A NUMERICAL SCHEME
FOR THE SCALAR PRESSURELESS GASES SYSTEM

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Abstract. In this work, we investigate the numerical solving of the one-dimensional pressureless gases system. After briefly recalling the mathematical framework of the duality solutions introduced by Bouchut and James [6], we point out that the upwind scheme for the density and momentum does not satisfy the one-sided Lipschitz (OSL) condition on the expansion rate required for the duality solutions. Then we build a diffusive scheme which allows to recover the OSL condition by following the strategy described in [9] for the continuous model.

1. Introduction

During the last two decades, there has been a lot of contributions dealing with the pressureless gases system. It seems natural to tackle the question of its discretization, to obtain a relevant numerical approximation of the solution to this system. Indeed, it appears as a degenerate hyperbolic system of conservation laws (the Jacobian is not diagonalizable), and it is interesting to investigate if the numerical schemes fitted to nondegenerate systems are also fitted to this system.

Let us first recall the one-dimensional system describing a pressureless gas. Let $T > 0$. The gas density $\varrho(t,x) \ge 0$ and the momentum $q(t,x) \in \mathbb{R}$ satisfy the following equations in $(0,T) \times \mathbb{R}$

$$\partial_t \varrho + \partial_x(\varrho u) = 0, \tag{1}$$

$$\partial_t q + \partial_x(q u) = 0. \tag{2}$$

One must define the velocity $u(t,x) \in \mathbb{R}$ as a quotient of q by ϱ, but it may not be possible, since ϱ can be nil. We shall discuss this issue below, by recalling the notion of duality solutions [5]. Both equations consist in a conservation law, (1) for mass and (2) for momentum. We also provide initial conditions

$$\varrho(0,\cdot) = \varrho^{\mathrm{in}}, \qquad q(0,\cdot) = q^{\mathrm{in}}, \tag{3}$$

in which the condition on the momentum can be replaced by an initial condition on the velocity $u(0,\cdot) = u^{\mathrm{in}}$, and then written again as $q(0,\cdot) = \varrho^{\mathrm{in}} u^{\mathrm{in}}$.

The previous system can be seen as a simplified model of the Euler equations to describe gas dynamics, where the pressure has been set to 0. It can

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describe either cold plasmas or galaxies' dynamics [25]. This system (1)–(2) and related problems (traffic models, magnetohydrodynamics, astrophysics, pressureless fluid equations...) have been widely studied, see, for instance, [4, 18, 15, 11, 23, 6, 9, 24, 1, 21, 7, 16, 14, 2, 22, 10]. Those references use a standard fluid setting or a kinetic one, involving the adhesion dynamics of the so-called sticky particles.

When one studies smooth solutions of the pressureless gases system, (2) can be replaced by the standard Burgers equation

$$\partial_t u + \partial_x \left(\frac{u^2}{2} \right) = \partial_t u + u \partial_x u = 0. \tag{4}$$

In that case, ϱ solves a plain transport equation, since u does not depend on ϱ. On the other hand, it is well-known that smooth initial data may eventually result in mass concentration, for example, when the velocity does not increase. In that case, the velocity cannot solve (4) anymore.

In [5], Bouchut and James introduced the notion of duality solution for one-dimensional transport equations and conservation laws. In [6], they proved that this framework was fitted the pressureless gases system. Let us briefly recall the results obtained there.

Definition 1. *A couple* $(\varrho, q) \in C(\mathbb{R}_+; \text{w*-}\mathcal{M}_{\text{loc}}(\mathbb{R}))^2$, *with* $\varrho \geq 0$, *is a duality solution to (1)–(2), if there exists a bounded Borel function* a *and* $\alpha \in L^1_{\text{loc}}(\mathbb{R}_+^*)$ *such that*

$$\partial_x a \leq \alpha, \qquad q = \alpha \varrho, \qquad \text{in } \mathbb{R}_+^* \times \mathbb{R},$$

and, in the duality sense on $(t_1, t_2) \times \mathbb{R}$, *for any* $0 < t_1 < t_2$,

$$\partial_t \varrho + \partial_x (\varrho a) = 0, \qquad \partial_t q + \partial_x (q a) = 0.$$

In that setting, u is defined ϱ-almost everywhere, and we have $u = a$ ϱ-a.e. Bouchut and James proved that duality solutions are stable, and also entropic, i.e. the following inequality holds, in the distributional sense,

$$\partial_t (\varrho S(u)) + \partial_x (\varrho u S(u)) \leq 0, \tag{5}$$

for any convex function S. Using those properties and the sticky particles dynamics, they obtained the following existence result.

Theorem 1. *Let* ϱ^{in}, $q^{\text{in}} \in \mathcal{M}_{\text{loc}}(\mathbb{R})$, *with* $\varrho^{\text{in}} \geq 0$ *and* $|q^{\text{in}}| \leq U \varrho^{\text{in}}$, $U \geq 0$. *Then there exists a duality solution to (1)–(3), and we have* $\|a\|_\infty \leq U$ *and* $\alpha(t) = 1/t$.

As proven in [19], the one-sided Lipschitz (OSL) condition on the expansion rate $\partial_x a \leq 1/t$, also known as the Oleinik entropy condition, is optimal for a convex scalar conservation law. In the proof of Theorem 1, it is clear that the standard convex entropy condition (5) is not enough, and the OSL condition is really required. Note that, when the solutions are smooth, this estimate can easily be proven, since the Burgers equation (4) lies in the class of convex scalar conservation laws [12, 20].

