# On a novel expression of the field scattered by an arbitrary constant impedance plane 

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Summary :
The electromagnetic field scattered by an impedance plane is generally given by its plane wave expansion (Fourier representation). Here we derive an alternative expression which is more suitable for point source illumination. For this, we consider an original expression of the Hertz potentials for the incident field and express the scattered potentials in a novel form. A special function then involved can be expressed in an integral which is in turn expanded in a convergent series. The expression presented also permits us to express complete asymptotics. Our development considers an arbitrary impedance, passive or active.
keywords : scattering, impedance plane, dipole, electromagnetism, analytical expression, passive or active material.

## 1) Introduction

In a recent paper [1], we considered the field scattered by a structure composed of several homogeneous and planar layers on a perfectly reflecting plane. We here apply our results to the particular case of an arbitrary constant impedance plane (passive or active) in the presence of bounded sources in electromagnetism. The impedance boundary conditions, that involve first order normal derivatives, are commonly used to model the scattering by imperfectly conducting objects of different geometries, such as an impedance cone studied in [2]. Concerning the impedance plane, the scattered field is generally given by its plane wave (Fourier representation) or Fourier-Bessel expansion [3]-[7], while other methods [8]-[13], developed for passive cases (for example the complex image method), are generally limited because of the presence of a branch cut as the impedance is active. In this paper, we give another approach to express the field for any impedance choice, which is more suitable for illumination by an arbitrary bounded source than the Fourier or Fourier-Bessel representations. Moreover, we give an original expression of the Hertz potentials for the incident field and express the scattered potential in a novel form.

In practice, the Fourier expansion is suitable in far field or for simple plane wave illuminations, but is particularly complex to use for non-plane, in particular spherical

[^0]incident waves near the scatterer. Indeed, even if double Fourier integrals can be reduced to simple Fourier-Bessel integrals for point source illumination in 3D, numerical integration is quite lengthy because of the highly oscillatory nature of the integral and the calculus of Bessel functions. Moreover, in far field, the steepest descent method (or saddle point method) that is currently used for this integral [3]-[7], leads us to an expansion that is not strictly convergent but asymptotic, and poles of the reflection coefficient near the steepest descent path can greatly complicate the calculus.
The analytic method developed here can be applied to the determination of coupling between antennas above an imperfectly reflective plane, or to the computation of Green's functions for planar lines printed on a multilayer. Other methods restricted to the passive case exist. So, for the radiation of a point source in 3D above a homogenous passive halfspace or a passive impedance plane in electromagnetism and in acoustics, we can notice [3]-[6] and [8]-[17], while for the determination of Green's functions for planar lines using asymptotics, we recommend the reading of [7].
The paper is organized as follows. The section 2 is devoted to novel expressions of the potentials fields for the impedance case. In section 2.1, we give a brief discussion on the representation of the field with potentials, and on boundary conditions. In section 2.2, we describe a novel form of the potentials attached to the field radiated by arbitrary bounded sources, equally valid for horizontal and vertical dipoles composing the sources. Then in section 2.3, we give a new compact expression of the potentials for the field scattered by an impedance plane for arbitrary sources. In section 3, we present convergent and asymptotic expansions of a special function involved in the development.

## 2) A novel expression of the field for arbitrary bounded sources above a passive or active impedance plane

## 2.1) Formulation of the problem

We consider the scattering by an imperfectly reflective plane when it is illuminated by the field radiated by a bounded source, composed of arbitrary electrical and magnetic currents $J$ and $M$ (see fig. 1). The plane is defined by $z=0$ in Cartesian coordinates $(x, y, z)$. A harmonic time dependence $e^{i \omega t}$, from now on assumed, is suppressed throughout. Each component of the scattered field is assumed to be regular in the domain $z>0$, and $O\left(e^{-\gamma|O P|}\right)$ at $P(x, y, z), \gamma>0$, as $z$ or $\rho \rightarrow \infty$, when $|\arg (i k)|<\pi / 2$. Following Harrington [18, p.131] in 1961 (see also Jones [3, p.19] in 1964), we can write the electric field $E$ and the magnetic field $H$ satisfying the Maxwell equations, with two
scalar potentials $\mathcal{E}$ and $\mathcal{H}$, following

$$
\begin{align*}
& E=-i k \operatorname{curl}(\mathcal{H} \widehat{z})+\left(\operatorname{grad}(\operatorname{div}(.))+k^{2}\right)(\mathcal{E} \widehat{z}) \\
& \sqrt{\frac{\mu_{0}}{\epsilon_{0}}} H=i k \operatorname{curl}(\mathcal{E} \widehat{z})+\left(\operatorname{grad}(\operatorname{div}(.))+k^{2}\right)(\mathcal{H} \widehat{z}) \tag{1}
\end{align*}
$$

where $\left(\Delta+k^{2}\right) \mathcal{E}=0$ and $\left(\Delta+k^{2}\right) \mathcal{H}=0$ outside the sources of radiation (i.e. outside $J, M$, and the scatterer), with $k=\omega \sqrt{\mu_{0} \epsilon_{0}}$, the constants $\epsilon_{0}$ and $\mu_{0}$ being respectively the permittivity and the permeability of the medium above the plane, $|\arg (i k)| \leq \pi / 2$. Following the theory of this representation, the constant vector $\widehat{z}$ can be chosen regardless of the sources, and $\mathcal{E}($ or $\mathcal{H}) \equiv e^{ \pm i k z}$ has no influence on $(E, H)$. Thereafter, we denote $\left(\mathcal{E}_{\text {inc }}, \mathcal{H}_{\text {inc }}\right)$ and $\left(\mathcal{E}_{s}, \mathcal{H}_{s}\right)$ the potentials corresponding to the incident field (incoming wave) and the scattered field (outgoing wave), and write (1) in the compact form $\left(E, \sqrt{\frac{\mu_{0}}{\epsilon_{0}}} H\right)=\mathcal{L}(\widehat{z} \mathcal{E}, \widehat{z} \mathcal{H})$.

figure 1) geometry : sources $(J, M)$ and observation point above the plane $z=0$

In [1], we considered the class of multimode boundary conditions on an isotropic plane,

$$
\begin{equation*}
\left.\prod_{j=1}^{N}\left(\frac{\partial}{\partial z}-i k g_{j}^{e}\right) E_{z, t o t}\right|_{z=0}=0,\left.\prod_{j=1}^{P}\left(\frac{\partial}{\partial z}-i k g_{j}^{h}\right) H_{z, t o t}\right|_{z=0}=0 \tag{2}
\end{equation*}
$$

which corresponds to the reflection coefficients of a plane wave for the principal polarizations $T M$ (components of electric field $E$ in the plane of incidence) and $T E$ (components of magnetic field $H$ in the plane of incidence),

$$
\begin{equation*}
R_{T M}(\beta)=\prod_{j=1}^{N} \frac{\cos \beta-g_{j}^{e}}{\cos \beta+g_{j}^{e}}, R_{T E}(\beta)=\prod_{j=1}^{P} \frac{\cos \beta-g_{j}^{h}}{\cos \beta+g_{j}^{h}} \tag{3}
\end{equation*}
$$

where $\beta$ is the angle of incidence with the normal $\widehat{z}[1], g_{j}^{(e, h)}$ are complex constants, $N$ and $P$ are two positive numbers. This class of problem corresponds to the reflection by a multilayer, composed of isotropic media, or more generally, of uniaxial anisotropic media with the principal axis along $z$, backed with a perfectly reflective plane. From the symmetry at normal incidence, we notice that the condition $R_{T E}(0)=-R_{T M}(0)$ has also to be satisfied. This implies that $g_{1}^{e}=1 / g_{1}^{h}$ for monomode conditions, when $N=P=1$.
We now restrict ourselves to this latter case, commonly named the impedance case, and take an arbitrary complex number $g_{1}^{e}=g^{e}$, corresponding to a passive ( $\operatorname{Re} g^{e}>0$ ) or active $\left(\operatorname{Re} g^{e}<0\right)$ plane. Using

