The Transparent Dead Leaves Process

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Abstract

This paper introduces the transparent dead leaves (TDL) process, a new germ-grain model in which the grains are combined according to a transparency principle. Informally, this model may be seen as the superimposition of infinitely many semi-transparent objects. Properties of this new model are established and a simulation algorithm is proposed. A central limit theorem is then proved, showing that when varying the transparency of the grain from opacity to total transparency, the TDL process ranges from the dead leaves model to a Gaussian random field.

Keywords: Germ-grain model; dead leaves model; transparency; occlusion; image modeling

1 Introduction

This paper deals with the stochastic modeling of physical transparency. The main contribution is the introduction and study of a new germ-grain model in which the grains are combined according to a transparency principle. To the best of our knowledge, this type of interaction between grains has not been studied before. Classical interactions between grains include *addition* for shot-noise processes [19, 11], *union* for Boolean models [23, 21], *occultation* for dead leaves models [17, 13, 3] or *multiplication* for compound Poisson cascades [2, 6].

The proposed process, that we call *transparent dead leaves* (TDL), is obtained from a collection of grains (random closed sets) indexed by time, as for the dead leaves process of G. Matheron. We assume that each grain is given a random gray level (intensity). Informally, the process may be seen as the superimposition of *transparent* objects associated with the grains. Each time a new grain is added, new values are obtained as a linear combination of former values and the intensity of the added grain, as illustrated in Figure 1. That is, when adding a grain X with gray level a, the current process $f : \mathbb{R}^2 \to \mathbb{R}$ is modified into g, defined for each $y \in \mathbb{R}^2$ as

$$g(y) = \begin{cases} \alpha a + (1-\alpha) f(y) & \text{if } y \in X, \\ f(y) & \text{otherwise,} \end{cases} \tag{1}$$

where $\alpha \in (0, 1]$ is a transparency coefficient. The process is then defined as the sequential superimposition of grains of a suitable Poisson process $\sum_i \delta_{(t_i, x_i, X_i, a_i)}$.

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(a) A disc (b) Initial scene (c) Resulting scene

Figure 1: Addition of a transparent object. The transparency coefficient of the disc is $\alpha = 0.5$.

The main motivation to define such a model originates from vision. Indeed, natural images are obtained from the light emitted by physical objects interacting in various ways. In the case of opaque objects, the main interaction is occlusion. That is, objects hide themselves depending on their respective positions with respect to the eye or the camera. A simple stochastic model for occlusion is given by the dead leaves model, which is therefore useful for the modeling of natural images [9, 5]. When objects are transparent, their interaction may be modeled by Formula (1). This is well known in the field of computer graphics, see [7] where the same principle is used for the creation of synthetic scenes. In this case, transparency is a source of heavy computations, especially in cases where objects are numerous (typically of the order of several thousands), e.g. in the case of grass, fur, smoke, fabrics, etc. The transparency phenomenon may also be encountered in other imaging modality where images are obtained through successive reflexion-transmission steps, as in microscopy or ultrasonic imaging. A related non-linear image formation principle is at work in the field of radiography. In such cases, it is useful to rely on accurate stochastic texture models in order to be able to detect abnormal images. The TDL may be an interesting alternative to Gaussian fields that are traditionally used, see e.g. [10, 20].

In this paper, we first define the transparent dead leaves model in Section 2 and give some elementary properties in Section 3, where we also address the problem of simulating the process and show some realizations. The TDL covariance is then computed in Section 4. Eventually, it is shown in Section 5 that the normalized TDL converges, as α tends to zero, to a Gaussian process having the same covariance function as the shot noise associated with the grain X and with intensity one. Thus the TDLs with varying transparency coefficient α provide us with a family of models ranging from the dead leaves model to Gaussian fields.

2 Definition of the TDL process

As explained in the introduction, the TDL process is obtained as the superimposition of transparent shapes. Formally it is defined from a marked Poisson point process, in a way similar to the dead leaves model [3]. Let $\mathcal{F}$ denote the set of closed sets of $\mathbb{R}^d$. On the state space

$$S := (-\infty, 0) \times \mathbb{R}^d \times \mathcal{F} \times \mathbb{R},$$

we define the point process

$$\Phi := \sum_i \delta_{(t_i, x_i, X_i, a_i)}, \qquad (2)$$

where

- $\{(t_i, x_i)\}$ is a stationary Poisson point process of intensity 1 in the half space $(-\infty, 0) \times \mathbb{R}^d$,
- $(X_i)_i$ is a sequence of i.i.d. random closed sets (RACS) with distribution P_X which is independent of the other random objects,
- $(a_i)_i$ is a sequence of i.i.d. real random variables (r.v.) with distribution P_a which is also independent of the other random objects.

Equivalently, Φ is a Poisson point process with intensity measure $\mu := \lambda \otimes \nu_d \otimes P_X \otimes P_a$, where λ denotes the restriction of the one-dimensional Lebesgue measure over $(-\infty, 0)$ and ν_d denotes the d-dimensional Lebesgue measure over $\mathbb{R}^d$.

Each point $(t_i, x_i, X_i, a_i) \in \Phi$ is called a *leaf*. Having fixed a transparency coefficient $\alpha \in (0, 1]$, the TDL process f is defined by sequentially combining the elements of Φ according to Formula (1).

Definition 1 (*Transparent Dead Leaves* process)**.** *The* Transparent Dead Leaves *process with transparency coefficient α associated to the Poisson process Φ defined by Equation (2) is the random field $f : \mathbb{R}^d \to \mathbb{R}$ defined by*

$$f(y) = \sum_{i \in \mathbb{N}} \mathbb{1}\left(y \in x_i + X_i\right) \alpha a_i \left(1 - \alpha\right)^{\left(\sum_{j \in \mathbb{N}} \mathbb{1}(t_j \in (t_i, 0) \text{ and } y \in x_j + X_j)\right)}. \qquad (3)$$

Let us justify that Formula (3) agrees with the informal description of the TDL process. Let y be a fixed point in $\mathbb{R}^d$, and let (t_i, x_i, X_i, a_i) be any leaf of the Poisson process Φ. If $y \notin x_i + X_i$ the contribution to $f(y)$ of the random shape $x_i + X_i$ is clearly 0. Otherwise, if $y \in x_i + X_i$ the contribution to $f(y)$ of the leaf (t_i, x_i, X_i, a_i) is αa_i multiplied by $(1 - \alpha)$ to the number of leaves fallen on the point y after the leaf (t_i, x_i, X_i, a_i), that is after time $t = t_i$. This number is exactly the exponent of $(1 - \alpha)$ in Equation (3):

$$\sum_{j \in \mathbb{N}} \mathbb{1}\left(t_j \in (t_i, 0) \text{ and } y \in x_j + X_j\right).$$

Since the distribution of the Poisson process Φ is invariant under shifts of the form $(t, x, X, a) \mapsto (t, x + y, X, a)$, the TDL process f is strictly stationary.

Remark 1 (Variable transparency)**.** *For the sake of simplicity, the transparency parameter α is assumed to be the same for all objects. However, one may attach a random transparency α_i to every objects in Definition 1 and generalize the results of Sections 3 and 4, as will be briefly commented thereafter.*

Before establishing further properties of the TDL process f, let us introduce some notations and specify several assumptions.

Notations: Define $\beta := 1 - \alpha$ and let X and a denote respectively a RACS with distribution P_X and a r.v. with distribution P_a which are both independent of all the other random objects. The expectation with respect to the distribution of Φ is denoted by $\mathbb{E}$ (e.g. $\mathbb{E}(f(y))$ whereas the expectation with respect to the distributions $P_X \otimes P_a$ of the marks (X, a) is denoted by E (e.g. $E(\nu_d(X))$, $E(a)$). Finally, γ_X denotes the mean geometric covariogram of the RACS X, that is the function defined by $\gamma_X(\tau) = E(\nu_d(X \cap \tau + X))$, $\tau \in \mathbb{R}^d$ (we refer to [18, 16, 8] for properties of the mean geometric covariogram).

Assumptions: Throughout the paper, it is assumed that

$$0 < E(\nu_d(X)) < +\infty.$$

This hypothesis ensures that each point $y \in \mathbb{R}^d$ is covered by a countable infinite number of leaves of Φ, whereas the number of leaves falling on y during a finite time interval $[s_1, s_2]$ is a.s. finite. We also assume that $E(|a|^2) < \infty$.

3 First-order distribution and simulation of the TDL process

In this section the distribution of the r.v. $f(y)$ is given and a simulation procedure is presented and illustrated.