Eventually, Bouchut and James also obtained uniqueness when ϱ^{in} is nonatomic (essentially meaning that ϱ^{in} is smooth).

In this work, we also need to consider the addition of a viscosity term in the pressureless gases system. For the study of this viscous system, as shown in [9], (2) is replaced by an equation on the velocity itself. Indeed, let us choose $\varepsilon > 0$. The gas density $\varrho(t,x) \geq 0$ and the velocity $u(t,x) \in \mathbb{R}$ satisfy, in $(0,T) \times \mathbb{R}$, Equation (1) and

$$\partial_t u + u\partial_x u = \frac{\varepsilon}{\varrho}\partial^2_{xx} u, \tag{6}$$

with the same set of initial conditions (3). That writing imposes that ϱ remains nonnegative, which is proven to be true in [9] if one assumes that the initial datum of ϱ is also nonnegative. Note that (6) is equivalent, when ε is fixed, to the following one

$$\partial_t u + \partial_x \left(\frac{u^2}{2}\right) = \frac{\varepsilon}{\varrho}\partial^2_{xx} u, \tag{7}$$

if we take into account the smoothness of the viscous velocity given in [9].

In fact, (6) or (7) can be rewritten as an equation on the momentum with a viscosity term $\varepsilon\partial^2_{xx} u$

$$\partial_t(\varrho u) + \partial_x(\varrho u^2) = \varepsilon\partial^2_{xx} u,$$

which yields (2) when ε goes to 0. In [9], the author proved the existence, in the sense of distributions, of solutions to the viscous system (1) and (3)–(7), and that the expansion rate satisfies a uniform (with respect to ε) upper bound $\partial_x u \leq A/(At+1)$, when $A = \max(\operatorname{ess\,sup} \partial_x u^{\text{in}}, 0)$ is finite. He also obtained the convergence of the viscous solutions towards the duality solutions to the pressureless gases system when ε vanishes. More precisely, the following convergence result holds.

Theorem 2. *Let $(\varrho^{\text{in}}_\varepsilon)$, $(u^{\text{in}}_\varepsilon)$ such that, for any $\varepsilon > 0$,*

$$\varrho^{\text{in}}_\varepsilon > 0, \qquad \varrho^{\text{in}}_\varepsilon \in L^\infty(\mathbb{R}), \qquad \left\|1/\varrho^{\text{in}}_\varepsilon\right\|_{L^\infty(\mathbb{R})} \leq C\varepsilon^{-1/4},$$
$$u^{\text{in}}_\varepsilon \in L^1 \cap L^\infty(\mathbb{R}), \qquad \left\|u^{\text{in}}_\varepsilon\right\|_{L^\infty(\mathbb{R})} \leq C,$$
$$\partial_x u^{\text{in}}_\varepsilon L^1 \cap L^2(\mathbb{R}), \qquad \operatorname{ess\,sup} \partial_x u^{\text{in}}_\varepsilon \leq C\varepsilon^{-1/2}.$$

We assume that $(\varrho^{\text{in}}_\varepsilon) \rightharpoonup \varrho^{\text{in}}$ and $(\varrho^{\text{in}}_\varepsilon u^{\text{in}}_\varepsilon) \rightharpoonup q^{\text{in}}$ in $\text{w*-}\mathcal{M}_{\text{loc}}(\mathbb{R})$. Then, up to a subsequence, $(\varrho_\varepsilon, \varrho_\varepsilon u_\varepsilon)$, given by the solutions to (1) and (7), with initial datum $(\varrho^{\text{in}}_\varepsilon, \varrho^{\text{in}}_\varepsilon u^{\text{in}}_\varepsilon)$, converges in $C_t(\text{w*-}\mathcal{M}_{\text{loc}}(\mathbb{R}))$ towards the duality solution (ϱ, q) of (1)–(3).

Both previous (viscous and inviscid) systems can also be studied in a periodic framework, i.e. we focus on the closed interval $[0,1]$ and impose that all the physical quantities have the same values at both $x = 0$ and $x = 1$, so that the solutions are 1-periodic.

This work is dedicated to the numerical solving of the inviscid and viscous pressureless gases system (1)–(2), where the latter may be replaced by (7). For readability reasons, we choose the periodic framework.

There are two possible methods to get *a priori* relevant schemes. The first one is to use the natural kinetic framework which underlies the pressureless gas dynamics, with kinetic schemes, as in [4, 8], or with particle methods [13]. The second one is related to the discretization of hyperbolic conservation laws. Gosse and James [17] pointed out the relevance of two families of numerical schemes: the upwind schemes and the Lax-Friedrichs schemes. In [3], the authors investigate a relaxation scheme for the pressureless gases system in one and two-dimensional settings.

As we already pointed out, the key condition to obtain the duality solution is that the velocity expansion rate must be upper-bounded by $1/t$, which can be discretized version, see [12]. In this work, we first investigate the upwind scheme associated to (1)–(2), and prove that it fails to ensure the OSL condition. Consequently, we try the upwind diffusive scheme associated to (1) and (7), and explain how we can obtain a good numerical approximation of the duality solution to the inviscid pressureless gases system using this scheme. We do not study the Lax-Friedrichs schemes described in [17]. Indeed, the numerical dissipation induced by those first order schemes is too significant. Since it is then natural to use higher order schemes, we recover the same kind of problems as in the diffusive upwind scheme we here propose, involving second order terms.