$$
\begin{equation*}
E_{z}=\frac{\partial^{2} \mathcal{E}}{\partial z^{2}}+k^{2} \mathcal{E}, \quad \sqrt{\frac{\mu_{0}}{\epsilon_{0}}} H_{z}=\frac{\partial^{2} \mathcal{H}}{\partial z^{2}}+k^{2} \mathcal{H} \tag{4}
\end{equation*}
$$

we will search the potentials $\mathcal{E}_{s}$ and $\mathcal{H}_{s}$, satisfying the Helmholtz equation as $z>0$, regular and vanishing as $z \rightarrow \infty$ when $|\arg (i k)|<\pi / 2$, that verify

$$
\begin{equation*}
\left.\left(\frac{\partial}{\partial z}-i k g^{e}\right)\left(\mathcal{E}_{s}+\mathcal{E}_{i n c}\right)\right|_{z=0}=0,\left.\left(\frac{\partial}{\partial z}-i k g^{h}\right)\left(\mathcal{H}_{s}+\mathcal{H}_{\text {inc }}\right)\right|_{z=0}=0 \tag{5}
\end{equation*}
$$

where $g^{h}=1 / g^{e}$. Therefore, we need first a correct definition of $\left(\mathcal{E}_{\text {inc }}, \mathcal{H}_{\text {inc }}\right)$.

## Remark 1 :

The boundary conditions on $z$-components can be simply deduced from boundary conditions on the tangential components perpendicular to the plane of incidence in the case of plane wave illumination, by derivating them along the line which is the intersection of the plane of incidence and the plane $z=0$.
remark 2 :
For a more general case where a line of discontinuity in the characteristics of the plane is present, contact conditions along this line (valid asymptotically for small layers) can be necessary to insure uniqueness. This point has been discussed independently by the author in [19, sect. 3]-[20] (electromagnetism) and Tuzhilin in [21] (acoustics).

## 2.2) An expression of $\left(\mathcal{E}_{i n c}, \mathcal{H}_{i n c}\right)$ for bounded sources $J$ and $M$

Let us consider the incident field $(E, H)$ at $r$ of coordinates $(x, y, z)$, radiated by arbitrary electric and magnetic bounded sources $J$ and $M$ [3],

$$
\begin{align*}
& E=\operatorname{curl}(G * M)+\frac{i}{\omega \epsilon_{0}}\left(\operatorname{grad}(\operatorname{div}(.))+k^{2}\right)(G * J) \\
& \sqrt{\frac{\mu_{0}}{\epsilon_{0}}} H=-\sqrt{\frac{\mu_{0}}{\epsilon_{0}}} \operatorname{curl}(G * J)+\frac{i}{k}\left(\operatorname{grad}(\operatorname{div}(.))+k^{2}\right)(G * M) \tag{6}
\end{align*}
$$

where $G(r)=-\frac{e^{-i k|r(x, y, z)|}}{4 \pi|r(x, y, z)|}$ with $|r|=\sqrt{x^{2}+y^{2}+z^{2}}$, and $*$ is the convolution product. The potentials $\left(\mathcal{E}_{\text {inc }}, \mathcal{H}_{\text {inc }}\right)$ for this field, satisfying the Helmholtz equation as $\pm z>0$, and vanishing at infinity when $|\arg (i k)|<\pi / 2$ as $\pm z \rightarrow \infty$, have a definite expression, particularly compact, that we develop in [1] for arbitrary sources. So, for $J$ and $M$ in the domain $\mp z>0$, we have as $\pm z>0$,

$$
\begin{align*}
& \left(\mathcal{E}_{\text {inc }}, \mathcal{H}_{\text {inc }}\right)=\frac{\widehat{z}}{8 \pi k^{2}}\left(\sqrt{\frac{\mu_{0}}{\epsilon_{0}}}\left(\operatorname{grad}(\operatorname{div}(J))+k^{2} J, i k \operatorname{curl}(J)\right)+\right. \\
& \left.+\left(-i k \operatorname{curl}(M), \operatorname{grad}(\operatorname{div}(M))+k^{2} M\right)\right) * \mathcal{W}=\frac{\widehat{z}}{8 \pi k^{2}} \mathcal{L}\left(\sqrt{\frac{\mu_{0}}{\epsilon_{0}}} J, M\right) * \mathcal{W} \tag{7}
\end{align*}
$$

where

$$
\begin{equation*}
\mathcal{W}(r)=\left(e^{i k|z|} E_{1}(i k(|r|+|z|))+e^{-i k|z|}\left(E_{1}(i k(|r|-|z|))+2 \ln \rho\right)\right) \tag{8}
\end{equation*}
$$

with $\rho=\sqrt{x^{2}+y^{2}}, E_{1}$ being the exponential integral function [23], and the notation $(A, B) * C \equiv(A * C, B * C)$. The reader can verify by inspection our expression, noticing that the conditions

$$
\begin{equation*}
\left(\frac{\partial^{2}}{\partial z^{2}}+k^{2}\right) \frac{\mathcal{W}(r)}{8 \pi i k}=G(r),\left(\Delta+k^{2}\right) \mathcal{W}(r)=0 \tag{9}
\end{equation*}
$$

are satisfied when $\pm z>0$, and that all derivatives of $\mathcal{W}$ are regular in these domains.

## 2.3) Expression of the potentials $\left(\mathcal{E}_{s}, \mathcal{H}_{s}\right)$ for an impedance plane

Using our expression of $\left(\mathcal{E}_{\text {inc }}, \mathcal{H}_{\text {inc }}\right)$ for the radiation of $J$ and $M$, we can express the potentials $\mathcal{E}_{s}$ and $\mathcal{H}_{s}$ which satisfy the impedance boundary conditions (5), from the method developed in [1]. So, letting $N=P=1$ in [1, prop. (5.2)], we obtain, as $z \geq 0$,

$$
\begin{align*}
& \mathcal{E}_{s}(x, y, z)=\mathcal{E}_{i n c}(x, y,-z)+\left(\left(\frac{\widehat{z}}{\omega \epsilon_{0}} \frac{\operatorname{grad}(\operatorname{div}(J))+k^{2} J}{8 \pi k}+\right.\right. \\
& \left.\left.+\frac{\widehat{z}}{k} \frac{(-i k \operatorname{curl}(M))}{8 \pi k}\right) * \sum_{\epsilon^{\prime}=-1,1} \frac{a^{e}}{\left(g^{e}-\epsilon^{\prime}\right)}\left(\mathcal{V}_{\epsilon^{\prime}}+\epsilon^{\prime} \mathcal{K}_{g^{e}}\right)\right)(x, y,-z) \\
& =\mathcal{E}_{\text {inc }}(x, y,-z)+\left(\left(\frac{\widehat{z}}{\omega \epsilon_{0}} \frac{\operatorname{grad}(\operatorname{div}(J))+k^{2} J}{8 \pi k}+\frac{\widehat{z}}{k} \frac{(-i k \operatorname{curl}(M))}{8 \pi k}\right) *\right. \\
& \left.* \sum_{\epsilon^{\prime}=-1,1}\left(\left(\frac{\epsilon^{\prime}+g^{e}}{\epsilon^{\prime}-g^{e}}-1\right) \mathcal{V}_{\epsilon^{\prime}}+\frac{\epsilon^{\prime} a^{e} \mathcal{K}_{g^{e}}}{\left(g^{e}-\epsilon^{\prime}\right)}\right)\right)(x, y,-z) \tag{10}
\end{align*}
$$