3.1 The Poisson process of the leaves intersecting a set

As one can observe from Equation (3), the only leaves which have a contribution to the sum defining $f(y)$ are the leaves (t_i, x_i, X_i, a_i) such that $y \in x_i + X_i$. When considering the restriction of f to a Borel set G the only leaves of interest are the ones intersecting G, *i.e.* the leaves (t_i, x_i, X_i, a_i) such that $x_i + X_i \cap G \neq \emptyset$. The next proposition gives the distribution of such leaves, a result to be used further in the paper. We first recall two notations: if A and B are two Borel sets then $\check{A} = \{-x : x \in A\}$ and $A \oplus B = \{x + y : x \in A \text{ and } y \in B\}$. Remark that $x + X \cap G \neq \emptyset \iff x \in G \oplus \check{X}$.

Proposition 1 (The Poisson process of the leaves intersecting a Borel set). *Let $G \subset \mathbb{R}^d$ be a Borel set such that $0 < E\left(\nu_d\left(X \oplus \check{G}\right)\right) < +\infty$ and let Φ be the Poisson process on $S = (-\infty, 0) \times \mathbb{R}^d \times \mathcal{F} \times \mathbb{R}$ with intensity measure $\mu = \lambda \otimes \nu_d \otimes P_X \otimes P_a$. Denote by Φ^G the point process of the leaves of Φ which intersect G, that is*

$$\Phi^G = \{(t, x, X, a) \in \Phi : x + X \cap G \neq \emptyset\},$$

and let us note $\mathcal{A}^G \subset \mathbb{R}^d \times \mathcal{F}$ the set $\mathcal{A}^G = \{(x, X) : x + X \cap G \neq \emptyset\}$. Then Φ^G is a Poisson process on S with intensity measure

$$\mu^G = \lambda \otimes (\nu_d \otimes P_X)_{\llcorner \mathcal{A}^G} \otimes P_a.$$

It is an independently marked Poisson process with ground process $\Pi^G = \left\{t : (t, x, X, a) \in \Phi^G\right\}$, an homogeneous Poisson process on $(-\infty, 0)$ of intensity $E\left(\nu_d\left(X \oplus \check{G}\right)\right)$, and with mark distribution

$$\frac{1}{E\left(\nu_d\left(X \oplus \check{G}\right)\right)} (\nu_d \otimes P_X)_{\llcorner \mathcal{A}^G} \otimes P_a.$$

Proof. Φ^G is the restriction of the Poisson process Φ to the measurable set

$$\left\{(t,x,X,a) \in (-\infty,0)\times\mathbb{R}^d\times\mathcal{F}\times\mathbb{R}:\ (x,X)\in\mathcal{A}^G\right\},$$

thus Φ^G is a Poisson process and its intensity measure μ^G is the restriction of μ to the above set. As for the interpretation of Φ^G as an independently marked one-dimensional Poisson process, it is based on the factorization of the intensity measure μ^G (see [1, Section 1.8] or [22, Section 3.5]). Indeed we have

$$0<\nu_d\otimes P_X\left(\mathcal{A}^G\right)=\int_{\mathcal{F}}\int_{\mathbb{R}^d}\mathbb{1}\left\{y\in G\oplus\check{Y}\right\}\nu_d(dy)P_X(dY)=E\left(\nu_d\left(X\oplus\check{G}\right)\right)<+\infty,$$

and thus we can write

$$\mu^G=E\left(\nu_d\left(X\oplus\check{G}\right)\right)\lambda\otimes\left[\frac{1}{E\left(\nu_d\left(X\oplus\check{G}\right)\right)}\left(\nu_d\otimes P_X\right)_{\llcorner\mathcal{A}^G}\otimes P_a\right],$$

where the measure between square brackets is a probability distribution. □

3.2 First-order distribution

Proposition 2 (First-order distribution)**.** *Let y be a point in $\mathbb{R}^d$. Then there exists a subsequence $(a(y,k))_{k\in\mathbb{N}}$ of i.i.d. r.v. with distribution P_a such that*

$$f(y)=\alpha\sum_{k=0}^{+\infty}a(y,k)\beta^k.$$

In particular we have $\mathbb{E}(f(y))=\mathbb{E}(a)$ and $\operatorname{Var}(f(y))=\dfrac{\alpha}{2-\alpha}\operatorname{Var}(a)$.

Proof. According to Proposition 1 the point process $\Phi^{\{y\}}$ of the leaves which cover y is an independently marked Poisson process, the ground process of which is a Poisson process on $(-\infty,0)$ with intensity $E(\nu_d(X))<+\infty$. Hence the falling times of the leaves of $\Phi^{\{y\}}$ are a.s. distinct and we can number the leaves

$$(t(y,k),x(y,k),X(y,k),a(y,k)),\ \ k\in\mathbb{N},$$

according to an anti-chronological order:

$$0>t(y,0)>t(y,1)>t(y,2)>\dots.$$

Proposition 1 also gives the distribution of the marks $(x(y,k),X(y,k),a(y,k))$, and in particular it shows that the r.v. $a(y,k)$, $k\in\mathbb{N}$ are i.i.d. with distribution P_a. As already mentioned, the only leaves involved in the sum which defines $f(y)$ are the leaves of $\Phi^{\{y\}}$. Besides, using the above numbering we have for all $k\in\mathbb{N}$

$$\sum_{(t_j,x_j,X_j,a_j)\in\Phi}\mathbb{1}\left(t_j\in(t(y,k),0)\text{ and }y\in x_j+X_j\right)=k.$$

Hence Equation (3) becomes

$$f(y)=\alpha\sum_{k=0}^{+\infty}a(y,k)\beta^k,$$

and the result follows. □

Remark 2 (Influence of the transparency coefficient α)**.** *Let us write f_α for the TDL process with transparency coefficient $\alpha \in (0,1]$. Proposition 2 shows that the expectation of f_α does not depend on α. As for the variance, $\mathrm{Var}\left(f_\alpha(y)\right) = \dfrac{\alpha}{2-\alpha}\mathrm{Var}(a)$ decreases as α decreases. Besides $\mathrm{Var}\left(f_\alpha(y)\right)$ tends to 0 as α tends to 0 (recall that the model is not defined for $\alpha = 0$). However, a central limit theorem for random geometric series [4] shows that for all $y \in \mathbb{R}^d$ the family of r.v. $\left(\dfrac{f_\alpha(y) - \mathbb{E}\left(f_\alpha\right)}{\sqrt{\mathrm{Var}\left(f_\alpha\right)}}\right)_\alpha$ converges in distribution to a standard normal distribution as α tends to 0. This pointwise convergence result will be extended in Section 5, where it will be shown that the family of normalized random fields $\left(y \mapsto \dfrac{f_\alpha(y) - \mathbb{E}\left(f_\alpha\right)}{\sqrt{\mathrm{Var}\left(f_\alpha\right)}}\right)_\alpha$ converges in the sense of finite-dimensional distributions.*

3.3 Simulation of the TDL process

In this section we draw on Proposition 2 to obtain a simulation algorithm for the restriction of the TDL process f to a bounded domain $U \subset \mathbb{R}^d$. The algorithm is based on a coupling from the past procedure, as the algorithm developed by Kendall and Thönnes [14] for simulating the dead leaves model (see also [13, 16]). This algorithm consists in sequentially superimposing transparent random objects but, contrary to the forward procedure described by Equation (1), each new object is placed *below* the former objects. In the case of the dead leaves model, this yields a perfect simulation algorithm. For the TDL process f, simulation is not perfect since the values $f(y)$ are the limits of convergent series. Nevertheless, supposing that the intensities a_i are bounded, we propose for any $\varepsilon > 0$ an algorithm which produces an approximation $\tilde{f}$ of f. This approximation satisfies

$$\mathbb{P}\left(\sup_{y\in U}\left|f(y) - \tilde{f}(y)\right| \leq \varepsilon\right) = 1$$

therefore providing a kind of perfect simulation with precision $\varepsilon > 0$.

In the remaining of this section we suppose that the colors a_i are a.s. bounded by $A > 0$. The control of the precision is based on the following elementary lemma.

Lemma 1 (Precision associated to the leaves layer)**.** *Let $y \in \mathbb{R}^d$ and let*

$$\tilde{f}_n(y) = \alpha\sum_{k=0}^{n-1} a(y,k)\beta^k$$

be the restriction of the sum defining $f(y)$ to the n latest leaves which have fallen on y. Then

$$\left|f(y) - \tilde{f}_n(y)\right| \leq A\beta^n.$$

Lemma 1 shows that to approximate $f(y)$ with a tolerance $\varepsilon > 0$ it is enough to cover the point y with (at least) $N(\varepsilon)$ leaves, where $N(\varepsilon)$ is the smallest integer n such that $A\beta^n \leq \varepsilon$, that is $N(\varepsilon) = \left\lceil \dfrac{\log\left(\varepsilon/A\right)}{\log(\beta)}\right\rceil$. This yields the following algorithm.