In the rest of the article, let Δt, $\Delta x > 0$ such that $N = T/\Delta t \in \mathbb{N}$ and $I = 1/\Delta x \in \mathbb{N}$, and set $\lambda = \Delta t/\Delta x$. We respectively denote ϱ_i^n, q_i^n and u_i^n the approximate values of ϱ, q and u at time $n\Delta t \in [0,T]$ and coordinate $(i+1/2)\Delta x \in [0,1)$, for $0 \le n \le N$ and $0 \le i < I$. Since we use a periodic framework, we define ϱ_i^n, q_i^n and u_i^n for any $i \in \mathbb{Z}$, by

$$\varrho_{i+pI}^n = \varrho_i^n, \qquad q_{i+pI}^n = q_i^n, \qquad u_{i+pI}^n = u_i^n, \qquad 0 \le i < I, \quad p \in \mathbb{Z}^*.$$

For the sake of readability, in the previous notations, we may drop the time iteration index n and replace $n+1$ by a prime symbol "$\,'\,$". For instance, the velocity at time $(n+1)\Delta t$ and coordinate $(i+1/2)\Delta x$ can be written as u_i' or u_i^{n+1}.

Apart from the density, momentum and velocity, the quantity of interest, which we name the numerical expansion rate, will be, for each time and space indices n and i,

$$w_i^n := n\lambda(u_{i+1}^n - u_i^n).$$

Indeed, the OSL condition at time $n\Delta t$ then reads $\max_i w_i^n \le 1$.

2. Upwind scheme

Let us first denote the positive and negative parts of $a \in \mathbb{R}$

$$a^+ = \max(0,a), \qquad a^- = \min(0,a).$$

The upwind scheme then writes, for any $0 \leq i \leq I-1$,

$$\begin{aligned} \varrho_i' &= \varrho_i - \lambda\left[\varrho_i(u_i)^+ - \varrho_{i-1}(u_{i-1})^+\right] - \lambda\left[\varrho_{i+1}(u_{i+1})^- - \varrho_i(u_i)^-\right], \\ q_i' &= q_i - \lambda\left[q_i(u_i)^+ - q_{i-1}(u_{i-1})^+\right] - \lambda\left[q_{i+1}(u_{i+1})^- - q_i(u_i)^-\right], \\ u_i' &= \frac{q_i'}{\varrho_i'}, \qquad \text{if } \varrho_i' > 0, \end{aligned}$$

and u_i' is not defined if $\varrho_i' = 0$. It is quite clear that the previous schemes on both ϱ and q are monotonic, if the standard Courant-Friedrichs-Lewy (CFL) condition $\lambda \max |u| \leq 1$ is satisfied. Hence, we only focus on the positive parts of the velocities, because we only choose positive initial data. The scheme is then the following one:

$$\varrho_i' = (1-\lambda u_i)\varrho_i + \lambda u_{i-1}\varrho_{i-1}, \tag{8}$$

$$q_i' = (1-\lambda u_i)q_i + \lambda u_{i-1}q_{i-1}, \tag{9}$$

$$u_i' = \frac{q_i'}{\varrho_i'}. \tag{10}$$

As we already stated, this last equality allows to define u_i' only when $\varrho_i' > 0$. This fits the mathematical setting of the pressureless gases system, since u can only be defined ϱ-almost everywhere. Nevertheless, it is not satisfying from a numerical viewpoint, since the computations should stop whenever the density becomes equal to 0. Nevertheless, it seems reasonable to impose whichever value we want, for instance, $u_i' = 0$, when $\varrho_i' = 0$. Indeed, since there is no matter at a given point, we do not care about the value of the velocity at this same point. But that implies that we shall not use those artificial nil values of u_i' to study the associated numerical expansion rate.

Thanks to (10), we immediately have

$$\varrho_i'\varrho_{i+1}'(u_{i+1}' - u_i') = \varrho_i' q_{i+1}' - \varrho_{i+1}' q_i',$$

which implies

$$\varrho_i'\varrho_{i+1}'\frac{w_i'}{(n+1)\lambda} = (1-\lambda u_{i+1})\varrho_{i+1}\varrho_i'\frac{w_i}{n\lambda} + \lambda u_{i-1}\varrho_{i-1}\varrho_{i+1}'\frac{w_{i-1}}{n\lambda}.$$

Under the CFL condition $\lambda \max |u| \leq 1$, if $(w_i)_{0\leq i<I}$ are negative, and if $(\varrho_i')_{0\leq i<I}$ are nonnegative, it is clear that the quantities (w_i') also remain negative. Unfortunately, if w_j is nonnegative for a given j, the OSL condition $w_i' \leq 1$ for all i may eventually not be satisfied, as it is proven in the following proposition.