and

$$
\begin{align*}
& \mathcal{H}_{s}(x, y, z)=\mathcal{H}_{i n c}(x, y,-z)+\left(\left(\frac{\widehat{z}}{\omega \epsilon_{0}} \frac{(i k \operatorname{curl}(J))}{8 \pi k}\right.\right. \\
& \left.\left.+\frac{\widehat{z}}{k} \frac{\left(\operatorname{grad}(\operatorname{div}(M))+k^{2} M\right)}{8 \pi k}\right) * \sum_{\epsilon^{\prime}=-1,1} \frac{a^{h}}{\left(g^{h}-\epsilon^{\prime}\right)}\left(\mathcal{V}_{\epsilon^{\prime}}+\epsilon^{\prime} \mathcal{K}_{g^{h}}\right)\right)(x, y,-z) \\
& =-\mathcal{H}_{i n c}(x, y,-z)+\left(\left(\frac{\widehat{z}}{\omega \epsilon_{0}} \frac{(i k \operatorname{curl}(J))}{8 \pi k}+\frac{\widehat{z}}{k} \frac{\left(\operatorname{grad}(\operatorname{div}(M))+k^{2} M\right)}{8 \pi k}\right) *\right. \\
& \left.* \sum_{\epsilon^{\prime}=-1,1}\left(\left(\frac{\epsilon^{\prime}+g^{h}}{\epsilon^{\prime}-g^{h}}+1\right) \mathcal{V}_{\epsilon^{\prime}}+\frac{\epsilon^{\prime} a^{h} \mathcal{K}_{g^{h}}}{\left(g^{h}-\epsilon^{\prime}\right)}\right)\right)(x, y,-z) \tag{11}
\end{align*}
$$

where $g^{e}=1 / g^{h}, a^{e, h}=-2 g^{e, h}$. In these expressions, the functions $\mathcal{V}_{\epsilon^{\prime}}, \mathcal{K}_{g}$ satisfy

$$
\begin{align*}
& \mathcal{V}_{\epsilon^{\prime}}(x, y, z)=e^{-\epsilon^{\prime} i k z}\left(E_{1}\left(i k\left(|r|-\epsilon^{\prime} z\right)\right)+\left(1-\epsilon^{\prime}\right) \ln \rho\right), \\
& \mathcal{K}_{g}(x, y, z)=e^{-i k g z} \mathcal{J}_{g}(\rho, z) \tag{12}
\end{align*}
$$

where $\rho=\sqrt{x^{2}+y^{2}}, g=g^{e}$ or $g=g^{h}$, and $\mathcal{J}_{g}(\rho,-z)$ is given by the integral

$$
\begin{equation*}
\mathcal{J}_{g}(\rho,-z)=\frac{e^{-i k g z}}{2} \int_{\mathcal{D}} \frac{H_{0}^{(2)}(k \rho \sin \beta) e^{-i k z \cos \beta}}{\cos \beta+g} \sin \beta d \beta \tag{13}
\end{equation*}
$$

with $\operatorname{Re}(i k \sin \beta)=0$ on $\mathcal{D}$ from $-i \infty-\arg (i k)$ to $i \infty+\arg (i k)$, which is a FourierBessel integral commonly encountered in scattering theory [5, p.234], also called a Sommerfeld-type integral [20]. Letting $g=\sin \theta_{1}$ with $\left|\operatorname{Re}\left(\theta_{1}\right)\right| \leq \pi / 2$, we notice the cut $\operatorname{Re}\left(i k \cos \theta_{1}\right)=0$ in active case $(\operatorname{Re} g<0)$, which is due to poles of $(\cos \beta+g)^{-1}$ that can go through $\mathcal{D}$. Thus, as discussed in [1], the transformations that improve the calculus in passive case $(\operatorname{Re} g>0)$ without considering the branch cut, can be wrong in active case. To avoid this restriction, we develop in [1] a novel expression for arbitrary $g=\sin \theta_{1}$ with $\left|\operatorname{Re}\left(\theta_{1}\right)\right| \leq \pi / 2$ as $|\arg (i k)| \leq \pi / 2$,

$$
\begin{equation*}
\mathcal{J}_{g}(\rho,-z)=i \int_{b}^{i \infty} e^{-a \cos \alpha} d \alpha \tag{14}
\end{equation*}
$$

where the parameters $a$ and $b$, with $|\operatorname{Re} b| \leq \pi, \operatorname{Re} a>0$, are defined following,

$$
\begin{align*}
& e^{\mp i b}=\frac{i k R}{a}\left(1 \pm \sin \theta_{1}\right)(1 \pm \cos \varphi), a=\epsilon i k R \sin \varphi \cos \theta_{1} \\
& -i a \sin b=i k R\left(\cos \varphi+\sin \theta_{1}\right), a \cos b=i k R\left(1+\sin \theta_{1} \cos \varphi\right) \tag{15}
\end{align*}
$$

and $z=R \cos \varphi, \rho=R \sin \varphi, R=\sqrt{\rho^{2}+z^{2}}, \epsilon=\operatorname{sign}\left(\operatorname{Re}\left(i k R \sin \varphi \cos \theta_{1}\right)\right)(\operatorname{Re}(a)=0$ being a limit case), $0<\varphi<\pi / 2$. So defined, we remark that $\operatorname{sign}(\operatorname{Re} b)=-\epsilon \operatorname{sign}\left(\operatorname{Im}\left(\sin \theta_{1}\right)\right)$, and $\operatorname{sign}(\operatorname{Im} a)=-\operatorname{sign}(\arg (i k))$ when $\epsilon=-1$.
We notice that, as $g$ varies in the complex plane, this expression has a correct cut as $\epsilon$ changes of sign for $\operatorname{Re} g \leq 0$, is singular for $g=-1$, and is regular elsewhere (note: for $\operatorname{Re} g>0$, the change of sign of $\epsilon$ does not induce a cut as $g$ varies).
Let us remark that this integral was also given in [12] for passive impedance case but it was with a definition of parameters which restricts its application (see details in section 2 of [1]). It is intimately related to the incomplete cylindrical function in the Poisson form [24] and to the leaky aquifer function (see $[25,26]$ and appendix B).
remark 3 :
The reader can verify by inspection that,

$$
\frac{\partial \mathcal{J}_{g}(\rho,-z)}{\partial z}=\frac{e^{-i k(R+g z)}}{R},\left(\Delta+k^{2}\right)\left(e^{i k g z} \mathcal{J}_{g}(\rho,-z)\right)=0
$$

when $z>0$.

## 3) A new expansion of $\mathcal{J}_{g}$

Several original expansions of $\mathcal{J}_{g}$ have been developed for an arbitrary complex $g$ in [1] (see appendix C) when multimode boundary conditions are considered. We present here a series of a new type. Using the error function, the expansion has both the properties to be convergent and to give the asymptotics.