Algorithm 1 (Simulation of the TDL process with tolerance $\varepsilon > 0$)**.** *Let $U \subset \mathbb{R}^d$ be a bounded set such that $0 < E\left(\nu_d\left(X \oplus \check{U}\right)\right) < +\infty$ and let $\varepsilon > 0$. Given a precision*

$\epsilon > 0$, *an approximation* $\tilde{f}$ *of the TDL process* f *is computed by controlling the number of leaves* L *at each point:*

- **Initialization:** *For all* $y \in U$, $\tilde{f}(y) \leftarrow 0$; $L(y) \leftarrow 0$;
- **Computation of the required number of leaves:** $N(\varepsilon) = \left\lceil \dfrac{\log(\varepsilon/A)}{\log(\beta)} \right\rceil$;
- **Iteration:** *While* $\left(\inf_{y \in U} L(y) < N(\varepsilon) \right)$ *add a new leaf:*

  1. **Draw a leaf** (x, X, a) **hitting** U**:**
     (a) *Draw* $X \sim P_X$*;*
     (b) *Draw* x *uniformly in* $U \oplus \check{X}$*;*
     (c) *Draw* $a \sim P_a$*;*
  2. **Add the leaf** (x, X, a) **to** $\tilde{f}$**:** *for all* $y \in U$, $\tilde{f}(y) \leftarrow \tilde{f}(y) + \mathbb{1}(y \in x + X)\, \alpha a \beta^{L(y)}$;
  3. **Update the leaves layer** L**:** *for all* $y \in U$, $L(y) \leftarrow L(y) + \mathbb{1}(y \in x + X)$;

Clearly Algorithm 1 a.s. converges if every point of U is covered by $N(\varepsilon)$ leaves in an a.s. finite time. This is always the case if U is a discrete set, since $E(\nu_d(X)) > 0$. It is also true for any bounded set U if there exists a non empty open ball B such that $E(\nu_d(X \ominus B)) > 0$ [3], where $X \ominus B = \{x \in X,\ x + B \subset X\}$ is the erosion of X by B [18, 16].

Several realizations of some TDL processes are represented in Fig. 2. Remark that as soon as $\alpha < 1$, the TDL process is not piecewise constant: any region is intersected by the boundaries of some leaves, producing discontinuities.

4 Covariance of the TDL process

This section is devoted to the computation of the covariance of the TDL. A classical way to achieve this would be to rely on Palm calculus, yielding relatively heavy computations in this case. Instead, we chose an alternative way relying on some no-memory property of the TDL, as explained below.

The following proposition is an extension of the fact that if $0 > t_0 > t_1 > t_2 > \dots$ is an homogeneous Poisson process on $(-\infty, 0)$ then the shifted process $0 > t_1 - t_0 > t_2 - t_0 > t_3 - t_0 > \dots$ is also a Poisson process with the same distribution [15, Chapter 4].

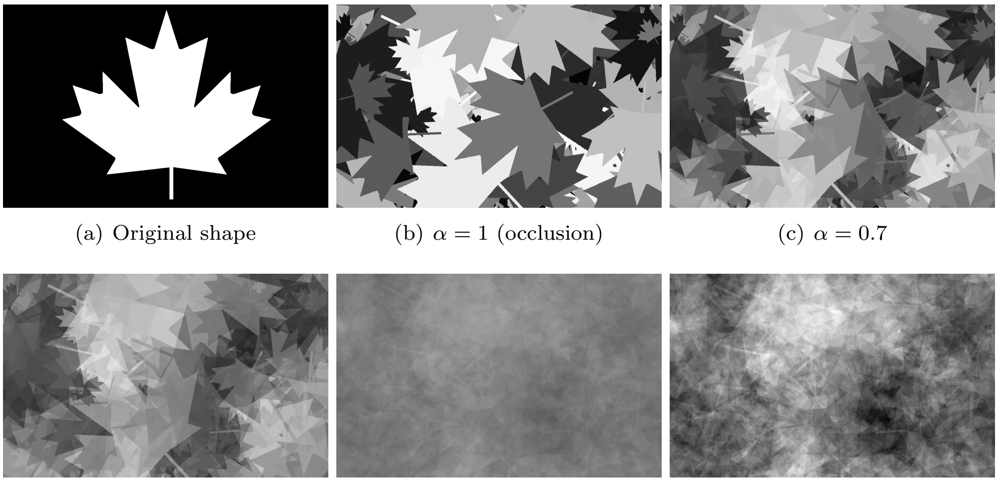
Proposition 3 (Last hitting leaf and the Poisson process preceding the last hit)**.** *Let* Ψ *be a Poisson process in* $(-\infty, 0) \times E$ *with intensity measure of the form* $\lambda \otimes \mu$ *where* λ *is the one-dimensional Lebesgue measure on* $(-\infty, 0)$ *and* μ *is a measure on* E*. Let* $A \subset E$ *be a measurable set such that* $0 < \mu(A) < +\infty$*. Define*

$$t_0 = \sup \{t_i | (t_i, y_i) \in \Psi \cap ((-\infty, 0) \times A)\},$$

y_0 *the a.s. unique* $y \in E$ *such that* $(t_0, y) \in \Psi \cap ((-\infty, 0) \times A)$*, and*

$$\Psi_{t_0} = \sum_{(t_i, y_i) \in \Psi} \mathbb{1}(t_i < t_0)\, \delta_{(t_i - t_0, y_i)}.$$

Then



(a) Original shape (b) $\alpha = 1$ (occlusion) (c) $\alpha = 0.7$

(d) $\alpha = 0.4$ (e) $\alpha = 0.05$ (f) $\alpha = 0.05$, enhanced contrast

Figure 2: TDL realizations with various transparency coefficients α. The RACS X_i are all obtained from the original shape of Fig. 2(a) in applying a rotation of angle $\theta \sim Unif(0, 2\pi)$ and a homothety of factor $r \sim Unif(0, 1)$, and $P_a = Unif(0, 255)$. For $\alpha = 1$, one obtains a colored dead leaves model. As soon as the leaves are transparent ($\alpha < 1$), one can distinguish several layers of leaves and not only the leaves on top. For $\alpha = 0.05$, the variance of the TDL process is nearly 0 (see Proposition 2). Enhancing the contrast of the image (Fig.2(f)) reveals the structure of the image.

- *t_0, y_0, and Ψ_{t_0} are mutually independent.*
- *$-t_0$ has an exponential distribution with parameter $\mu(A)$.*
- *y_0 has distribution Q_A defined for all $B \in \mathcal{B}(E)$ by $Q_A(B) = \dfrac{\mu(B \cap A)}{\mu(A)}$.*
- *Ψ_{t_0} is a Poisson process with intensity measure $\lambda \otimes \mu$,* i.e. *Ψ_{t_0} has the same distribution as Ψ.*

In the following of this section, Proposition 3 will be applied to the Poisson process Φ of the colored leaves to compute some statistics of the TDL process f. As a first example, let us reobtain the expectation of f by using Proposition 3. Let $y \in \mathbb{R}^d$ and let us note (t_0, x_0, X_0, a_0) the leaf which hits y at the maximal time t_0. Then one can decompose $f(y)$ into

$$f(y) = \alpha a_0 + \beta f_{t_0}(y), \tag{4}$$

where f_{t_0} is the TDL process associated to the time-shifted point process Φ_{t_0}. According to Proposition 3, a_0 has distribution P_a and both point processes Φ and Φ_{t_0} have the same distribution. Consequently, $f(y)$ and $f_{t_0}(y)$ also have the same distribution, and in particular the same expectation. Hence the above decomposition of $f(y)$ yields to the equation

$$\mathbb{E}\left(f(y)\right) = \alpha E(a) + \beta \mathbb{E}\left(f(y)\right),$$

which gives $\mathbb{E}\left(f(y)\right) = E(a)$, in accordance with Proposition 2.

The very same method is used below to compute the covariance of f. This method will also be applied in Section A.2.3 to derive a technical result useful for the central limit theorem of Section 5.

Recall that $\gamma_X(\tau) = E(\nu_d(X \cap \tau + X))$ is the mean covariogram of X.