Proposition 3. *Let λ, $U > 0$ such that $\lambda < 1$ and $\lambda U < 1$, and choose an integer $I > 2 + 1/\lambda$. We consider the following set of numerical initial data*

$$\varrho_i^0 = 1, \quad 0 \leq i \leq I-1, \qquad u_0^0 = U, \quad u_i^0 = 0, \quad 1 \leq i \leq I-1. \tag{11}$$

Then the upwind scheme (8)–(10) *does not satisfy the OSL condition. More precisely, we have*

$$\max_i w_i^{I-2} > U. \tag{12}$$

The assumption on the Courant number $0 < \lambda < 1$ is standard and is a natural consequence of the CFL condition $\lambda U \le 1$ when U is large.

Proof. It is easy to simultaneously prove, by induction on the time step $0 \le j < I-1$, that

$$\varrho_i^j > 0, \qquad 0 \le i \le I-1,$$

and

$$u_0^j = U, \qquad u_{j+1}^j > 0, \qquad u_i^j = 0, \quad j+2 \le i \le I-1.$$

We must emphasize that the nil values of u_i^j are obtained because $q_i^j = 0$ and $\varrho_i^j > 0$, and not because of the artificial choice of nil velocity when the density equals 0.

Then we can write that

$$w_{I-1}^{I-2} = (I-2)\lambda(u_0^{I-2} - u_{I-1}^{I-2}) = (I-2)\lambda U.$$

That implies

$$\max_i w_i^{I-2} \ge (I-2)\lambda U > U,$$

by choice of I. Note that the first inequality is in fact an equality, but we do not need to prove it here. □

Remark 1. As we already pointed out, the standard numerical version of the OSL condition reads $\max_i w_i^n \le 1$. It may have been relaxed into $\max_i w_i^n \le K$, where K is a nonnegative constant, which does not depend on the initial data. But (12) implies that the quantity $\max_i w_i^{I-2}$ can be as large as we want, depending on the choice of the value of U.

Proposition 3 means in particular that, if the space step Δx is refined enough, the numerical OSL condition cannot be satisfied anymore, with a set of initial data given by (11). Moreover, we must point out that, whatever the final time is, one can find a discretization for which the upwind scheme cannot satisfy the OSL condition, because the choice of I does not depend on T.

The initial datum chosen for u in the previous proposition is not smooth, and one may argue that the smoothness may provide an upwind scheme satisfying the OSL condition. In fact, even if the initial data are smooth (and remain periodic), the OSL condition may not hold either. That will be numerically shown in Section 4.

3. Adding artificial viscosity

We now put a small artificial viscosity to the problem, as it was done in [9], and we give a discretization of (1) and (7). First of all, we have to choose periodic initial data $u^{\text{in}} \ge 0$, $\varrho^{\text{in}} \ge 0$, and regularize them so that (keeping the same notations for both) u^{in}, $\varrho^{\text{in}} \in C^1(\mathbb{R}; \mathbb{R}_+^*)$ remain periodic. We must emphasize that the regularized ϱ^{in} must lie in $\mathbb{R}_+^*$, since the continuous diffusive model involves a division by ϱ.

Let us then consider $\varepsilon > 0$. Note that the regularized initial data do not depend on ε. In the following, we set

$$U = \max_{[0,1]} u^{\text{in}} > 0, \qquad V = \min_{[0,1]} u^{\text{in}} > 0,$$
$$A = \max(0, \max_{[0,1]} (u^{\text{in}})') \geq 0, \qquad R = \min_{[0,1]} \varrho^{\text{in}} > 0.$$

Let $\Delta t, \Delta x > 0$, and set

$$\lambda = \frac{\Delta t}{\Delta x}, \qquad \sigma = \frac{\Delta t}{\Delta x^2}.$$

For the rest of the article, we shall make the following assumptions on the time and space steps:

$$0 < \quad \Delta x \quad \leq \frac{2V}{1+A}, \tag{13}$$
$$0 < \quad \Delta t \quad \leq \min\left(\frac{1}{4A+1}, \frac{1}{4U}\Delta x, \frac{R}{4\varepsilon(1+AT)}\Delta x^2\right). \tag{14}$$

In fact, (13) is not restrictive, since, eventually, Δx will go to 0.

With the same notations for quantities at times $n\Delta t$ and $(n+1)\Delta t$ as in Section 2, we now focus on the following scheme, corresponding to the discretization of (1) and (7).

$$u_i' \quad = \quad u_i - \lambda\left(\frac{{u_i}^2}{2} - \frac{{u_{i-1}}^2}{2}\right) + \frac{\varepsilon\sigma}{\varrho_i}(u_{i-1} + u_{i+1} - 2u_i), \tag{15}$$
$$\varrho_i' \quad = \quad (1 - \lambda u_i')\varrho_i + \lambda u_{i-1}' \varrho_{i-1}. \tag{16}$$

Note that (15) is obtained from (7), which is written under a conservative form, as suggested in [12].

If we choose $u^{\text{in}} \equiv 1$, we can note that both upwind and diffusive schemes give $u_i^n = 1$ for any i and n, which is reassuring: in that case, and when ϱ remains nonnegative, the velocity satisfies the Burgers equation, which implies, at least formally, that u remains constant.

Remark 2. The velocity terms which appear in (16) are the ones at time $(n+1)\Delta t$. They must not be at time $n\Delta t$ to ensure the lower bound on ϱ, as we shall see in the proof of Theorem 4 below.