## 3.1) A convergent expansion of $\mathcal{J}_{g}$ which also gives the asymptotics

For this, we consider the definition of $\epsilon$ given in (15), and thus, the following property,

$$
\begin{equation*}
\left|\tan ^{2}\left(\frac{b}{2}\right)\right|^{\epsilon}=\left|\frac{\cos b-\epsilon}{\cos b+\epsilon}\right|=\left|\frac{1+\sin \left(\theta_{1}-\varphi\right)}{1+\sin \left(\theta_{1}+\varphi\right)}\right|=\left|\frac{e^{\operatorname{Im} \theta_{1}} e^{i\left(\frac{\pi}{2}-\operatorname{Re} \theta_{1}\right)}+e^{-i \varphi}}{e^{\operatorname{Im} \theta_{1}} e^{i\left(\frac{\pi}{2}-\operatorname{Re} \theta_{1}\right)}+e^{i \varphi}}\right|^{2} \leq 1 \tag{16}
\end{equation*}
$$

in order to expand the expression (14) of $\mathcal{J}_{g}$, with $|R e b| \leq \pi$, $\operatorname{Re} a>0$. Let us notice that, from this inequality, $\frac{1-\epsilon}{4} \pi \leq|\operatorname{Re} b| \leq \frac{3-\epsilon}{4} \pi$.
After some particular development given in appendix A, we can write: if $\epsilon=1$,

$$
\begin{equation*}
\mathcal{J}_{g}(\rho,-z)=i\left(\int_{0}^{i \infty} e^{-a \cos \alpha} d \alpha-\int_{0}^{b} e^{-a \cos \alpha} d \alpha\right)=-\left(K_{0}(a)+\mathcal{M}(a, b)\right) \tag{17}
\end{equation*}
$$

where,

$$
\begin{align*}
& \mathcal{M}(a, b)=\left.i e^{-a \cos b} \tan \left(\frac{b}{2}\right) \sum_{m \geq 0} c_{m}\left(\tan \left(\frac{b}{2}\right)\right)^{2 m} \frac{\partial^{m}}{\partial s^{m}}\left(\frac{\sqrt{\pi} e^{s}}{s^{1 / 2}} \operatorname{erf}\left(s^{1 / 2}\right)\right)\right|_{s=-2 a \sin ^{2}\left(\frac{b}{2}\right)} \\
& =i e^{-a}\left(\left[\tan \left(\frac{b}{2}\right) \sum_{m \geq 0} c_{m}\left(\tan \left(\frac{b}{2}\right)\right)^{2 m}\left(W_{m}\left(-2 a \sin ^{2}\left(\frac{b}{2}\right)\right)-W_{m}(0)\right)\right]+b\right) \tag{18}
\end{align*}
$$

while, if $\epsilon=-1$, with $v=\operatorname{sign}(\operatorname{Re} b)$,

$$
\begin{align*}
& \mathcal{J}_{g}(\rho,-z)=-i\left(\int_{v \pi}^{b} e^{-a \cos \alpha} d \alpha-\int_{v \pi}^{0} e^{-a \cos \alpha} d \alpha-\int_{0}^{i \infty} e^{-a \cos \alpha} d \alpha\right) \\
& =-\left(\mathcal{N}(a, b)+i \pi v I_{0}(a)+K_{0}(a)\right) \\
& =-\left(\left(\mathcal{N}(a, b)-v \operatorname{sign}(\arg (i k)) K_{0}(-a)\right)+(1+v \operatorname{sign}(\arg (i k))) K_{0}(a)\right) \tag{19}
\end{align*}
$$

where $\mathcal{N}(a, b)=\mathcal{M}(-a, b-v \pi)$.
In these expressions, $c_{m}=(-1)^{m}(2 m)!/\left((m!)^{2} 2^{2 m}\right)$ is the binomial coefficient for the function $(1+t)^{-1 / 2}, W_{m}(s)=\frac{e^{-s} \partial^{m}}{\partial s^{m}}\left(\frac{e^{s} \operatorname{erf}\left(s^{1 / 2}\right)}{s^{1 / 2}}\right)$ and erf is the error function [23]. We have used that the modified Bessel functions $I_{0}$ and $K_{0}$ [23] satisfies $\operatorname{sign}(\operatorname{Im} a) i \pi I_{0}(a)=K_{0}(-a)-K_{0}(a)$ and that $\operatorname{sign}(\operatorname{Im} a)=-\operatorname{sign}(\arg (i k))$ when $\epsilon=-1$.
Concerning asymptotics when $a$ (and thus if $k R$ ) is large, we let $\operatorname{erf}(z)=1-\operatorname{erfc}(z)$, then we derive (see appendix A) that, for $\epsilon=1$,

$$
\begin{align*}
& \mathcal{J}_{g}(\rho,-z) \sim-\left((1-\delta) K_{0}(a)-i \sqrt{\pi} e^{-a \cos b} \tan \left(\frac{b}{2}\right)\right. \\
& \left.\times\left.\sum_{m \geq 0} c_{m}\left(\tan \left(\frac{b}{2}\right)\right)^{2 m} \frac{\partial^{m}}{\partial s^{m}}\left(\frac{e^{s}}{s^{1 / 2}} \operatorname{erfc}\left(s^{1 / 2}\right)\right)\right|_{s^{1 / 2}=-i \delta \sqrt{2 a \sin \left(\frac{b}{2}\right)}}\right), \tag{20}
\end{align*}
$$

and, for $\epsilon=-1$,

$$
\begin{align*}
& \mathcal{J}_{g}(\rho,-z) \sim-\left((1+v \operatorname{sign}(\arg (i k))) K_{0}(a)-(v \operatorname{sign}(\arg (i k))+\delta v) K_{0}(-a)\right. \\
& \left.+\left.i \sqrt{\pi} e^{-a \cos b} \cot \left(\frac{b}{2}\right) \sum_{m \geq 0} c_{m}\left(\cot \left(\frac{b}{2}\right)\right)^{2 m} \frac{\partial^{m}}{\partial s^{m}}\left(\frac{e^{s}}{s^{1 / 2}} \operatorname{erfc}\left(s^{1 / 2}\right)\right)\right|_{s^{1 / 2}=-\delta \sqrt{2 a c o s}\left(\frac{b}{2}\right)}\right), \tag{21}
\end{align*}
$$

where $\delta$ is chosen + or -1 , so that $\operatorname{Re} s^{1 / 2}>0$ and $\frac{\partial^{m}}{\partial s^{m}}\left(\frac{e^{s}}{s^{1 / 2}} \operatorname{erfc}\left(s^{1 / 2}\right)\right)=O\left(\frac{1}{s^{m+1}}\right)$. Considering the definitions of $a$ and $b$ given in (15), it is worth noticing that

$$
\begin{align*}
& \left(-i \tan \left(\frac{b}{2}\right)\right)^{\epsilon}=\frac{1+\sin \left(\theta_{1}-\varphi\right)}{\cos \varphi+\sin \theta_{1}}=\frac{\cos \varphi+\sin \theta_{1}}{1+\sin \left(\theta_{1}+\varphi\right)} \\
& -\left.\epsilon 2 a\right|_{\cos ^{2}\left(\frac{b}{2}\right) \text { when } \epsilon=-1} ^{\sin ^{2}\left(\frac{b}{2}\right) \text { when } \epsilon=1}=a \cos b-\epsilon a=i k R\left(1+\sin \left(\theta_{1}-\varphi\right)\right) \tag{22}
\end{align*}
$$


figure 2: definition of $S$ and $\Omega_{g}^{\varphi}$ for $\arg (i k)>0$

So, $\delta$ changes when we cross $S$ with $i k\left(1+\sin \left(\theta_{1}-\varphi\right)\right) \leq 0 \quad$ i.e. $\operatorname{Re} \theta_{1}=\varphi-\frac{\pi}{2}+\mathcal{G}\left(\operatorname{Im} \theta_{1}\right), \mathcal{G}(x)=2 \arctan \left(\tan \left(\frac{\arg (i k)}{2}\right) \tanh \left(\frac{x}{2}\right)\right)$ [1]. It follows that, when $\epsilon=1$ (resp. -1 ), $\delta=1$ (resp. $\delta=v$ ) on the right of $S$, and $\delta=-1($ resp. $\delta=-v$ ) on the left of $S$, with $v=\operatorname{sign}(\operatorname{Re} b)=-\epsilon \operatorname{sign}\left(\operatorname{Im}\left(\sin \theta_{1}\right)\right),(v \operatorname{sign}(\arg (i k))+\delta v)=2$ in $\Omega_{g}^{\varphi}$ (figure 2). The reader will notice that, in passive case $\left(\operatorname{Re} \theta_{1}>0\right)$, we recover the asymptotics given by Thomasson in [14].