Proposition 4 (Covariance of the TDL process)**.** *The TDL process f is a square-integrable stationary random field and its covariance is given by*

$$\mathrm{Cov}(f)(\tau) = \frac{\alpha\gamma_X(\tau)}{2E(\nu_d(X)) - \alpha\gamma_X(\tau)}\,\mathrm{Var}(a),\ \ \tau \in \mathbb{R}^d.$$

Proof. Let y and z be such that $z - y = \tau$. Let us note (t_0, x_0, X_0, a_0) the last leaf which hits y or z at the maximal time t_0, and let Φ_{t_0} be the corresponding time-shifted Poisson process. According to Proposition 3, (x_0, X_0, a_0) is independent of Φ_{t_0}. In addition $\Phi_{t_0} \overset{d}{=} \Phi$, and consequently, noting f_{t_0} the TDL associated with Φ_{t_0}, $(f_{t_0}(y), f_{t_0}(z)) \overset{d}{=} (f(y), f(z))$. Proposition 3 also shows that a_0 has distribution P_a. As for the distribution of (x_0, X_0), a straightforward computation shows that

$$\nu_d \otimes P_X\left(\{(x,X),\ \{y,z\} \cap x + X \neq \emptyset\}\right) = E\left(\nu_d\left(X \oplus \{-y,-z\}\right)\right) = 2\gamma_X(0) - \gamma_X(\tau)$$

and

$$\nu_d \otimes P_X\left(\{(x,X),\ \{y,z\} \subset x + X\}\right) = E\left(\nu_d\left(-y + X \cap -z + X\right)\right) = \gamma_X(\tau).$$

Hence we have

$$\mathbb{P}\left(\{y,z\} \subset x_0 + X_0\right) = \frac{\nu_d \otimes P_X\left(\{(x,X),\ \{y,z\} \subset x + X\}\right)}{\nu_d \otimes P_X\left(\{(x,X),\ \{y,z\} \cap x + X \neq \emptyset\}\right)} = \frac{\gamma_X(\tau)}{2\gamma_X(0) - \gamma_X(\tau)}, \tag{5}$$

and by symmetry and complementarity

$$\mathbb{P}\left(y \in x_0 + X_0 \text{ and } z \notin x_0 + X_0\right) = \mathbb{P}\left(z \in x_0 + X_0 \text{ and } y \notin x_0 + X_0\right) = \frac{\gamma_X(0) - \gamma_X(\tau)}{2\gamma_X(0) - \gamma_X(\tau)}.$$

As a shorter notation we write $m = \mathbb{E}(a) = \mathbb{E}(f)$. We have to compute $\mathrm{Cov}\left(f(y), f(z)\right) = \mathbb{E}\left(\left(f(y) - m\right)\left(f(z) - m\right)\right)$. Conditioning with respect to the coverage of the last leaf (t_0, x_0, X_0, a_0) we have

$$\begin{aligned}
&\mathbb{E}\left(\left(f(y) - m\right)\left(f(z) - m\right)\right) \\
&= \mathbb{E}\left(\left(f(y) - m\right)\left(f(z) - m\right)\middle|\{y,z\} \subset x_0 + X_0\right) \frac{\gamma_X(\tau)}{2\gamma_X(0) - \gamma_X(\tau)} \\
&\quad + \mathbb{E}\left(\left(f(y) - m\right)\left(f(z) - m\right)\middle| y \in x_0 + X_0 \text{ and } z \notin x_0 + X_0\right) \frac{\gamma_X(0) - \gamma_X(\tau)}{2\gamma_X(0) - \gamma_X(\tau)} \\
&\quad + \mathbb{E}\left(\left(f(y) - m\right)\left(f(z) - m\right)\middle| z \in x_0 + X_0 \text{ and } y \notin x_0 + X_0\right) \frac{\gamma_X(0) - \gamma_X(\tau)}{2\gamma_X(0) - \gamma_X(\tau)}.
\end{aligned}$$

By symmetry it is clear that the two last terms of the above sum are equal. On the event $\{\{y,z\} \subset x_0 + X_0\}$ we have

$$f(y) - m = \alpha(a_0 - m) + \beta\left(f_{t_0}(y) - m\right) \quad \text{and} \quad f(z) - m = \alpha(a_0 - m) + \beta\left(f_{t_0}(z) - m\right),$$

so that

$$\begin{aligned}
\left(f(y) - m\right)\left(f(z) - m\right) = \alpha^2(a_0 - m)^2 &+ \beta^2\left(f_{t_0}(y) - m\right)\left(f_{t_0}(z) - m\right) \\
&+ \alpha\beta(a_0 - m)\left(\left(f_{t_0}(y) - m\right) + \left(f_{t_0}(z) - m\right)\right).
\end{aligned}$$

By Proposition 3, a_0, (x_0, X_0), and $(f_{t_0}(y), f_{t_0}(z))$ are mutually independent, hence

$$\begin{aligned}
&\mathbb{E}\left((f(y)-m)\left(f(z)-m\right)|\{y,z\}\subset x_0+X_0\right)\\
&=\alpha^2\mathbb{E}\left((a_0-m)^2\right)+\beta^2\mathbb{E}\left((f_{t_0}(y)-m)\left(f_{t_0}(z)-m\right)\right)\\
&=\alpha^2\operatorname{Var}(a)+\beta^2\operatorname{Cov}\left(f(y),f(z)\right).
\end{aligned}$$

On the event $\{y \in x_0 + X_0$ and $z \notin x_0 + X_0\}$ we have

$$f(y)-m=\alpha(a_0-m)+\beta\left(f_{t_0}(y)-m\right)\quad\text{and}\quad f(z)-m=f_{t_0}(z)-m.$$

Using the above arguments,

$$\mathbb{E}\left((f(y)-m)\left(f(z)-m\right)|y\in x_0+X_0\text{ and }z\notin x_0+X_0\right)=\beta\operatorname{Cov}\left(f(y),f(z)\right).$$

Coming back to the above decomposition of $\mathbb{E}\left((f(y)-m)\left(f(z)-m\right)\right)$, one obtains an equation involving the covariance $\operatorname{Cov}\left(f(y),f(z)\right)$, the values $\gamma_X(0)$ and $\gamma_X(\tau)$ of the mean covariogram of X, and the variance $\operatorname{Var}(a)$. Simplifying this equation one obtains the enunciated formula. □

Remark 3 (Variable transparency and second order property)**.** *The technique used in this section enables to generalize second order formulas to the case where the transparency parameter α is assumed to be different for each object, that is, when it is assumed that each object X_i is assigned a transparency α_i distributed as a random variable α and independent of other objects. First, it is straightforward to show that in this case we still have $\mathbb{E}(f(y)) = E(a)$. Then, a simple application of Formula (4) yields $\operatorname{Var} f(y) = E(\alpha^2)\operatorname{Var}(a)(2E(\alpha) - E(\alpha^2))^{-1}$. Observe that a direct computation starting from the definition of f would be much more painful. Eventually, applying the same technique enables to show that the covariance of the model with variable transparency satisfies, for $\tau \in \mathbb{R}^d$,*

$$\operatorname{Cov}(f)(\tau)=\frac{E(\alpha^2)\gamma_X(\tau)}{2E(\alpha)E(\nu_d(X))-E(\alpha^2)\gamma_X(\tau)}\operatorname{Var}(a).$$

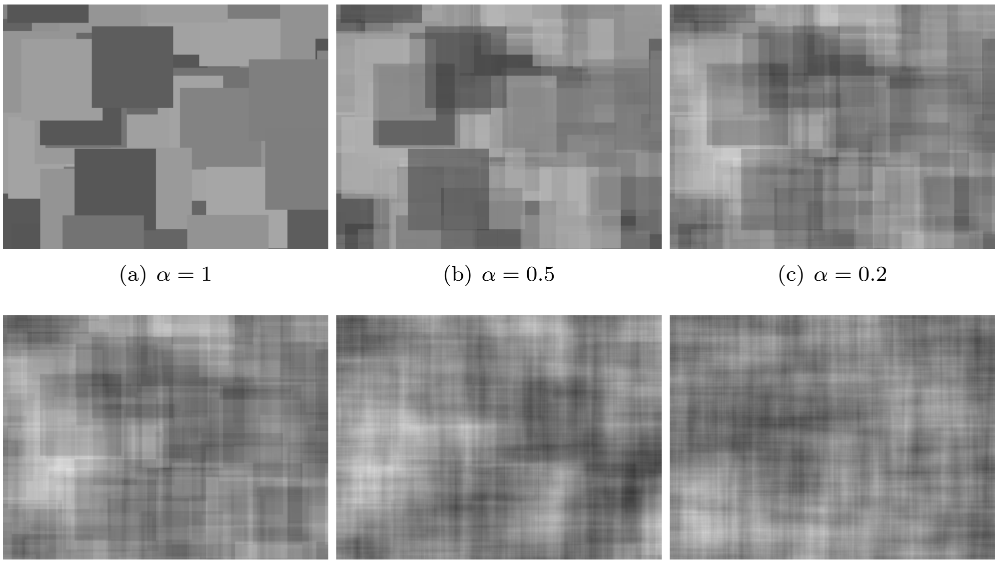
5 Gaussian convergence as the objects tend to be fully transparent

Recall that the TDL process with transparency coefficient α is denoted f_α.