The following theorem states that the scheme (15)–(16) is L^∞-stable, consistent, monotonic, and that it satisfies the OSL condition. Consequently, this scheme can eventually provide a good approximation of a solution of the inviscid pressureless gases system, if one chooses ε small enough, and regularized initial data close to the original ones. Indeed, since the numerical scheme (15)–(16) satisfies Theorem 4, that ensures that it converges, when both Δt and Δx go to 0 under assumptions (13)–(14), towards the solution of the viscous pressureless gases system. Thanks to [9], we know that the latter solution converges to the solution of the (inviscid) pressureless gases system, when ε goes to 0.

Theorem 4. *We assume that* (13)–(14) *hold. Then we have, for any i and $n \geq 0$,*

$$V \leq u_i^n \leq U, \tag{17}$$

$$u_i^n - u_{i-1}^n \leq \frac{A\Delta x}{1 + An\Delta t}, \tag{18}$$

$$\varrho_i^n \geq \frac{R}{1 + An\Delta t} \geq \frac{R}{1 + AT} > 0. \tag{19}$$

Moreover, the discrete total mass is conserved, i.e., *for any $n \geq 0$,*

$$\sum_i \varrho_i^n \Delta x = \sum_i \varrho_i^0 \Delta x. \tag{20}$$

Finally, the scheme (15)–(16) *is consistent of order 1 in time and space, and is monotonic.*

Equations (17) and (19) respectively correspond to the maximum principles on the velocity and the density, (18) stands for the discrete version of the OSL condition.

Remark 3. The assumptions (14) on Δt ensure the stability of the scheme. More precisely, the second one is induced by the CFL condition and the third one is similar to standard stability conditions for explicit diffusive schemes. The first one is needed for the required properties of the scheme, as it will be detailed in the proof of Theorem 4.

Proof. We proceed by induction on $n \in \mathbb{N}$, and first investigate the case when $n = 0$. Equations (17) and (19) are obviously satisfied by definitions of U, V and R, and thanks to (13). The fact that (18) holds comes from the fact that u^{in} is smooth, and consequenly satisfies the intermediate values inequality.

In the rest of the proof, we shall suppose that $A > 0$. The case when $A = 0$ can easily be treated. Let us assume that (17)–(19) hold for a fixed n, and prove them for $n+1$. Equation (15) can be rewritten under the form

$$u_i' = \left(1 - \lambda\frac{u_i + u_{i-1}}{2} - \frac{2\varepsilon\sigma}{\varrho_i}\right) u_i + \frac{\varepsilon\sigma}{\varrho_i} u_{i+1} + \left(\lambda\frac{u_i + u_{i-1}}{2} + \frac{\varepsilon\sigma}{\varrho_i}\right) u_{i-1}. \tag{21}$$

Under this form, u_i' is a convex combination of u_{i-1}, u_i and u_{i+1}, since the corresponding coefficients in (21) live in $[0,1]$ and their sum equals 1. Indeed, we clearly have, thanks to (14) and (19),

$$0 \leq \frac{2\varepsilon\sigma}{\varrho_i} \leq \frac{1}{2},$$

and, thanks to (14) and (17),

$$0 \leq \lambda\frac{u_i + u_{i-1}}{2} \leq \frac{1}{4}.$$

Then it is easy to check that u_i' satisfies (17).

Let us now define, for any i,

$$\delta_i = u_{i+1} - u_i - \frac{A\Delta x}{1 + An\Delta t},$$

which we know it is negative, and prove that δ_i' is also negative, for any i. Thanks to (21), we can write

$$\begin{aligned} u_{i+1}' - u_i' = &\left[1 - \frac{\lambda}{2}(u_{i+1} + u_i) - \frac{\varepsilon\sigma}{\varrho_i} - \frac{\varepsilon\sigma}{\varrho_{i+1}}\right](u_{i+1} - u_i) \\ &+ \frac{\varepsilon\sigma}{\varrho_{i+1}}(u_{i+2} - u_{i+1}) + \left[\frac{\lambda}{2}(u_i + u_{i-1}) + \frac{\varepsilon\sigma}{\varrho_i}\right](u_i - u_{i-1}). \end{aligned}$$

Then we have

$$\begin{aligned} \delta_i' = &\frac{\varepsilon\sigma}{\varrho_{i+1}}\delta_{i+1} + \left[\frac{\lambda}{2}(u_i + u_{i-1}) + \frac{\varepsilon\sigma}{\varrho_i} - \frac{A\lambda\Delta x}{2(1 + An\Delta t)}\right]\delta_{i-1} \\ &+ \left[1 - \frac{\lambda}{2}(u_{i+1} + u_i) - \frac{\varepsilon\sigma}{\varrho_i} - \frac{\varepsilon\sigma}{\varrho_{i+1}} - \frac{A\lambda\Delta x}{2(1 + An\Delta t)}\right]\delta_i \\ &+ \frac{A\Delta x}{1 + An\Delta t}\left(1 - \frac{A\Delta t}{1 + An\Delta t}\right) - \frac{A\Delta x}{1 + A(n+1)\Delta t}. \end{aligned}$$

The coefficient before δ_{i+1} is clearly positive. Let us check that the ones before δ_{i-1} and δ_i are positive too. We have

$$\frac{A\lambda\Delta x}{2(1 + An\Delta t)} \leq \frac{\lambda}{2}(u_i + u_{i-1})$$

and

$$\frac{\lambda}{2}(u_{i+1} + u_i) + \frac{\varepsilon\sigma}{\varrho_i} + \frac{\varepsilon\sigma}{\varrho_{i+1}} + \frac{A\Delta t}{2(1 + An\Delta t)} \leq 1,$$

because of (13)–(19). Since the (δ_i) are all negative, we still have to prove that the remaining term is negative to get $\delta_i' \leq 0$. After simplifying by $A\Delta x$, which has no influence on the sign, we write

$$\frac{1 + A(n-1)\Delta t}{(1 + An\Delta t)^2} - \frac{1}{1 + A(n+1)\Delta t} = \frac{-(A\Delta t)^2}{(1 + An\Delta t)^2(1 + A(n+1)\Delta t)},$$

which is clearly negative, and ensures that (18) holds for $n + 1$.