## 3.2) Numerical applications

For the calculus of our series with the error functions, it is worth noticing, from [23], that

$$
\begin{align*}
& W_{m}(s)=\frac{e^{-s} \partial^{m}}{\partial s^{m}}\left(\frac{e^{s} \operatorname{erf}\left(s^{1 / 2}\right)}{s^{1 / 2}}\right)=\int_{0}^{1} \frac{e^{-s t}(1-t)^{m}}{\sqrt{\pi} t^{1 / 2}} d t=\frac{m!M\left(\frac{1}{2}, m+\frac{3}{2},-s\right)}{\Gamma\left(m+\frac{3}{2}\right)}, \\
& \frac{\partial^{m}}{\partial s^{m}}\left(\frac{e^{s} \operatorname{erfc}\left(s^{1 / 2}\right)}{m!s^{1 / 2}}\right)=\frac{(-1)^{m}}{\sqrt{\pi}} U\left(m+1, m+\frac{3}{2}, s\right)=\frac{(-1)^{m}}{\sqrt{\pi} s^{m+1}}\left(1+O\left(s^{-1}\right)\right) \tag{23}
\end{align*}
$$

where $M$ and $U$ are confluent hypergeometric functions [23]. These special functions are well-tabulated, and we remark in particular that, for large $m \gg|s|$,

$$
\begin{equation*}
W_{m}(s)=W_{m}(0)+O\left(s / m^{3 / 2}\right)=m^{-1 / 2}+O\left(m^{-3 / 2}\right) \tag{24}
\end{equation*}
$$

since $M(1 / 2, m+3 / 2,-s)=1+O(s / m)$ and $\frac{\Gamma(m+1)}{\Gamma(m+3 / 2)}=m^{-1 / 2}+O\left(m^{-3 / 2}\right)$.
Numerical examples are given in figures 3 and 4 for the modulus, and in figure 5 for the phase. The comparison of results given by our series (17)-(19) with the one given by numerical evaluation of the Fourier-Bessel expression (13) (very costly in computer time) is excellent for passive and active case. Notice the presence of a discontinuity as we let vary $\operatorname{Re} g$ when $\operatorname{Im} g>0$. It is due to a cut when $g$ is active, in the quarter plane $\operatorname{Re} g<0, \arg (i k) \operatorname{Im} g>0$, of the complex plane.


figure 3) Comparison of $\left|\mathcal{J}_{g}\right|$ given by our series ( $-\square-$ ) and by Fourier-Bessel expansion ( $-\circ-$ ), when Reg varies, as $\operatorname{Im}(g)=0.4$ and $k=1-.01 i$; left : $z=0.1, \rho=0.2$; right : $z=1$., $\rho=2$.

$$
\left|\mathcal{J}_{g}\right|
$$


figure 4) Comparison of $\left|\mathcal{J}_{g}\right|$ given by our series ( $-\square-$ ) and by Fourier-Bessel expansion ( $-\circ-$ ), when $\operatorname{Re} g$ varies, as $\operatorname{Im}(g)=-0.4$ and $k=1-.01 i ;$ left $: z=0.1, \rho=0.2 ;$ right $: z=1 ., \rho=2$.

figure 5) Comparison of $\arg \left(\mathcal{J}_{g}\right)$ given by our series $(-\square-)$ and by Fourier-Bessel expansion ( $-\circ-$ ), when $\operatorname{Re} g$ varies, as $\operatorname{Im}(g)=0.4$ and $k=1-.01 i, z=0.1, \rho=0.2$.

## 4) Conclusion

We have developed simple exact expressions of the field scattered by an impedance plane which can be active or passive for arbitrary bounded sources. In general, difficulties restrict the use in active case of expressions known for a passive plane, and we give here a novel exact expression for arbitrary parameters, in a compact form derived from the application of our works on multimode plane in [1]. A new expansion of the special function involved in the expression is given which has the particularities to be convergent and to also give the asymptotics.

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## Appendix A : expansion of $\mathcal{J}_{g}$

We give here the proof of the results given in section 3, considering that $\left|\tan ^{2}\left(\frac{b}{2}\right)\right|^{\epsilon} \leq 1$. In the case $\epsilon=1$, we use $\cos \alpha=1-2(\sin (\alpha / 2))^{2}$ in (14), and obtain,

$$
\begin{align*}
& \mathcal{M}(a, b)=i \int_{0}^{b} e^{-a \cos \alpha} d \alpha=-\int_{0}^{(-i \sin (b / 2))^{2}} \frac{e^{-a} e^{-2 a t}}{t^{1 / 2}(1+t)^{1 / 2}} d t \\
& =-\int_{0}^{1} \frac{e^{-a \cos b} e^{2 a(-i \sin (b / 2))^{2} t}(-i \sin (b / 2))}{(1-t)^{1 / 2}\left(1+(-i \sin (b / 2))^{2}(1-t)\right)^{1 / 2}} d t \\
& =e^{-a \cos b} \int_{0}^{1} \frac{i \tan \left(\frac{b}{2}\right) e^{2 a(-i \sin (b / 2))^{2} t}}{(1-t)^{1 / 2}\left(1+\tan ^{2}\left(\frac{b}{2}\right) t\right)^{1 / 2}} d t \tag{25}
\end{align*}
$$

so that, using the binomial expansion of $\left(1+\tan ^{2}\left(\frac{b}{2}\right) t\right)^{-1 / 2}$, we have

$$
\begin{align*}
& \mathcal{M}(a, b)=i \tan \left(\frac{b}{2}\right) \sum_{m \geq 0} c_{m}\left(\tan ^{2}\left(\frac{b}{2}\right)\right)^{m} e^{-a \cos b} \int_{0}^{1} \frac{e^{2 a(-i \sin (b / 2))^{2} t} t^{m}}{(1-t)^{1 / 2}} d t \\
& =i \tan \left(\frac{b}{2}\right) e^{-a \cos b} \sum_{m \geq 0} c_{m}\left(\tan \left(\frac{b}{2}\right)\right)^{2 m} \frac{\partial^{m}}{\partial s^{m}}\left(\left.\frac{\sqrt{\pi} e^{s}}{s^{1 / 2}}\left(\operatorname{erf}\left(s^{1 / 2}\right)\right)\right|_{s=-2 a \sin ^{2}\left(\frac{b}{2}\right)}\right. \tag{26}
\end{align*}
$$