Theorem 1 (Normal convergence of the TDL process)**.** *Suppose that $\operatorname{Var}(a) > 0$. Then, as the transparency coefficient α tends to zero, the family of random fields $\left(\dfrac{f_\alpha-\mathbb{E}\left(f_\alpha\right)}{\sqrt{\operatorname{Var}\left(f_\alpha\right)}}\right)_\alpha$ converges in the sense of finite-dimensional distributions to a stationary Gaussian random field with covariance function*

$$C(\tau)=\frac{\gamma_X(\tau)}{E(\nu_d(X))}=\frac{\gamma_X(\tau)}{\gamma_X(0)}.$$

The proof of Theorem 1 is postponed to the appendix. It relies on a central limit theorem for non independent sequences due to Janson (See Section A.2.1 in the appendix or [12, Theorem 2]). This theorem involves families of r.v. having a controlled



(a) $\alpha = 1$ (b) $\alpha = 0.5$ (c) $\alpha = 0.2$

(d) $\alpha = 0.1$ (e) $\alpha = 0.01$ (f) Limit Gaussian r.f.

Figure 3: From colored dead leaves to Gaussian random fields: Visual illustration of the normal convergence of the normalized TDL processes $\left(\frac{f_\alpha - \mathbb{E}(f_\alpha)}{\sqrt{\mathrm{Var}(f_\alpha)}}\right)_\alpha$ (see Theorem 1). As α decreases to 0 the normalized TDL realizations look more and more similar to the Gaussian texture 3(f).

dependency structure. Here this control is basically obtained from the obvious observation that a leaf covers at most once each considered points (see Appendix A for the details).

The normal convergence of the normalized family of r.v. $\left(\frac{f_\alpha - \mathbb{E}(f_\alpha)}{\sqrt{\mathrm{Var}(f_\alpha)}}\right)_\alpha$ is illustrated by Fig. 3. The five first images are normalized TDL realizations obtained from the same random colored leaves but with various transparency coefficients α. The last image is a realization of the limit Gaussian random field given by Theorem 1. Observe that this Gaussian field is also the limit of the normalized shot noise associated with X when the intensity of germs tends to infinity [11].

A Proof of Theorem 1

A.1 Notation and plan of the proof

Let $p \in \mathbb{N}$, $p \geq 2$, $y_1, \dots, y_p$ be p distinct points of $\mathbb{R}^d$ and $w_1, \dots, w_p$ be p non null real coefficients. It must be shown that the linear combination

$$\sum_{j=1}^{p} w_j \frac{f_\alpha(y_j) - \mathbb{E}(f_\alpha)}{\sqrt{\mathrm{Var}(f_\alpha)}}$$

converges in distribution when $\alpha \to 0$ to a Gaussian r.v. with mean 0 and variance V where

$$V := \sum_{j=1}^{p} \sum_{i=1}^{p} w_j w_i \frac{\gamma_X(y_i - y_j)}{\gamma_X(0)}. \tag{6}$$

Note that V is strictly positive since the covariogram γ_X is strictly positive-definite [8].

Let (α_n) be any sequence such that $\alpha_n \in (0,1]$ and $\alpha_n \to 0$. For all $n \in \mathbb{N}$, define

$$Y_n = \sum_{j=1}^{p} w_j f_{\alpha_n}(y_j).$$

With these new notation, the goal of the proof is to show that

$$\frac{Y_n - \mathbb{E}(Y_n)}{\sqrt{\mathrm{Var}(f_{\alpha_n})}} \xrightarrow[n \to +\infty]{\mathcal{D}} \mathcal{N}(0, V). \tag{7}$$

From the expression of the covariance of the TDL process (Proposition 4),

$$\mathrm{Var}(Y_n) = \mathrm{Var}(a) \sum_{j=1}^{p} \sum_{i=1}^{p} w_j w_i \frac{\alpha_n \gamma_X(y_i - y_j)}{2\gamma_X(0) - \alpha_n \gamma_X(y_i - y_j)} \underset{n \to +\infty}{\sim} \frac{V}{2} \mathrm{Var}(a) \alpha_n \tag{8}$$

and

$$\mathrm{Var}(f_{\alpha_n}) = \frac{\alpha_n}{2 - \alpha_n} \mathrm{Var}(a) \underset{n \to +\infty}{\sim} \frac{1}{2} \mathrm{Var}(a) \alpha_n.$$

Hence

$$\lim_{n \to +\infty} \frac{\mathrm{Var}(Y_n)}{\mathrm{Var}(f_{\alpha_n})} = V$$

and the normal convergence (7) is equivalent to

$$\frac{Y_n - \mathbb{E}(Y_n)}{\sqrt{\mathrm{Var}(Y_n)}} \xrightarrow[n \to +\infty]{\mathcal{D}} \mathcal{N}(0, 1). \tag{9}$$

Following the notation of Proposition 2, for each point y_j we order the sequence of leaves falling on y_j in anti-chronological order: $t(y_j, 0) > t(y_j, 1) > t(y_j, 2) > \dots$, and the intensities a_i associated to each falling time $t(y_j, k)$ are denoted by $a(y_j, k)$. Noting $\beta_n = (1 - \alpha_n)$ for all $n \in \mathbb{N}$, we have for all $j \in \{1, \dots, p\}$, $f_{\alpha_n}(y_j) = \sum_{k=0}^{+\infty} \alpha_n a(y_j, k) \beta_n^k$. Hence

$$Y_n = \sum_{j=1}^{p} w_j \left(\sum_{k=0}^{+\infty} \alpha_n a(y_j, k) \beta_n^k \right) = \sum_{k=0}^{+\infty} \alpha_n \left(\sum_{j=1}^{p} w_j a(y_j, k) \right) \beta_n^k.$$

To prove (9), we introduce an intermediary sequence (S_n) which is obtained from (Y_n) by truncating the intensities $a(y_j, k)$ and by restricting the summation over k to a finite number of terms. More precisely let us consider a sequence (A_n) such that $A_n > 0$ and $\lim_n A_n = +\infty$. For all $n \in \mathbb{N}$, let T_n be the truncation operator defined for all $b \in \mathbb{R}$ by

$$T_n(b) = \begin{cases} b & \text{if } b \in [-A_n, A_n], \\ A_n & \text{if } b > A_n, \\ -A_n & \text{if } b < -A_n. \end{cases}$$

Let us also consider a sequence of integers (N_n) such that $N_n \to +\infty$. Given the two sequences (A_n) and (N_n), for all $n \in \mathbb{N}$, $k \in \{0, \ldots, N_n\}$, define

$$Z_{n,k} = \alpha_n \left(\sum_{j=1}^{p} w_j T_n \left(a(y_j, k) \right) \right) \beta_n^k, \ n \in \mathbb{N}, \ k \in \{0, \ldots, N_n\}, \tag{10}$$

and

$$S_n = \sum_{k=0}^{N_n} Z_{n,k}.$$

The proof of (9), and thus of Theorem 1, consists in two main steps. First, for some well-chosen sequences (A_n) and (N_n), depending only on α_n, it will be shown that

$$\frac{S_n - \mathbb{E}(S_n)}{\sqrt{\mathrm{Var}(S_n)}} \xrightarrow{\mathcal{D}} \mathcal{N}(0,1)$$

(see Lemma 7). Second, it will be shown that, with the same sequences (A_n) and (N_n), we have

$$\frac{Y_n - \mathbb{E}(Y_n)}{\sqrt{\mathrm{Var}(Y_n)}} - \frac{S_n - \mathbb{E}(S_n)}{\sqrt{\mathrm{Var}(S_n)}} \xrightarrow{L^2} 0$$

(see Lemma 8). Thanks to Slutsky's theorem, these two results implies the normal convergence (9).

A.2 Normal convergence of the partial sums

We decompose the proof of the normal convergence of the sequence (S_n) in several lemmas. The intent of those technical lemmas is to show that a normal convergence theorem due to Janson [12, Theorem 2] applies for some well-chosen sequences (A_n) and (N_n). First let us recall Janson's theorem.

A.2.1 Janson's normal convergence theorem

We first recall the definition of dependency graph of a finite family of r.v.. A graph $\Gamma = (\{1, \ldots, N\}, \mathcal{E})$ with vertices $\{1, \ldots, N\}$ and edges $\mathcal{E}$ is a *dependency graph* for the finite family of r.v. $Z_1, \ldots, Z_N$ if for any pair of disjoint sets of vertices V_1 and $V_2 \subset \{1, \ldots, N\}$ such that no edge in $\mathcal{E}$ has one endpoint in V_1 and the other in V_2, the corresponding sets of random variables $\{Z_k, \ k \in V_1\}$ and $\{Z_k, \ k \in V_2\}$ are independent. Let us also recall that the *maximal degree* of a graph is the maximal number of edges incident to a single vertex.

