_p^{n+1} = X_p^n + \Delta t \boldsymbol{U}_p^{n+1/2}$

3- Compute V_c^{n+1}, ρ_c^{n+1}

4- Compute $\varepsilon_c^{n+1} = \varepsilon_c^n + \Delta t\, f(\boldsymbol{U}_p^{n+1/2}, P_c^*)$

5- Compute P_c^{n+1}

Stability

Scheme #1

1- Predict $\boldsymbol{U}_p^{n+*} = \boldsymbol{U}_p^n + \Delta t\, f(P_c^n)$, and $\boldsymbol{U}^{n+1/2} = \frac{1}{2}(\boldsymbol{U}^n + \boldsymbol{U}^{n+*})$
2- Predict $\boldsymbol{X}_p^{n+*} = X_p^n + \Delta t \boldsymbol{U}_p^{n+1/2}$
3- Compute V_c^{n+*}, ρ_c^{n+*}
4- Predict $\varepsilon_c^{n+*} = \varepsilon_c^n + \Delta t\, f(\boldsymbol{U}_p^{n+1/2}, P_c^n)$
5- Predict $P_c^* \equiv \alpha P_c^{n+*} + (1-\alpha) P_c^n$

1- Compute $\boldsymbol{U}_p^{n+1} = \boldsymbol{U}_p^n + \Delta t\, f(P_c^*)$, and $\boldsymbol{U}^{n+1/2} = \frac{1}{2}(\boldsymbol{U}^n + \boldsymbol{U}^{n+1})$
2- compute $\boldsymbol{X}_p^{n+1} = X_p^n + \Delta t \boldsymbol{U}_p^{n+1/2}$
3- Compute V_c^{n+1}, ρ_c^{n+1}
4- Compute $\varepsilon_c^{n+1} = \varepsilon_c^n + \Delta t\, f(\boldsymbol{U}_p^{n+1/2}, P_c^*)$
5- Compute P_c^{n+1}

Scheme #2

1-
2- Predict $\boldsymbol{X}_p^{n+*} = X_p^n + \Delta t \boldsymbol{U}_p^n$
3- Compute V_c^{n+*}, ρ_c^{n+*}
4- Predict $\varepsilon_c^{n+*} = \varepsilon_c^n + \Delta t\, f(\boldsymbol{U}_p^n, P_c^n)$
5- Predict $P_c^* \equiv \alpha P_c^{n+*} + (1-\alpha) P_c^n$

1- Compute $\boldsymbol{U}_p^{n+1} = \boldsymbol{U}_p^n + \Delta t\, f(P_c^*)$, and $\boldsymbol{U}^{n+1/2} = \frac{1}{2}(\boldsymbol{U}^n + \boldsymbol{U}^{n+1})$
2- compute $\boldsymbol{X}_p^{n+1} = X_p^n + \Delta t \boldsymbol{U}_p^{n+1/2}$
3- Compute V_c^{n+1}, ρ_c^{n+1}
4- Compute $\varepsilon_c^{n+1} = \varepsilon_c^n + \Delta t\, f(\boldsymbol{U}_p^{n+1/2}, P_c^*)$
5- Compute P_c^{n+1}

2D wave equation (as a model)

$$\frac{du}{dt} = \frac{\partial p}{\partial x}, \qquad \frac{dv}{dt} = \frac{\partial p}{\partial y}, \qquad \frac{dp}{dt} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial \widetilde{y}}.$$

Prelims

Rectangular scheme, periodic BCs, staggered placement of variables ; cell centered $p_{i+1/2,j+1/2}$ and nodal $u_{i,j}$, $v_{i,j}$. Mid-edge values are interpolated values $p_{i+\frac{1}{2},j+1} = \frac{1}{2}\left(p_{i+\frac{1}{2},j+\frac{3}{2}} + p_{i+\frac{1}{2},j-\frac{1}{2}}\right)$, and $u_{i+\frac{1}{2},j+1} = \frac{1}{2}\left(u_{i,j+1} + u_{i+1,j+1}\right)$, $\lambda_x = \Delta t/\Delta x$ and
Any variable w defined at two time levels $t_{n+1} > t_n$ on a point or in a cell, we define at an intermediate time $n+\kappa$

$$w^{n+\kappa} = \kappa w^{n+1} + (1-\kappa)\, w^n, \quad 0 \le \kappa \le 1.$$

Stability

von Neumann stability study

Fully implicit staggered scheme

$$u_{i,j}^{n+1} = u_{i,j}^{n} + \lambda_x \left(p_{i+\frac{1}{2},j}^{n+\alpha} - p_{i-\frac{1}{2},j}^{n+\alpha} \right), \quad v_{i,j}^{n+1} = v_{i,j}^{n} + \lambda_y \left(p_{i,j+\frac{1}{2}}^{n+\alpha} - p_{i,j-\frac{1}{2}}^{n+\alpha} \right),$$