We now focus on the properties of ϱ. We successively have, thanks to (19) for n and (18) for $n + 1$,

$$\varrho_i' \geq \left[1 - \frac{A\Delta t}{1 + A(n+1)\Delta t}\right]\frac{R}{1 + An\Delta t} = \frac{R}{1 + A(n+1)\Delta t},$$

which concludes the induction. Note that, as we pointed out in Remark 2, if (16) only involved velocities at time $n\Delta t$, the previous inequality would not hold, and we would not get any maximum principle on $1/\varrho$.

We easily notice that in the equality

$$\sum_i \varrho_i'\Delta x = \sum_i \varrho_i\Delta x - \lambda\Delta x\sum_i \varrho_i u_i' + \lambda\Delta x\sum_i \varrho_{i-1}u_{i-1}',$$

the last two terms cancel, which ensures the discrete total mass conservation.

Finally, let us investigate some basic properties of the scheme (15)–(16). The consistency is quite clear. Moreover, if we study u_i' as a function of u_{i-1}, u_i and u_{i+1}, we immediately have

$$\frac{\partial u_i'}{\partial u_{i-1}} = \lambda u_{i-1} + \frac{\varepsilon\sigma}{\varrho_i} \geq 0, \quad \frac{\partial u_i'}{\partial u_i} = 1 - \frac{2\varepsilon\sigma}{\varrho_i} - \lambda u_i \geq 0, \quad \frac{\partial u_i'}{\partial u_{i+1}} = \frac{\varepsilon\sigma}{\varrho_i} \geq 0,$$

which ensures the required property of monotonicity for (15), whereas it is clear for (16).

That ends the proof of Theorem 4. □

Remark 4. Let us check the behaviour of the numerical total momentum. Indeed, in its continuous version (7), the total momentum is conserved, since all the terms besides the time derivative of ϱu are partial derivatives in x. Unfortunately, the scheme does not ensure the exact conservation of the total momentum. Nevertheless, we can write

$$\sum_i q_i' = \sum_i \varrho_i u_i' + \lambda \sum_i \varrho_i u_i'(u_{i+1}' - u_i'),$$

which implies the following inequalities

$$[1 - \lambda(U-V)] \sum_i \varrho_i u_i' \leq \sum_i q_i' \leq \left[1 + \min\left(\frac{1}{n+1}, \lambda(U-V)\right)\right] \sum_i \varrho_i u_i'.$$

Then we have to study the behaviour of the quantity

$$\sum_i \varrho_i u_i' = \sum_i q_i - \frac{\lambda}{2} \sum_i \varrho_i ({u_i}^2 - {u_{i-1}}^2),$$

for which we have

$$\sum_i q_i - U \min\left(\frac{1}{n}, \lambda(U-V)\right) \sum_i \varrho_i^0 \leq \sum_i \varrho_i u_i' \leq \sum_i q_i + \lambda V(U-V) \sum_i \varrho_i^0.$$

We eventually can write

$$\begin{aligned} \sum_i q_i' &\geq [1 - \lambda(U-V)] \left[\sum_i q_i - U \min\left(\frac{1}{n}, \lambda(U-V)\right) \sum_i \varrho_i^0\right], \\ \sum_i q_i' &\leq \left[1 + \min\left(\frac{1}{n+1}, \lambda(U-V)\right)\right] \left[\sum_i q_i + \lambda V(U-V) \sum_i \varrho_i^0\right], \end{aligned}$$

which is not really satisfactory. Nevertheless, since the time and space steps satisfy (14), we have

$$\lambda \leq \frac{R}{4\varepsilon(1+AT)} \Delta x,$$

which ensures that λ is small when both Δx and Δt go to 0, and $\varepsilon > 0$ is fixed. Of course, that will not prevent the numerical total momentum to vary, but, at least, from one time step to the next one, the variations have to remain small. It is interesting to note that, in the examples of the next section, the total momentum conservation almost holds, meaning that the previous estimates may be improved in some cases.

4. Numerical examples

As we already pointed out, a significant drawback of our scheme (15)–(16) is that it does not ensure the exact conservation of the total momentum, since it involves a scheme on the velocity which does not conserve the momentum. Moreover, initial data with some vacuum need to be regularized since our scheme cannot stand nil values of ϱ. In this section, apart from checking that the OSL condition is satisfied (or not, if studying the behaviour of the upwind scheme), we shall also study the behaviour of the total momentum.

Of course, the time and space steps in the following tests are chosen such as the CFL condition is satisfied when using the upwind scheme, and (13)–(14) when using the diffusive scheme.