Theorem 2 (Janson's normal convergence theorem [12])**.** *Suppose that for each $n \in \mathbb{N}$, $\{Z_{n,k}, \ k \in \{0, \ldots, N_n\}\}$, is a family of bounded r.v. and let $B_n > 0$ such that $|Z_{n,k}| \leq B_n$. Suppose further that $Z_{n,k}$ admits a dependency graph the maximal degree of which is less than $M_n \geq 1$. Let*

$$S_n = \sum_{k=0}^{N_n} Z_{n,k}.$$

If there exists an integer m such that

$$\frac{N_n^{\frac{1}{m}} M_n^{1-\frac{1}{m}} B_n}{\sqrt{\mathrm{Var}(S_n)}} \underset{n \to +\infty}{\longrightarrow} 0,$$

then the normalized sequence $\left(\dfrac{S_n - \mathbb{E}(S_n)}{\sqrt{\mathrm{Var}(S_n)}}\right)$ *converges in distribution to* $\mathcal{N}(0,1)$.

A.2.2 Maximal degree

Lemma 2. *For all* n*, the family of r.v.*

$$Z_{n,k} = \alpha_n \left(\sum_{j=1}^{p} w_j T_n \left(a(y_j, k) \right) \right) \beta_n^k, \; k \in \{0, \dots, N_n\},$$

admits a dependency graph which has a maximal degree inferior or equal to $p(p-1)$.

Proof. Let us define a graph $\Gamma_n = (\{1,\dots,N_n\}, \mathcal{E}_n)$, where $\mathcal{E}_n$ is the set of edges $\{k,l\}$, $k \neq l$, such that there exist two points y_j and y_i for which the intensities $a(y_j,k)$ and $a(y_i,l)$ correspond to the same leaf. In other words, the edge $\{k,l\}$ is in $\mathcal{E}_n$ as soon as the sums defining $Z_{n,k}$ and $Z_{n,l}$ involve one common r.v. a_i. Let us justify that Γ_n is a dependency graph for $Z_{n,k}$, $k \in \{0,\dots,N_n\}$. If V_1 and $V_2 \subset \{1,\dots,N\}$ are such that no edge in $\mathcal{E}_n$ has one endpoint in V_1 and the other in V_2, then the two sets of r.v. $\mathcal{A}_1 = \{a(y_j,k),\; j \in \{1,\dots,p\},\; k \in V_1\}$ and $\mathcal{A}_2 = \{a(y_j,k),\; j \in \{1,\dots,p\},\; k \in V_2\}$ correspond to two disjoint sets of leaves. But since the intensities a_i of the leaves form an i.i.d. sequence, the sets of r.v. $\mathcal{A}_1$ and $\mathcal{A}_2$ are independent. To conclude the proof let us now bound the maximal degree of Γ_n. Let $k \in \{1,\dots,N_n\}$. Let $j \in \{1,\dots,p\}$ and let us consider the r.v. $a(y_j,k)$. Since the leaf associated to $a(y_j,k)$ covers at most once each of the $p-1$ points $\{y_i,\; i \neq j\}$, the r.v. $a(y_j,k)$ appears at most in $p-1$ sums $Z_{n,l}$, $l \neq k$. Hence the r.v. $a(y_j,k)$ induces at most $p-1$ edges incident to the vertex k. Since $Z_{n,k}$ is the sum of p r.v. $a(y_j,k)$, the number of edges incident to k is less than $p(p-1)$. This is valid for all vertex k, and thus we deduce that the maximal degree of Γ_n is inferior or equal to $p(p-1)$.

□

A.2.3 Determining an equivalent of the sequence $(\mathrm{Var}\,(S_n))$

The goal of the following technical lemmas is to establish an equivalent of the sequence $(\mathrm{Var}\,(S_n))$.

Lemma 3 (An expression of $\mathrm{Var}\,(S_n)$)**.**

$$\mathrm{Var}(S_n) = \mathrm{Var}\left(T_n(a)\right) \sum_{j=1}^{p} \sum_{i=1}^{p} w_j w_i \left(\alpha_n^2 \sum_{k=0}^{N_n} \sum_{l=0}^{N_n} \beta_n^k \beta_n^l \mathbb{P}\left(t(y_j,k) = t(y_i,l)\right) \right).$$

Proof. Recall that

$$S_n = \sum_{k=0}^{N_n} Z_{n,k} = \sum_{k=0}^{N_n} \alpha_n \left(\sum_{j=1}^{p} w_j T_n \left(a(y_j,k) \right) \right) \beta_n^k.$$

By linearity we have

$$S_n - \mathbb{E}\left(S_n\right) = \alpha_n \sum_{k=0}^{N_n} \left(\sum_{j=1}^{p} w_j \left(T_n\left(a(y_j,k)\right) - \mathbb{E}\left(T_n(a)\right) \right) \right) \beta_n^k.$$

Hence

$$\operatorname{Var}(S_n) = \alpha_n^2 \sum_{k=0}^{N_n} \sum_{l=0}^{N_n} \sum_{j=1}^{p} \sum_{i=1}^{p} w_j w_i \beta_n^k \beta_n^l \operatorname{Cov}\left(T_n\left(a(y_j,k)\right), T_n\left(a(y_i,l)\right)\right).$$

Each r.v. a_i has distribution P_a and is independent of the others r.v. a_i. Hence, the two r.v. $T_n\left(a(y_j,k)\right)$ and $T_n\left(a(y_i,l)\right)$ are either identical or independent. Thus

$$\operatorname{Cov}\left(T_n\left(a(y_j,k)\right), T_n\left(a(y_i,l)\right)\right) = \operatorname{Var}(T_n(a))\mathbb{P}\left(t(y_j,k) = t(y_i,l)\right).$$

□

The next lemma computes the probability $\mathbb{P}\left(t(y,k) = t(z,l)\right)$, $k,l \in \mathbb{N}$, that is the probability that the $(k+1)$-th leaf covering y is the $(l+1)$-th leaf covering z.

Lemma 4 (Simultaneous covering)**.** *Let $y,z \in \mathbb{R}^d$ and note $\tau = z - y$. Let us note*

$$\kappa = \frac{\gamma_X(\tau)}{2\gamma_X(0) - \gamma_X(\tau)} \quad \text{and} \quad \omega = \frac{\gamma_X(0) - \gamma_X(\tau)}{2\gamma_X(0) - \gamma_X(\tau)}.$$

Then, for all $k,l \in \mathbb{N}$,

$$\mathbb{P}\left(t(y,k) = t(z,l)\right) = \kappa \sum_{q=0}^{\min(k,l)} \frac{(k+l-q)!}{q!(k-q)!(l-q)!} \kappa^q \omega^{k+l-2q}. \tag{11}$$

Proof. Formula (11) is proved by induction over $N = k + l$. For $N = 0$, as already shown by Equation (5) in the proof of Proposition 4 we have $\mathbb{P}\left(t(y,0) = t(z,0)\right) = \kappa$. Let $(k,l) \in \mathbb{N}^2$, $k + l \geq 1$. Like for the proof of Proposition 4, conditioning with respect to the coverage of the last leaf (t_0, x_0, X_0, a_0) which hits y or z we have

$$\begin{aligned}\mathbb{P}\left(t(y,k) = t(z,l)\right) = \mathbb{P}\left(t(y,k-1) = t(z,l-1)\right)\kappa \\ + \mathbb{P}\left(t(y,k-1) = t(z,l)\right)\omega \\ + \mathbb{P}\left(t(y,k) = t(z,l-1)\right)\omega\end{aligned}$$

(with the convention that if $k - 1 = -1$ or $l - 1 = -1$ the probability involving $t(y,k-1)$ or $t(z,l-1)$ is null, which is consistent with Formula (11)). Using the induction hypothesis followed by a change of index $q = q + 1$ we have

$$\begin{aligned}\mathbb{P}\left(t(y,k-1) = t(z,l-1)\right)\kappa &= \kappa \sum_{q=0}^{\min(k-1,l-1)} \frac{(k+l-q-2)!}{q!(k-q-1)!(l-q-1)!} \kappa^{q+1} \omega^{k+l-2q-2} \\ &= \kappa \sum_{q=1}^{\min(k,l)} \frac{(k+l-q-1)!}{(q-1)!(k-q)!(l-q)!} \kappa^q \omega^{k+l-2q} \\ &= \kappa \sum_{q=0}^{\min(k,l)} \frac{q(k+l-q-1)!}{q!(k-q)!(l-q)!} \kappa^q \omega^{k+l-2q},\end{aligned}$$

where in the last step a null term was added in the sum. Similarly, by the induction hypothesis and in possibly adding a null term in the sum

$$\begin{aligned}\mathbb{P}\left(t(y,k-1) = t(z,l)\right)\omega &= \kappa \sum_{q=0}^{\min(k-1,l)} \frac{(k+l-q-1)!}{q!(k-q-1)!(l-q)!} \kappa^q \omega^{k+l-2q} \\ &= \kappa \sum_{q=0}^{\min(k,l)} \frac{(k-q)(k+l-q-1)!}{q!(k-q)!(l-q)!} \kappa^q \omega^{k+l-2q},\end{aligned}$$

and by symmetry

$$\mathbb{P}\left(t(y,k)=t(z,l-1)\right)\omega=\kappa\sum_{q=0}^{\min(k,l)}\frac{(l-q)(k+l-q-1)!}{q!(k-q)!(l-q)!}\kappa^q\omega^{k+l-2q}.$$