$$p_{i+\frac{1}{2},j+\frac{1}{2}}^{n+1} = p_{i+\frac{1}{2},j+\frac{1}{2}}^{n} + \lambda_x \left(u_{i+1,j+\frac{1}{2}}^{n+\beta} - u_{i,j+\frac{1}{2}}^{n+\beta} \right) + \lambda_y \left(v_{i+\frac{1}{2},j+1}^{n+\beta} - v_{i+\frac{1}{2},j}^{n+\beta} \right).$$

$$\boldsymbol{M} = \begin{pmatrix} 0 & 0 & Q_x \\ 0 & 0 & Q_y \\ -Q_x^* & -Q_y^* & 0 \end{pmatrix}, \quad \boldsymbol{\Lambda} = \begin{pmatrix} \lambda_x & 0 & 0 \\ 0 & \lambda_y & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

$$(Q_x p)_{i,j} = \frac{1}{2} \left(p_{i+\frac{1}{2},j+\frac{1}{2}} + p_{i+\frac{1}{2},j-\frac{1}{2}} - p_{i-\frac{1}{2},j+\frac{1}{2}} - p_{i-\frac{1}{2},j-\frac{1}{2}} \right)$$

$$(Q_x^* u)_{i+\frac{1}{2},j+\frac{1}{2}} = \frac{1}{2} \left(u_{i,j} + u_{i,j+1} - u_{i+1,j} - u_{i+1,j+1} \right).$$

Hence the implicit scheme also writes

$$\boldsymbol{w}^{n+1} = \boldsymbol{w}^{n} + \boldsymbol{\Lambda} \boldsymbol{M} \boldsymbol{\Lambda} \boldsymbol{w}^{\alpha,\beta}.$$

Theorem

The fully implicit scheme is stable for any $\lambda_{x,y}$ is $\alpha \geq 2$ anb $\beta \geq 2$.

P/C staggered scheme #1

Predictor step :

$$\begin{aligned}
\widetilde{u}^{\,n+1}_{i,j} &= u^n_{i,j} + \lambda_x \left(Q_x p^n\right)_{i,j},\\
\widetilde{v}^{\,n+1}_{i,j} &= v^n_{i,j} + \lambda_y \left(Q_y p^n\right)_{i,j},\\
\widetilde{p}^{\,n+1}_{i+\frac12,j+\frac12} &= p^n_{i+\frac12,j+\frac12} - \lambda_x \left(Q_x^* u^{n+\beta}\right)_{i+\frac12,j+\frac12}\\
&\quad -\lambda_y \left(Q_y^* v^{n+\beta}\right)_{i+\frac12,j+\frac12}.
\end{aligned}$$

Corrector step :

$$\begin{aligned}
u^{n+1}_{i,j} &= u^n_{i,j} + \lambda_x \left(Q_x p^{n+\alpha}\right)_{i,j},\\
v^{n+1}_{i,j} &= v^n_{i,j} + \lambda_y \left(Q_y p^{n+\alpha}\right)_{i,j},\\
p^{n+1}_{i+\frac12,j+\frac12} &= p^n_{i+\frac12,j+\frac12} - \lambda_x \left(Q_x^* u^{n+\beta}\right)_{i+\frac12,j+\frac12}\\
&\quad -\lambda_y \left(Q_y^* v^{n+\beta}\right)_{i+\frac12,j+\frac12}.
\end{aligned}$$

von Neumann analysis : $p^n_{i+\frac12,j+\frac12} \longmapsto p_0 e^{\theta(n\Delta t)+\mathrm{i}\left(2\delta\left((i+\frac12)\Delta x\right)+2\gamma\left((j+\frac12)\Delta y\right)\right)}$, θ complex, δ, γ reals

$$\mathbf{S} = \begin{pmatrix}
1-\alpha\Phi_x^2 & -\alpha\Phi_x\Phi_y & \mathrm{i}\Phi_x\left(1-\alpha\beta(\Phi_x^2+\Phi_y^2)\right)\\
-\alpha\Phi_x\Phi_y & 1-\alpha\Phi_y^2 & \mathrm{i}\Phi_y\left(1-\alpha\beta\left(\Phi_x^2+\Phi_y^2\right)\right)\\
\mathrm{i}\Phi_x\left(1-\alpha\beta(\Phi_x^2+\Phi_y^2)\right) & \mathrm{i}\Phi_y\left(1-\alpha\beta(\Phi_x^2+\Phi_y^2)\right) & 1+\alpha\beta^2(\Phi_x^2+\Phi_y^2)^2-\beta\left(\Phi_x^2+\Phi_y^2\right)
\end{pmatrix}.$$

Setting $\Phi_x = 2\lambda_x \sin\xi\cos\eta$ and $\Phi_y = 2\lambda_y\sin\eta\cos\xi$, we further study the boundness of numerical radius $R(\mathbf{S}) = \sup_{\mathbf{w}} \mid\langle \mathbf{Sw},\mathbf{w}\rangle\mid$, with $\langle\mathbf{w},\mathbf{w}\rangle = 1$.