Letting $W_{m}(s)=\frac{e^{-s} \partial^{m}}{\partial s^{m}}\left(\frac{e^{s} \operatorname{erf}\left(s^{1 / 2}\right)}{s^{1 / 2}}\right)$ and $W_{m}(s)=\left(W_{m}(s)-W_{m}(0)\right)+W_{m}(0)$, the convergence can be improved, noticing that the sum with $W_{m}(0)$ is $e^{-a} \mathcal{M}(0, b)=i b e^{-a}$. Concerning asymptotics for large $a$, we let $\frac{\operatorname{erf}(u)}{u}=\frac{1-\operatorname{erfc}(\delta u)}{\delta u}$ with $\operatorname{Re}(\delta u)>0$, and write

$$
\begin{align*}
& \mathcal{M}(a, b)=\lim _{M \rightarrow \infty}\left[i e^{-a} \tan \left(\frac{b}{2}\right) \sum_{m=0}^{M} c_{m}\left(\tan \left(\frac{b}{2}\right)\right)^{2 m}\left(e^{-s} \frac{\partial^{m}}{\partial s^{m}}\left(\frac{\sqrt{\pi} e^{s}}{s^{1 / 2}}\right)\right)\right. \\
& \left.-i e^{-a \cos b} \tan \left(\frac{b}{2}\right) \sum_{m=0}^{M} c_{m}\left(\tan \left(\frac{b}{2}\right)\right)^{2 m} \frac{\partial^{m}}{\partial s^{m}}\left(\frac{\sqrt{\pi} e^{s}}{s^{1 / 2}} \operatorname{erfc}\left(s^{1 / 2}\right)\right)\right]_{s^{1 / 2}=-i \delta \sqrt{2 \operatorname{asin}\left(\frac{b}{2}\right)}} \tag{27}
\end{align*}
$$

The terms under summation signs have no exponential dependence, and we remark that the term with $e^{-a}$ factor has the same asymptotics as $-\delta K_{0}(a)$. Indeed, using that

$$
\begin{equation*}
\frac{\partial^{m}}{\partial s^{m}}\left(\frac{e^{s}}{s^{1 / 2}}\right)=\frac{e^{s}}{s^{1 / 2}} \sum_{p=0}^{m} \frac{(-1)^{p+1} m!(-1 \times 1 \times \ldots \times(2 p-1))}{2^{p} s^{p} p!(m-p)!} \tag{28}
\end{equation*}
$$

we can verify by inspection that, for large $a$ and $M$,

$$
\begin{align*}
& -\left.i \sqrt{\pi} e^{-a} \tan \left(\frac{b}{2}\right) \sum_{m=0}^{M} c_{m}\left(\tan ^{2}\left(\frac{b}{2}\right)\right)^{m}\left(e^{-z} \frac{\partial^{m}}{\partial z^{m}}\left(\frac{e^{z}}{z^{1 / 2}}\right)\right)\right|_{z^{1 / 2}=-i \sqrt{2 a} \sin \left(\frac{b}{2}\right)} \\
& \sim-i \sqrt{\pi} e^{-a} \tan \left(\frac{b}{2}\right) \sum_{p=0}^{M}\left(\tan ^{2}\left(\frac{b}{2}\right)\right)^{p} \frac{(-1)^{p+1}(-1 \times \ldots \times(2 p-1))}{2^{p}\left(-i \sqrt{2 a} \sin \left(\frac{b}{2}\right)\right)^{2 p+1}} \\
& \times \frac{1}{p!} \sum_{m \geq p}^{M} c_{m} \frac{m!}{(m-p)!}\left(\tan ^{2}\left(\frac{b}{2}\right)\right)^{m-p} \sim-i \sqrt{\pi} e^{-a} \tan \left(\frac{b}{2}\right) \\
& \times\left.\sum_{p \geq 0}\left(\tan ^{2}\left(\frac{b}{2}\right)\right)^{p} \frac{(-1)^{p+1}(-1 \times \ldots \times(2 p-1))}{2^{p} p!\left(-i \sqrt{2 a} \sin \left(\frac{b}{2}\right)\right)^{2 p+1}} \frac{\partial^{p}}{\partial v^{p}}\left(\frac{1}{(1+v)^{1 / 2}}\right)\right|_{v=\tan ^{2}\left(\frac{b}{2}\right)} \\
& =\sqrt{\pi} e^{-a} \frac{\sqrt{\cos ^{2}\left(\frac{b}{2}\right)}}{\cos \left(\frac{b}{2}\right)} \sum_{p \geq 0} \frac{(-1)^{p}(-1 \times \ldots \times(2 p-1))^{2}}{2^{2 p}(2 a)^{p+1 / 2} p!} \tag{29}
\end{align*}
$$

which is equal to the known asymptotics of $K_{0}(a)$ [23] as $|\operatorname{Re} b|<\pi$. Besides, since $\operatorname{Re}\left(s^{1 / 2}\right)>0, \frac{\partial^{m}}{\partial s^{m}}\left(\frac{e^{s}}{s^{1 / 2}} \operatorname{erfc}\left(s^{1 / 2}\right)\right)=O\left(\frac{1}{s^{m+1}}\right)$ which implies the remaining sum is also asymptotic.
Concerning the case with $\epsilon=-1$, we can use that

$$
\begin{align*}
& \mathcal{N}(a, b)=i \int_{v \pi}^{b} e^{-a \cos \alpha} d \alpha=i \int_{0}^{b-v \pi} e^{a \cos \alpha} d \alpha \\
& =\left.\frac{e^{-a \cos b}}{i} \cot \left(\frac{b}{2}\right) \sum_{m \geq 0} c_{m}\left(\cot \left(\frac{b}{2}\right)\right)^{2 m} \frac{\partial^{m}}{\partial s^{m}}\left(\frac{\sqrt{\pi} e^{s}}{s^{1 / 2}} \operatorname{erf}\left(s^{1 / 2}\right)\right)\right|_{s=2 a \cos ^{2}\left(\frac{b}{2}\right)} \tag{30}
\end{align*}
$$

and consider previous development, changing $b$ for $b-v \pi$ and $a$ for $-a$.

## Remark 4 :

We can also apply our development for a convergent expansion of $I_{0}(a)$,

$$
\begin{equation*}
\pi I_{0}(a)=\left(\int_{0}^{\frac{\pi}{2}}-\int_{\pi}^{\frac{\pi}{2}}\right) e^{-a \cos \alpha} d \alpha=\sum_{m \geq 0} c_{m} \sum_{ \pm} \frac{\partial^{m}}{\partial s^{m}}\left(\left.\frac{\sqrt{\pi} e^{s}}{s^{1 / 2}}\left(\operatorname{erf}\left(s^{1 / 2}\right)\right)\right|_{s=\mp a}\right. \tag{31}
\end{equation*}
$$

and consider $\operatorname{erf}(x)=1-\operatorname{erfc}(x)$ to obtain its asymptotics.

## Remark 5 :

Defining $\epsilon$ as $\epsilon=\operatorname{sign}\left(\ln \left|\cot \left(\frac{b}{2}\right)\right|\right.$ ), we can apply our series (and those of [1]) for any $a, b$ with $\operatorname{Re} a>0,|\operatorname{Re} b|<\pi$ (other definition $\epsilon=\operatorname{sign}(\operatorname{Re}(\cos b))$ or $\frac{1-\epsilon}{4} \pi \leq|\operatorname{Re} b| \leq \frac{3-\epsilon}{4} \pi$ ).

Remark 6 :
Letting $V_{m}(s)=W_{m}(s)-W_{m}(0)$ with $W_{m}(0)=\frac{\Gamma(m+1)}{\Gamma(m+3 / 2)}$, we derive, from [23],

$$
\begin{equation*}
2 s V_{m+1}(s)+V_{m}(s)(1-2 s+2 m)-2 m V_{m-1}(s)=\frac{s \Gamma(m+1)}{\Gamma(m+5 / 2)} \tag{32}
\end{equation*}
$$

## Appendix B : some notes on leaky aquifer function, incomplete Bessel function and asymptotics

A.2.1) About leaky aquifer function, incomplete Bessel fuction and a new original development of Temme.