The three obtained sums are similar except for their numerators. Summing them, on the numerator we have $[q+(k-q)+(l-q)]\,(k+l-q-1)!=(k+l-q)!$. Hence we obtain Formula (11), which completes the proof by induction. □

Lemma 5. *Let $y,z\in\mathbb{R}^d$ and $\tau=z-y$. Let $(K_n)_n$ and $(L_n)_n$ be two sequences of $\mathbb{N}\cup\{+\infty\}$ such that*

$$\alpha_nK_n\underset{n\to+\infty}{\longrightarrow}+\infty\quad\textit{and}\quad\alpha_nL_n\underset{n\to+\infty}{\longrightarrow}+\infty.$$

Define $\Gamma_n=\Gamma_n(y,z,(K_n),(L_n))$ to be the sequence

$$\Gamma_n=\alpha_n^2\sum_{k=0}^{K_n}\sum_{l=0}^{L_n}\beta_n^k\beta_n^l\,\mathbb{P}\left(t(y,k)=t(z,l)\right).$$

Then

$$\Gamma_n\underset{n\to+\infty}{\sim}\frac{\gamma_X(\tau)}{2\gamma_X(0)}\alpha_n.$$

Proof. Starting from Lemma 4 and changing the order of summation we have

$$\begin{aligned}\Gamma_n&=\alpha_n^2\sum_{k=0}^{K_n}\sum_{l=0}^{L_n}\beta_n^k\beta_n^l\kappa\sum_{q=0}^{\min(k,l)}\frac{(k+l-q)!}{q!(k-q)!(l-q)!}\kappa^q\omega^{k+l-2q}\\&=\alpha_n^2\kappa\sum_{q=0}^{\min(K_n,L_n)}\sum_{k=q}^{K_n}\sum_{l=q}^{L_n}\beta_n^k\beta_n^l\frac{(k+l-q)!}{q!(k-q)!(l-q)!}\kappa^q\omega^{k+l-2q}.\end{aligned}$$

Changing the indices in the triple sum in $r=k-q$, $s=l-q$ gives

$$\Gamma_n=\alpha_n^2\kappa\sum_{q=0}^{\min(K_n,L_n)}\sum_{r=0}^{K_n-q}\sum_{s=0}^{L_n-q}\frac{(r+s+q)!}{q!r!s!}\left(\beta_n^2\kappa\right)^q\left(\beta_n\omega\right)^r\left(\beta_n\omega\right)^s.$$

One recognizes the multinomial formula of order 3:

$$(\lambda_1+\lambda_2+\lambda_3)^\sigma=\sum_{\substack{k_1,k_2,k_3\\k_1+k_2+k_3=\sigma}}\frac{\sigma!}{k_1!k_2!k_3!}\lambda_1^{k_1}\lambda_2^{k_2}\lambda_3^{k_3}.$$

However the subdomain of $\mathbb{N}^3$

$$\mathcal{A}_n=\left\{(q,r,s)\in\mathbb{N}^3\,|q\in\{0,\dots,\min(K_n,L_n)\},\ r\in\{0,\dots,K_n-q\},\ s\in\{0,\dots,L_n-q\}\right\},$$

over which the sum is done is not well-adapted for the multinomial formula. Let us introduce

$$\mathcal{A}_n^-=\bigcup_{\sigma=0}^{\min(K_n,L_n)}\left\{(q,r,s)\in\mathbb{N}^3\,|q+r+s=\sigma\right\},$$

and

$$\mathcal{A}_n^+ = \bigcup_{\sigma=0}^{K_n+L_n} \left\{ (q,r,s) \in \mathbb{N}^3 \, | q+r+s=\sigma \right\}.$$

It is straightforward to observe that $\mathcal{A}_n^- \subset \mathcal{A}_n \subset \mathcal{A}_n^+$. Defining Γ_n^- and Γ_n^+ by

$$\Gamma_n^{\pm} = \alpha_n^2 \kappa \sum_{(q,r,s)\in\mathcal{A}_n^{\pm}} \frac{(r+s+q)!}{q!r!s!} \left(\beta_n^2\kappa\right)^q \left(\beta_n\omega\right)^r \left(\beta_n\omega\right)^s,$$

the inclusions $\mathcal{A}_n^- \subset \mathcal{A}_n \subset \mathcal{A}_n^+$ imply that $\Gamma_n^- \leq \Gamma_n \leq \Gamma_n^+$. The end of the proof consists in showing that both sequences $(\Gamma_n^-)_n$ and $(\Gamma_n^+)_n$ are equivalent to $\dfrac{\gamma_X(\tau)}{2\gamma_X(0)}\alpha_n$. We have

$$\begin{aligned}
\Gamma_n^- &= \alpha_n^2 \kappa \sum_{(q,r,s)\in\mathcal{A}_n^{\pm}} \frac{(r+s+q)!}{q!r!s!} \left(\beta_n^2\kappa\right)^q \left(\beta_n\omega\right)^r \left(\beta_n\omega\right)^s \\
&= \alpha_n^2 \kappa \sum_{\sigma=0}^{\min(K_n,L_n)} \left(\beta_n^2\kappa + 2\beta_n\omega\right)^\sigma \\
&= \alpha_n^2 \kappa \frac{1-\left(\beta_n^2\kappa + 2\beta_n\omega\right)^{\min(K_n,L_n)+1}}{1-\left(\beta_n^2\kappa + 2\beta_n\omega\right)}.
\end{aligned}$$

Using the expressions of κ, ω and $\beta_n = 1-\alpha_n$,

$$1-\left(\beta_n^2\kappa + 2\beta_n\omega\right) = \frac{\alpha_n\left(2\gamma_X(0)-\alpha_n\gamma(\tau)\right)}{2\gamma_X(0)-\gamma_X(\tau)}.$$

Hence

$$\Gamma_n^- = \frac{\alpha_n\gamma_X(\tau)}{2\gamma_X(0)-\alpha_n\gamma_X(\tau)} \left(1-\left(1-\frac{\alpha_n\left(2\gamma_X(0)-\alpha_n\gamma(\tau)\right)}{2\gamma_X(0)-\gamma_X(\tau)}\right)^{\min(K_n,L_n)+1}\right).$$

Recalling that by hypothesis we have $\lim\limits_{n\to+\infty} \alpha_n \min(K_n,L_n) = +\infty$, we obtain the announced equivalent for the sequence $(\Gamma_n^-)_n$:

$$\begin{aligned}
\Gamma_n^- &= \frac{\alpha_n\gamma_X(\tau)}{2\gamma_X(0)-\alpha_n\gamma_X(\tau)} \left(1-e^{(\min(K_n,L_n)+1)\ln\left(1-\alpha_n\frac{2\gamma_X(0)-\alpha_n\gamma(\tau)}{2\gamma_X(0)-\gamma_X(\tau)}\right)}\right) \\
&\underset{n\to+\infty}{\sim} \frac{\alpha_n\gamma_X(\tau)}{2\gamma_X(0)} \left(1-e^{-\min(K_n,L_n)\alpha_n\frac{2\gamma_X(0)}{2\gamma_X(0)-\gamma_X(\tau)}}\right) \\
&\underset{n\to+\infty}{\sim} \frac{\gamma_X(\tau)}{2\gamma_X(0)}\alpha_n.
\end{aligned}$$

Similarly, since $\lim\limits_{n\to+\infty} (K_n+L_n)\,\alpha_n = +\infty$ we have

$$\begin{aligned}
\Gamma_n^+ &= \frac{\alpha_n\gamma_X(\tau)}{2\gamma_X(0)-\alpha_n\gamma_X(\tau)} \left(1-\left(1-\frac{\alpha_n\left(2\gamma_X(0)-\alpha_n\gamma(\tau)\right)}{2\gamma_X(0)-\gamma_X(\tau)}\right)^{K_n+L_n+1}\right) \\
&\underset{n\to+\infty}{\sim} \frac{\gamma_X(\tau)}{2\gamma_X(0)}\alpha_n.
\end{aligned}$$