4.1. Almost nil velocity everywhere except in 0.5. This test is related to the one described in Proposition 3 to prove that the OSL condition was eventually not satisfied by the upwind scheme. We first choose $\varepsilon = 10^{-6}$. The space step is set to $\Delta x = 10^{-4}$ on $[0, 1]$, i.e. $I = 10^4$, and the Courant number to $\lambda = 0.25$, so that $\Delta t = 2.5\ 10^{-5}$, and perform 40 iterations in time, i.e. $T = 10^{-3}$ s. We cannot use the initial data from Proposition 3 as they are, since the assumptions of both Theorems 2 and 4 must hold. Nevertheless, since u cannot be nil as in (11), it is important to lower the density everywhere else from the maximal value of u.

Consequently, we set, for any $i \neq I/2$,

$$\varrho_i^0 = \frac{\varepsilon^{1/4}}{2} \simeq 0.0158, \qquad u_i^0 = \frac{2\Delta x}{\sqrt{\varepsilon}} = 0.2,$$

and $\varrho_{I/2}^0 = 1$, $u_{I/2}^0 = 1$. In that case, the interesting value of x is no longer 0 or 1, it is $x = 0.5$.

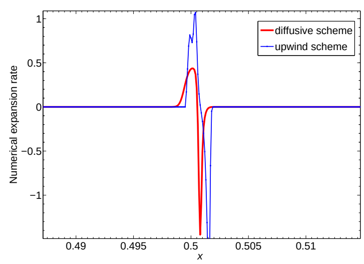

Figure 1. Numerical expansion rate near 0.5 at final time T

4.2. **Piecewise linear velocity.** There are other situations when the upwind scheme does not satisfy the OSL condition. For instance, let us consider the following set of initial data

$$\varrho^{\mathrm{in}}(x) = 1, \qquad u^{\mathrm{in}}(x) = 1 - x \geq 0, \qquad \forall x \in [0,1), \tag{22}$$

extended by 1-periodicity on $\mathbb{R}$. In both tests, we choose $T = 1.2$ and $\Delta x = 10^{-4}$.

4.2.1. *Using the upwind scheme.* Using the upwind scheme implies choosing the Courant number λ so that the CFL condition holds. We set $\lambda = 0.1$, which ensures $\lambda \max u < 1$. Then, on Figure 2, the positive part of the numerical expansion rate w is plotted on $[0,1]$.

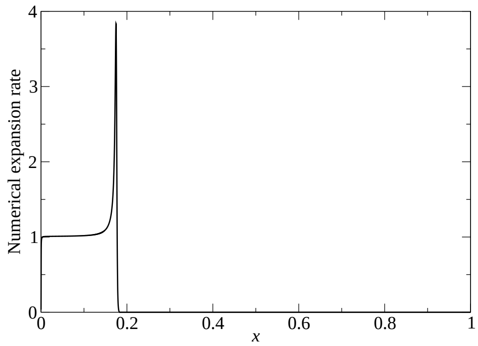

FIGURE 2. "Upwind" plot of w^+ at $t = 0.2\,\mathrm{s}$ with initial data (22)

It is then clear that there are some values of i such that $w_i > 1$, and, in anticipation of the next paragraph, we must point out that, of course, choosing a lower Courant number does not have any effect on the behaviour of the numerical expansion rate.

4.2.2. *Using the diffusive scheme.* We here choose $\varepsilon = 0.001$. As explained in Section 3, the initial data must be regularized: both ϱ and q must be $C^1(\mathbb{R};\mathbb{R}_+^*)$. The initial datum u is regularized near 0 in order to have a reasonable periodic agreement with the value in 1, and satisfies $u^{\mathrm{in}} \geq \Delta x/2$ for any i. Since (14) must hold, it is possible to check that $\lambda = 0.01$ (and $\Delta t = 10^{-6}$) is a relevant choice.

This time, the OSL condition is satisfied, as one can see on Figure 3 at $x = 0.1$, where the upwind scheme experiences trouble with the expansion rate for times smaller than 0.2.

Eventually, to investigate the total numerical momentum, on Figure 4, we show its behaviour with respect to t, till T, and the result is quite convincing. On the same figure, we also show the total numerical mass, which is of course exactly conserved.

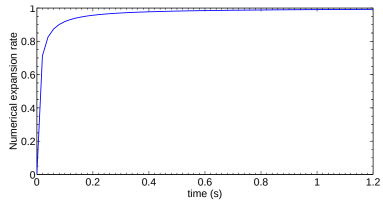

FIGURE 3. "Diffusive" plot of w^+ at $x = 0.1$ with regularized initial data (22)

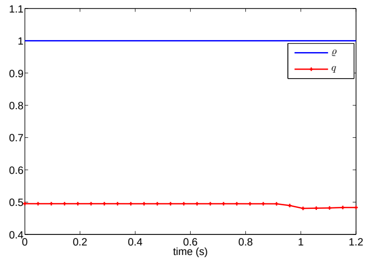

FIGURE 4. Numerical total mass and momentum

4.3. **Continuous velocity and piecewise constant density.** Of course, the upwind scheme may often provide a numerical solution satisfying the OSL condition. It is then interesting to check the behaviour of both upwind and diffusive schemes, which should have similar behaviours. Let $\varepsilon = 10^{-12}$. We choose $T = 2$, $\Delta x = 0.002$, and $\lambda = 0.1$ for both upwind and diffusive cases. Then we consider the following initial data for the density

$$\varrho^{\text{in}}(x) = 1, \quad 0 \leq x < 0.2, \qquad \varrho^{\text{in}}(x) = 0.5, \quad 0.2 \leq x < 1,$$

and for the velocity

$$u^{\text{in}}(x) = 0.5(1 - \cos(10\pi x)) + 16\Delta x, \qquad 0 \leq x < 1,$$

extended by 1-periodicity on $\mathbb{R}$.