The function $\mathcal{J}_{g}$ is intimately connected to the incomplete Bessel function [24], and in his recent paper, Harris [25] notices the connection between this class of functions and leaky aquifer functions. We had some discussion with Nico Temme [26] on this subject during June 2010, in connection with my new developments in [1]. On this occasion, I found the development given in section 3, Temme recovered my expansions found in [1] and discovered another remarkable series. I now give here some of his original developments with his permission :
"Let us consider the leaky aquifer function of zero order $K_{0}(x, y)$. We can write

$$
\begin{equation*}
K_{0}(x, y)=\int_{1}^{\infty} e^{-x t-y / t} d t=\int_{\beta}^{\infty} e^{-a \cosh w} d w \tag{33}
\end{equation*}
$$

This form is obtained by substituting $t=\sqrt{y / x} e^{w}$ and using

$$
\begin{equation*}
a=2 \sqrt{x y}, \beta=\ln \sqrt{x / y} \tag{34}
\end{equation*}
$$

Next, transform $s=2 \sinh \frac{1}{2} w$ to obtain

$$
\begin{equation*}
K_{0}(x, y)=e^{-a} \int_{\sigma}^{\infty} e^{-\frac{1}{2} a s^{2}} \frac{d s}{\sqrt{1+\frac{1}{4} s^{2}}} \tag{35}
\end{equation*}
$$

where

$$
\begin{equation*}
\sigma=2 \sinh \left(\frac{1}{2} \ln \sqrt{x / y}\right)=\left(\frac{x}{y}\right)^{1 / 4}-\left(\frac{y}{x}\right)^{1 / 4} . \tag{36}
\end{equation*}
$$

Assume $\sigma>0$ (that is $x>y$ ) and substitute $s=\sigma \sqrt{1+u}$. This gives

$$
\begin{equation*}
K_{0}(x, y)=\frac{1}{2} \sigma e^{-x-y} \int_{0}^{\infty} e^{-\frac{1}{2} a \sigma^{2} u} f(u) d u \tag{37}
\end{equation*}
$$

where

$$
\begin{equation*}
f(u)=\frac{1}{\sqrt{1+\frac{1}{4} \sigma^{2}(1+u)}} \frac{1}{\sqrt{1+u}} \tag{38}
\end{equation*}
$$

Expand

$$
f(u)=\frac{1}{1+u} \sum_{k=0}^{\infty} c_{k}\left(\frac{u}{1+u}\right)^{k} .
$$

Then, with $p=u /(1+u)$ it follows that

$$
\begin{equation*}
(1+u) f(u)=\frac{1}{\sqrt{1+\frac{1}{4} \sigma^{2}-p}} \Rightarrow c_{k}=(-1)^{k}\binom{-\frac{1}{2}}{k}\left(\frac{1}{1+\frac{1}{4} \sigma^{2}}\right)^{k+\frac{1}{2}} \tag{39}
\end{equation*}
$$

which permits to recover the expansion found in section 3.3.2 of [1] and to write,

$$
\begin{equation*}
K_{0}(x, y)=\frac{1}{2} \sigma e^{-x-y} \sum_{k=0}^{\infty} c_{k} k!U\left(k+1,1, \frac{1}{2} a \sigma^{2}\right) . \tag{40}
\end{equation*}
$$

Another option is using the expansion

$$
\begin{equation*}
f(u)=\frac{1}{\sqrt{1+u}} \sum_{k=0}^{\infty} d_{k}\left(\frac{u}{1+u}\right)^{k} . \tag{41}
\end{equation*}
$$

Using again $p=u /(1+u)$ it follows that

$$
\begin{equation*}
\sqrt{1+u} f(u)=\frac{\sqrt{1-p}}{\sqrt{1+\frac{1}{4} \sigma^{2}-p}} \Rightarrow d_{k}=\sum_{j=0}^{k}(-1)^{j}\binom{\frac{1}{2}}{j} c_{k-j} \tag{42}
\end{equation*}
$$

which gives us the original expansion

$$
\begin{equation*}
K_{0}(x, y)=\frac{1}{2} \sigma e^{-x-y} \sum_{k=}^{\infty} d_{k} k!U\left(k+1, \frac{3}{2}, \frac{1}{2} a \sigma^{2}\right) \tag{43}
\end{equation*}
$$

These expansions are convergent for positive arguments $x$ and $y$ with $\sigma>0$, and have an asymptotic character when $\frac{1}{2} a \sigma^{2} \rightarrow \infty$."

## A.2.2) About the asymptotic character of some expansions

Let us consider the expansion $W_{N}(z)=\sum_{n=0}^{N} a_{n} w_{n}(z)$ at order $N$ of a function $w(z)$. We will consider that this development has an asymptotic character for $z \rightarrow \infty$, if

$$
\begin{equation*}
z^{N \alpha}\left(w(z)-W_{N}(z)\right) \rightarrow 0 \tag{44}
\end{equation*}
$$

where $\alpha$ is a constant. In our case $\alpha=1$, but the choice of the function $w_{n}$ can differ with the authors.
For example, G.D. Maliuzhinets [8] and A.D. Rawlins [15] found the complete development of the problem in acoustics for passive impedance with $w_{n}(z)=z^{-n}$. Let us notice that, the asymptotics given in [15] can also be derived from (14) if we consider the asymptotic expansion of incomplete cylindrical function given in [24, chap. 2 sect. 10].
In contrary, our developments (for multimode case in [1], and in the present paper), the ones given in [14] or by Temme (see previous section), are different: $w_{n}$ are then special functions, which permits the expansion to be more accurate.

## Appendix C : some expansions of $\mathcal{J}_{g}$ given in [1]

C.1) Exact series rapidly convergent for small $a$ (prop. 3.2 and prop. 3.3 of [1]).

An exact expansion of $\mathcal{J}_{g}(\rho,-z)$ for arbitrary $g=\sin \theta_{1}$ is given by,

$$
\begin{align*}
& \mathcal{J}_{g}(\rho,-z)=-E_{1}\left(\frac{a e^{-i b}}{2}\right)-1_{\Omega_{g}} 2\left(K_{0}(a)-K_{0}(-a)\right) \\
& -\sum_{p \geq 1} \frac{\left(-\frac{a e^{i b}}{2}\right)^{p}}{p!} E_{p+1}\left(\frac{a e^{-i b}}{2}\right) \tag{45}
\end{align*}
$$

where $1_{\Omega_{g}}$ is the indicator function of the region $\Omega_{g}$, where $\epsilon=-1, \operatorname{Re}\left(\sin \theta_{1}\right)<0$ (or $\operatorname{Im}\left(\sin \theta_{1}\right) \arg (i k)>0$ or $\left.\operatorname{Re}(b) \operatorname{Im}(a)<0\right)$, bounded by the cut of $E_{p+1}\left(\frac{a e^{-i b}}{2}\right)$, following

$$
\begin{equation*}
1_{\Omega_{g}}=\frac{1-\epsilon}{2} U\left(-\operatorname{Re}\left(\sin \theta_{1}\right)\right) U\left(\operatorname{Re} \theta_{1}-\left(-\frac{\pi}{2}+\mathcal{G}\left(\operatorname{Im} \theta_{1}\right)\right)\right) \tag{46}
\end{equation*}
$$

with $\mathcal{G}(x)=2 \arctan \left(\tan \left(\frac{\arg (i k)}{2}\right) \tanh \left(\frac{x}{2}\right)\right)$, and $U(x)$ being the unit step function.