Stability

von Neumann stability study

P/C staggered scheme #2

Predictor step :

$$\begin{aligned}
\widetilde{u}^{\,n+1}_{i,j} &= u^n_{i,j}, \\
\widetilde{v}^{\,n+1}_{i,j} &= v^n_{i,j}, \\
\widetilde{p}^{\,n+1}_{i+\frac12,j+\frac12} &= p^n_{i+\frac12,j+\frac12} - \lambda_x \left(Q^*_x u^{n+\beta}\right)_{i+\frac12,j+\frac12} \\
&\quad -\lambda_y \left(Q^*_y v^{n+\beta}\right)_{i+\frac12,j+\frac12}.
\end{aligned}$$

Corrector step :

$$\begin{aligned}
u^{n+1}_{i,j} &= u^n_{i,j} + \lambda_x \left(Q_x p^{n+\alpha}\right)_{i,j}, \\
v^{n+1}_{i,j} &= v^n_{i,j} + \lambda_y \left(Q_y p^{n+\alpha}\right)_{i,j}, \\
p^{n+1}_{i+\frac12,j+\frac12} &= p^n_{i+\frac12,j+\frac12} - \lambda_x \left(Q^*_x u^{n+\beta}\right)_{i+\frac12,j+\frac12} \\
&\quad -\lambda_y \left(Q^*_y v^{n+\beta}\right)_{i+\frac12,j+\frac12}.
\end{aligned}$$

von Neumann analysis : $p^n_{i+\frac12,j+\frac12} \longmapsto p_0 e^{\theta(n\Delta t)+\mathrm{i}\left(2\delta\left((i+\frac12)\Delta x\right)+2\gamma\left((j+\frac12)\Delta y\right)\right)}$, θ complex, δ, γ reals

$$\mathbf{S} = \begin{pmatrix} 1-\alpha\Phi_x^2 & -\alpha\Phi_x\Phi_y & \mathrm{i}\Phi_x \\ -\alpha\Phi_x\Phi_y & 1-\alpha\Phi_y^2 & \mathrm{i}\Phi_y \\ \mathrm{i}\Phi_x\left(1-\alpha\beta(\Phi_x^2+\Phi_y^2)\right) & \mathrm{i}\Phi_y\left(1-\alpha\beta(\Phi_x^2+\Phi_y^2)\right) & 1+\alpha\beta^2(\Phi_x^2+\Phi_y^2)^2-\beta\left(\Phi_x^2+\Phi_y^2\right) \end{pmatrix}.$$

Setting $\Phi_x = 2\lambda_x \sin\xi\cos\eta$ and $\Phi_y = 2\lambda_y \sin\eta\cos\xi$, we further study the boundness of numerical radius $R(\mathbf{S}) = \sup_{\mathbf{w}} |\langle \mathbf{Sw}, \mathbf{w}\rangle|$, with $\langle \mathbf{w}, \mathbf{w}\rangle = 1$.

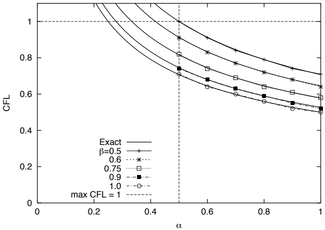
Stability

von Neumann stability study

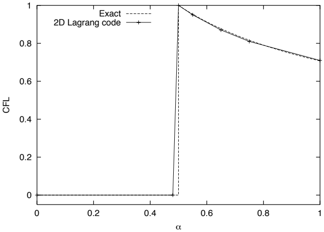
Theorem

The 2D staggered rectangular scheme #1 and #2 are stable if $\alpha \geq \frac{1}{2}$, $\beta \geq \frac{1}{2}$ and $4\alpha\beta \max\left(\lambda_x^2, \lambda_y^2\right) \leq 1$ and unstable if $\alpha < \frac{1}{2}$ and $\beta < \frac{1}{2}$.

Numerical tests



2D wave equations



2D Euler equations, $\beta = 1/2$,

On a 100×100 mesh one runs 10^5 cycles and compute the total kinetic energy $K^\lambda(t^n) = \frac{1}{2}\sum\left[\left(u_{i,j}^n\right)^2 + \left(v_{i,j}^n\right)^2\right]$ for a given CFL number λ at a given time t^n. It must remain at the square of machine precision, about $10^{-28} \sim 10^{-30}$

Conclusion and Perspectives

Conclusions

- Compatible staggered Lagrangian scheme is old and venerable but presents some features that need to be pointed out
- Inconsistency of cell volume definition can be overcome by iterations but seems to be a second-order error
- Particular stability diagram can be deduced from analysis and numerics

Perspectives

Moot points :

- subcells are Lagrangian object ?
- P/C scheme is 2nd order ? What about GRP, ADER type of schemes (one step second order scheme) ?
- impact of artificial viscosity always difficult to analyse.

A votre avis

A priori ou a posteriori ?

- *a priori* on pense savoir ce que nos schémas d'ordre élevés/complexes font,
- *a posteriori* on s'apperçoit qu'ils ne le font pas
  - Erreurs : bugs, mauvaise init, paramètres hors normes...
  - Comportements bizarres "explicables" : numériques (ou physiques)
  - ou inexplicables

Tester et réparer a posteriori *vs* Prédire (théorie du pire) et agir (princip. précaution) a priori ?

Acknowledgments

THANK YOU !

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