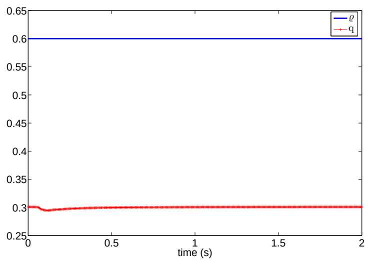

FIGURE 5. Numerical total mass and momentum computed with the diffusive scheme

First, we check on Figure 5 that the numerical total momentum is still well conserved by the diffusive scheme.

Let us get into some more details of the behaviour of both schemes with respect to time. For small times, one can check on Figures 6–7 that both schemes give very similar results for ϱ, u and w. If we accurately study Figure 6b, we can see that the upwind scheme has very small variations with respect to the diffusive scheme near some points, which are in fact the jump points of the density, see Figure 8a.

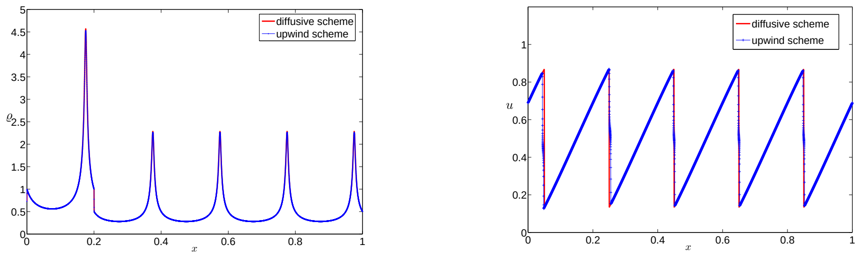

FIGURE 6. (a) Density at 0.04 s, (b) velocity at 0.2 s

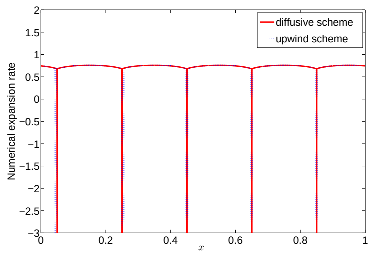

FIGURE 7. Numerical expansion rate at time 0.2 s

Hence, when time grows, the behaviours of both schemes become more and more different, as seen on Figures 8–10, for quite small times for the density, later for the velocity and the numerical expansion rate.

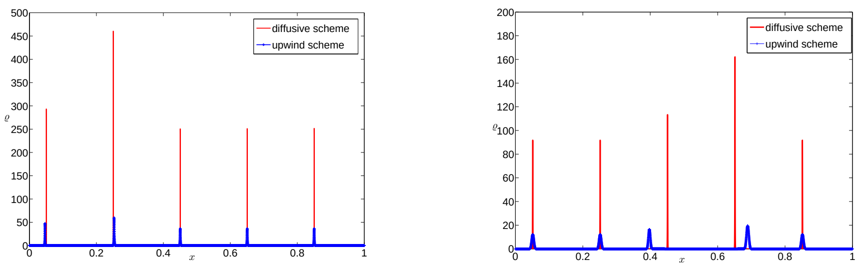

FIGURE 8. Density at times (a) 0.2 s, and (b) 1 s

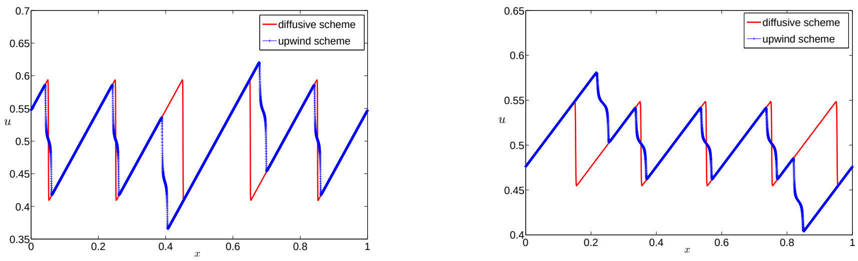

FIGURE 9. Velocity at times (a) 1 s, and (b) 2 s

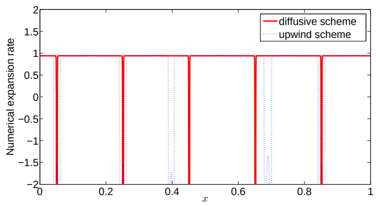

FIGURE 10. Numerical expansion rate at time 1 s

It is important to note that the numerical expansion rates are still upper bounded by 1, for both schemes. The differences between the numerical solutions is consequently not related to the OSL condition. In fact, we believe that the diffusive scheme is more trustworthy. Indeed, the upwind scheme has natural numerical diffusion, which is responsible for the variations. This numerical diffusion seems to be fully avoided by the diffusive scheme: it is absorbed by the artificial viscosity inserted in the scheme, and its effect cannot numerically appear.

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