A second expression of $\mathcal{J}_{g}(\rho,-z)$ with a better convergence in the vicinity of $g=-1$, is given by

$$
\begin{align*}
& \mathcal{J}_{g}(\rho,-z)=E_{1}\left(\frac{a e^{i b}}{2}\right)-2 K_{0}(a)+1_{\Omega_{-g}} 2\left(K_{0}(a)-K_{0}(-a)\right) \\
& +\sum_{p \geq 1} \frac{\left(-\frac{a e^{-i b}}{2}\right)^{p}}{p!} E_{p+1}\left(\frac{a e^{i b}}{2}\right) \tag{47}
\end{align*}
$$

C.2) a series rapidly convergent when $\left|\frac{2}{\cos b+\epsilon}\right|<1$, and asymptotic for large $a$ (from prop. 3.4 of [1])

The function $\mathcal{J}_{g}(\rho,-z)$ can be developed following,

$$
\begin{align*}
& \mathcal{J}_{g}(\rho,-z)=-e^{-\epsilon a} E_{1}(a \cos b-\epsilon a)\left(-i \tan \left(\frac{b}{2}\right)\right)^{\epsilon} \\
& -\left(1-\delta_{\epsilon}\right) K_{0}(a)-1_{\Omega_{g}^{\varphi}} 2 \pi i \operatorname{sign}(\arg (i k)) I_{0}(a)-\sum_{p=1}^{n-1} h_{p}-H_{n} \tag{48}
\end{align*}
$$

where $h_{p}$ and $H_{n}$ are, respectively, the term of the series,

$$
\begin{equation*}
h_{p}=\left(\frac{2 \epsilon}{\cos b+\epsilon}\right)^{p} \frac{(1 \times \ldots \times(2 p-1))}{2^{p} p!} e^{-\epsilon a} v_{p+1}(a \cos b-\epsilon a)\left(-i \tan \left(\frac{b}{2}\right)\right)^{\epsilon} \tag{49}
\end{equation*}
$$

and the remaining integral term,

$$
\begin{equation*}
H_{n}=-i \epsilon^{n} e^{-\epsilon a} \delta_{\epsilon} \frac{(1 \times \ldots \times(2 n-1))}{(n-1)!} \int_{\delta_{\epsilon} b}^{i \infty+s} \frac{v_{n}(a(\cos \alpha-\epsilon))}{(\cos \alpha+\epsilon)^{n}} d \alpha \tag{50}
\end{equation*}
$$

with $s=1_{\Omega_{g}^{\varphi}} w 2 \pi$. In this expression, $1_{\Omega_{g}^{\varphi}}$ is the indicator function of the subregion $\Omega_{g}^{\varphi}$, with $\epsilon=-1, \operatorname{Re}\left(\sin \theta_{1}\right)<0$ (or $\operatorname{Im}\left(\sin \theta_{1}\right) \arg (i k)>0$ or $\operatorname{Re}(b) \operatorname{Im}(a)<0$ ), that is bounded by the cut of $E_{p+1}(a \cos b-\epsilon a)$, following

$$
\begin{equation*}
1_{\Omega_{g}^{\varphi}}=\frac{1-\epsilon}{2} U\left(-\operatorname{Re}\left(\sin \theta_{1}\right)\right) U\left(\operatorname{Re} \theta_{1}-\varphi-\left(-\frac{\pi}{2}+\mathcal{G}\left(\operatorname{Im} \theta_{1}\right)\right)\right) \tag{51}
\end{equation*}
$$

and $2 \delta_{\epsilon}=(1+\epsilon) \delta_{1}+(1-\epsilon) \delta$ with

$$
\begin{equation*}
\delta=\operatorname{sign}(\operatorname{Im}(b)), \delta_{1}=\operatorname{sign}\left(\operatorname{Re} \theta_{1}-\varphi-\left(-\frac{\pi}{2}+\mathcal{G}\left(\operatorname{Im} \theta_{1}\right)\right)\right) \tag{52}
\end{equation*}
$$

and $\delta w=\operatorname{sign}(\operatorname{Re}(b))$ which is equal to $\operatorname{sign}(\arg (i k))$ in $\Omega_{g}^{\varphi}, U(x)$ being the unit step function. The function $v_{n}(z)$, also denoted $U(n, 1, z)$ [23] (detailed in [1]), is given by $v_{n}(t)=\sum_{m=0}^{n-1} \frac{(-1)^{m}(n-1)!}{m!(n-1-m)!} E_{m+1}(t)$.
When $\left|\frac{2 \sin \varphi \cos \theta_{1}}{\left(1+\sin \left(\theta_{1}+\varphi\right)\right)}\right|=\frac{2}{|\cos b+\epsilon|}<1$ and as $n \rightarrow \infty, \operatorname{Im}\left(\delta_{\epsilon} b\right)>0$ and the term $H_{n}$ vanishes, and the expansion becomes an absolutely convergent series. Moreover, for large $a, a^{n} H_{n}$ is small, and the expansion is asymptotic, except when $b=0$ and $\cos \varphi+\sin \theta_{1}=0$.
C.3) a series rapidly convergent when $\frac{2}{|\cos b+\epsilon|}>1$ (prop.3.5 in [1])

When $\left|\frac{\cos b-\epsilon}{2}\right|<1$, and thus as $\left|\frac{\cos b+\epsilon}{2}\right|=\left|\frac{\left(1+\sin \left(\theta_{1}+\varphi\right)\right)}{2 \sin \varphi \cos \theta_{1}}\right|<1$ since $\left|\frac{\cos b-\epsilon}{\cos b+\epsilon}\right| \leq 1$, two convergent expansions apply. If $\epsilon=1$, we are in vicinity of $b=0$ and we can write

$$
\begin{equation*}
\mathcal{J}_{g}(\rho,-z)=-\left(K_{0}(a)+i \int_{0}^{b} e^{-a \cos \alpha} d \alpha\right) \tag{53}
\end{equation*}
$$

where,

$$
\begin{equation*}
\int_{0}^{b} e^{-a \cos \alpha} d \alpha=\sum_{m \geq 0} \frac{i c_{m}}{(2 a)^{m+1 / 2}} e^{-a} \gamma\left(m+1 / 2,(-i \sqrt{2 a} \sin (b / 2))^{2}\right) \tag{54}
\end{equation*}
$$

while, if $\epsilon=-1$, in vicinity of $b=v \pi, v=\operatorname{sign}(\operatorname{Re} b)$, we have

$$
\begin{equation*}
\mathcal{J}_{g}(\rho,-z)=-\left(i \pi v I_{0}(a)+K_{0}(a)+i \int_{v \pi}^{b} e^{-a \cos \alpha} d \alpha\right) \tag{55}
\end{equation*}
$$

where,

$$
\begin{equation*}
\int_{v \pi}^{b} e^{-a \cos \alpha} d \alpha=-v \sum_{m \geq 0} \frac{(-1)^{m} c_{m}}{(2 a)^{m+1 / 2}} e^{a} \gamma\left(m+1 / 2,(\sqrt{2 a} \cos (b / 2))^{2}\right) \tag{56}
\end{equation*}
$$

In these expressions, $c_{m}=(-1)^{m}(2 m)!/\left((m!)^{2} 2^{2 m}\right)$ is the binomial coefficient of the function $(1+t)^{-1 / 2}$ for $|t|<1$, and $\gamma(m+1 / 2, z)$ is the incomplete gamma function [23, p.262], related to error function $\operatorname{erf}(x)$ by

$$
\begin{equation*}
\gamma\left(1 / 2, x^{2}\right)=\sqrt{\pi} \operatorname{erf}(x), \quad \gamma\left(\alpha+1, x^{2}\right)=\alpha \gamma\left(\alpha, x^{2}\right)-x^{2 \alpha} e^{-x^{2}} \tag{57}
\end{equation*}
$$

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