This concludes the proof. □

Lemma 6 (An equivalent of $\mathrm{Var}(S_n)$)**.** *Suppose that the sequences $(A_n)_n$ and $(N_n)_n$ satisfy*

$$\lim_{n\to+\infty} A_n = +\infty \quad \textit{and} \quad \lim_{n\to+\infty} \alpha_n N_n = +\infty.$$

Then

$$\mathrm{Var}(S_n) \underset{n\to+\infty}{\sim} \frac{V}{2}\,\mathrm{Var}(a)\alpha_n.$$

Proof. Using the expression of Lemma 3 and the notation of Lemma 5, we have

$$\mathrm{Var}(S_n) = \mathrm{Var}\left(T_n(a)\right)\sum_{j=1}^{p}\sum_{i=1}^{p} w_j w_i \Gamma_n(y_j, y_i, (N_n)_n, (N_n)_n).$$

Thus

$$\begin{aligned}\lim_{n\to+\infty}\frac{\mathrm{Var}(S_n)}{\alpha_n} &= \lim_{n\to+\infty}\mathrm{Var}\left(T_n(a)\right)\sum_{j=1}^{p}\sum_{i=1}^{p} w_j w_i \frac{\Gamma_n(y_j, y_i, (N_n)_n, (N_n)_n)}{\alpha_n}\\ &= \mathrm{Var}(a)\sum_{j=1}^{p}\sum_{i=1}^{p} w_j w_i \frac{\gamma_X(y_j - y_i)}{2\gamma_X(0)} = \frac{V}{2}\,\mathrm{Var}(a).\end{aligned}$$

□

A.2.4 Applying Janson's theorem

In the following $\lfloor x \rfloor$ denotes the integral part of a real x.

Lemma 7 (Normal convergence of the partial sums)**.** *Let s_1 and s_2 be real numbers satisfying $s_1 \in (0, 1/2)$ and $s_2 > 1$. For all $n \in \mathbb{N}$ define*

$$A_n = \alpha_n^{-s_1} \quad \textit{and} \quad N_n = \left\lfloor \alpha_n^{-s_2} \right\rfloor.$$

Then the corresponding sequence $\left(\dfrac{S_n - \mathbb{E}\left(S_n\right)}{\sqrt{\mathrm{Var}\left(S_n\right)}}\right)$ converges in distribution to $\mathcal{N}(0,1)$.

Proof. This lemma is proved by applying Janson's theorem [12], reproduced above as Theorem 2. For all $n \in \mathbb{N}$ and $k \in \{0, \ldots, N_n\}$, we have $|T_n\left(a(y_j, k)\right)| \leq A_n$, hence

$$|Z_{n,k}| \leq \alpha_n \left(\sum_{j=1}^{p} |w_j|\right) A_n \beta_n^k \leq \left(\sum_{j=1}^{p} |w_j|\right) A_n \alpha_n.$$

According to Lemma 2, for all $n \in \mathbb{N}$ the family $(Z_{n,k})_{k=0,\ldots,N_n}$ admits a dependency graph which has a maximal degree inferior or equal to $p(p-1)$. Hence Janson's theorem should be applied with $N_n = \lfloor \alpha_n^{-s_2} \rfloor$, $M_n = p(p-1)$, and $B_n = \left(\sum_{j=1}^{p} |w_j|\right) A_n \alpha_n$. Since $s_1 > 0$ and $s_2 > 1$, we have $\lim_n A_n = +\infty$ and $\lim_n \alpha_n N_n = +\infty$, and thus Lemma 6 ensures that

$$\mathrm{Var}(S_n) \underset{n\to+\infty}{\sim} \frac{V}{2}\,\mathrm{Var}(a)\alpha_n.$$

Combining all these results, for all integers m we have

$$\frac{N_n^{\frac{1}{m}} M_n^{1-\frac{1}{m}} B_n}{\sqrt{\mathrm{Var}\left(S_n\right)}} \underset{n\to+\infty}{\sim} C\alpha_n^{-\frac{s_2}{m}} \alpha_n^{1-s_1} \alpha_n^{-\frac{1}{2}} \underset{n\to+\infty}{\sim} C\alpha_n^{\frac{1}{2}-s_1-\frac{s_2}{m}},$$

where C is a constant independent of n. Since $\frac{1}{2} - s_1 > 0$, there exists an integer $m \geq 3$ large enough so that $\frac{1}{2} - s_1 - \frac{s_2}{m} > 0$. For this value of m we get

$$\frac{N_n^{\frac{1}{m}} M_n^{1-\frac{1}{m}} B_n}{\sqrt{\mathrm{Var}\,(S_n)}} \underset{n\to+\infty}{\longrightarrow} 0.$$

Hence Janson's theorem applies and ensures the announced normal convergence. □

Remark 4. *Note that the choice of values of (A_n) and (N_n) depends only on the sequence (α_n). Besides, in the above proof one can choose s_1 and s_2 such that $\frac{1}{2} - s_1 - \frac{s_2}{m} > 0$ with $m = 3$ (take $(s_1, s_2) = \left(\frac{1}{9}, \frac{10}{9}\right)$ for example). Hence there is no real need to use Janson's result on normal convergence by higher cumulants [12, Theorem 1]. The key result here is the upper bound of semi-invariants for sums of random variables admitting a dependency graph with a known maximal degree [12, Lemma 4].*

A.3 Convergence in L^2 of the difference of the normalized sequences

Lemma 8. *Suppose that the sequences $(A_n)_n$ and $(N_n)_n$ satisfy*

$$\lim_{n\to+\infty} A_n = +\infty \quad \textit{and} \quad \lim_{n\to+\infty} \alpha_n N_n = +\infty.$$

Then

$$\frac{Y_n - \mathbb{E}\,(Y_n)}{\sqrt{\mathrm{Var}\,(Y_n)}} - \frac{S_n - \mathbb{E}\,(S_n)}{\sqrt{\mathrm{Var}\,(S_n)}} \xrightarrow{L^2} 0.$$

Proof. Since

$$\mathbb{E}\left(\left(\frac{Y_n - \mathbb{E}\,(Y_n)}{\sqrt{\mathrm{Var}\,(Y_n)}} - \frac{S_n - \mathbb{E}\,(S_n)}{\sqrt{\mathrm{Var}\,(S_n)}}\right)^2\right) = 2 - 2\frac{\mathrm{Cov}(Y_n, S_n)}{\sqrt{\mathrm{Var}\,(Y_n)}\sqrt{\mathrm{Var}\,(S_n)}}$$

it is equivalent to show that

$$\mathrm{Cov}(Y_n, S_n) \underset{n\to+\infty}{\sim} \sqrt{\mathrm{Var}\,(Y_n)}\sqrt{\mathrm{Var}\,(S_n)}.$$

Equation (8) shows that $\mathrm{Var}\,(Y_n) \underset{n\to+\infty}{\sim} \frac{V}{2}\mathrm{Var}(a)\alpha_n$, where the constant V is defined in (6). Besides Lemma 6 ensures that $\mathrm{Var}\,(S_n) \underset{n\to+\infty}{\sim} \frac{V}{2}\mathrm{Var}(a)\alpha_n$. As for computing an equivalent of $\mathrm{Cov}(Y_n, S_n)$, this is done using techniques similar to the ones of the proof of Lemma 6. First, observe that

$$\mathrm{Cov}(Y_n, S_n) = \mathrm{Cov}(a, T_n(a)) \sum_{j=1}^{p}\sum_{i=1}^{p} w_j w_i \sum_{k=0}^{+\infty}\sum_{l=0}^{N_n} \alpha_n^2 \beta_n^k \beta_n^l \mathbb{P}\left(t(y,k) = t(z,l)\right).$$

Using the notation of Lemma 5 we get

$$\mathrm{Cov}(Y_n, S_n) = \mathrm{Cov}\left(a, T_n(a)\right) \sum_{j=1}^{p}\sum_{i=1}^{p} w_j w_i \Gamma_n(y_j, y_i, (+\infty)_n, (N_n)_n).$$

According to Lemma 5, $\lim_{n\to 0} \frac{1}{\alpha_n}\Gamma_n(y_j, y_i, (+\infty)_n, (N_n)_n) = \frac{\gamma_X(y_j - y_i)}{2\gamma_X(0)}$. Thus

$$\lim_{n\to 0} \frac{1}{\alpha_n} \operatorname{Cov}(Y_n, S_n) = \frac{V}{2} \operatorname{Var}(a).$$

This shows that $\operatorname{Cov}(Y_n, S_n)$ and $\sqrt{\operatorname{Var}(Y_n)}\sqrt{\operatorname{Var}(S_n)}$ are equivalent and concludes the proof. □

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