Riemann invariant solutions of the isentropic fluid flow equations[1]

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Abstract

A new version of the conditional symmetry method is used to obtain rank-k solutions expressed in terms of Riemann invariants of the isentropic compressible ideal fluid flow in $(3+1)$ dimensions. A detailed description of the procedure for constructing bounded solutions in terms of elliptic Weierstrass $\wp$-function is presented.

1 Conditional symmetry method for the isentropic fluid flow

In this section we present a brief description of a procedure detailed in [4] for constructing rank-k solutions in terms of Riemann invariants of a compressible ideal fluid in $(3+1)$ dimensions

$$u_t^\alpha + \sum_{\beta=1}^{4}\sum_{i=1}^{3} \mathcal{A}^{i\alpha}_{\ \beta}(u) u_i^\beta = 0, \quad \alpha = 1,2,3,4, \tag{1.1}$$

where $\mathcal{A}^1, \mathcal{A}^2$ and $\mathcal{A}^3$ are 4×4 real-valued matrix functions of the form

$$\mathcal{A}^i = \begin{pmatrix} u^i & \delta_{i1}\kappa^{-1}a & \delta_{i2}\kappa^{-1}a & \delta_{i3}\kappa^{-1}a \\ \delta_{i1}\kappa a & u^i & 0 & 0 \\ \delta_{i2}\kappa a & 0 & u^i & 0 \\ \delta_{i3}\kappa a & 0 & 0 & u^i \end{pmatrix}, \quad i=1,2,3.$$

The independent and dependent variables are denoted by $x = (t, x^1, x^2, x^3) \in X \subset \mathbb{R}^4$ and $u = (a, \vec{u}) \in U \subset \mathbb{R}^4$, respectively, and u_i stands for the first order partial derivatives of u, i.e. $u_i^\alpha \equiv \partial u^\alpha / \partial x^i$. Here a stands for the velocity of sound in the medium and $\vec{u}$ is the velocity vector field of the flow. Throughout this paper, we adopt the summation convention over repeated lower and upper indices, except in the case in which one index is taken in brackets. The purpose of this article is to obtain rank-k solutions of system (1.1) expressible in terms of Riemann invariants. To this end, we seek solutions $u(x)$ of (1.1) defined implicitly by the following set of relations between the variables u^α, r^A and x^i

$$u = f(r^1(x,u), \ldots, r^k(x,u)), \quad r^A(x,u) = \lambda_i^A(u) x^i, \quad \ker(\lambda_0^A \mathcal{I}_4 + \mathcal{A}^i(u)\lambda_i^A) \neq 0, \tag{1.2}$$

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for some function $f : \mathbb{R}^k \to \mathbb{R}^4$ and $A = 1, \ldots, k \leq 3$. Such a solution is called a rank-k solution if $\mathrm{rank}(u_i^\alpha) = k$. The functions $r^A(x, u)$ are called the Riemann invariants associated with the wave vectors $\lambda^A = (\lambda_0^A, \vec{\lambda}^A) \in \mathbb{R}^4$ of the system (1.1). Here, $\vec{\lambda}^A = (\lambda_1^A, \lambda_2^A, \lambda_3^A)$ denotes a direction of wave propagation and the eigenvalue λ_0^A is a phase velocity of the considered wave. Two types of admissible wave vectors for the isentropic equations (1.1) are obtained by solving the dispersion relation

$$\det(\lambda_0(u)\mathcal{I}_4 + \lambda_i(u)\mathcal{A}^i(u)) = [(\lambda_0 + \vec{u} \cdot \vec{\lambda})^2 - a^2\vec{\lambda}^2](\lambda_0 + \vec{u} \cdot \vec{\lambda})^2 = 0. \tag{1.3}$$

They are called the entropic (E) and acoustic (S) wave vectors and are defined by

$$\text{i) } \lambda^E = (\epsilon a + \vec{u} \cdot \vec{e}, -\vec{e}), \quad \epsilon = \pm 1, \quad \text{ii) } \lambda^S = (\det(\vec{u}, \vec{e}, \vec{m}), -\vec{e} \times \vec{m}), \quad |\vec{e}|^2 = 1, \tag{1.4}$$

where $\vec{e}$ and $\vec{m}$ are unit and arbitrary vectors, respectively.

The construction of rank-k solutions through the conditional symmetry method is achieved by considering an overdetermined system, consisting of the original system (1.1) in p independent variables together with a set of compatible first order differential constraints (DCs),

$$\xi_a^i(u)u_i^\alpha = 0, \quad \lambda_i^A \xi_a^i = 0, \quad a = 1, \ldots, p-k, \tag{1.5}$$

for which a symmetry criterion is automatically satisfied. Under the above circumstances, the following result holds.

Proposition 1. *A nondegenerate quasilinear hyperbolic system of first order PDEs (1.1) in p independent and q dependent variables admits a $(p-k)$-dimensional conditional symmetry algebra L if and only if there exists a set of $(p-k)$ linearly independent vector fields*

$$X_a = \xi_a^i(u)\frac{\partial}{\partial x^i}, \quad a = 1, \ldots, p-k, \quad \ker\left(\mathcal{A}^i(u)\lambda_i^A\right) \neq 0, \quad \lambda_i^A \xi_a^i = 0, \quad A = 1, \ldots, k,$$

which satisfy on some neighborhood of $(x_0, u_0) \in X \times U$ the trace conditions

$$\text{i)} \quad \mathrm{tr}\left(\mathcal{A}^\mu \frac{\partial f}{\partial r}\lambda\right) = 0, \quad \text{ii)} \quad \mathrm{tr}\left(\mathcal{A}^\mu \frac{\partial f}{\partial r}\eta_{(a_1}\frac{\partial f}{\partial r} \cdots \eta_{a_s)}\frac{\partial f}{\partial r}\lambda\right) = 0, \quad \mu = 1, \ldots, l, \tag{1.6}$$

where

$$\lambda = (\lambda_i^A) \in \mathbb{R}^{k \times p}, \quad r = (r^1, \ldots, r^k) \in \mathbb{R}^k, \quad \frac{\partial f}{\partial r} = \left(\frac{\partial f^\alpha}{\partial r^A}\right) \in \mathbb{R}^{q \times k},$$

$$\eta_{a_s} = \left(\frac{\partial \lambda_{a_s}^A}{\partial u^\alpha}\right) \in \mathbb{R}^{k \times q}, \quad s = 1, \ldots, k-1,$$

$\mathcal{I}_q$ is the q by q identity matrix and $(a_1, \ldots, a_s)$ denotes the symmetrization over all indices in the bracket. Solutions of the system which are invariant under the Lie algebra L are precisely rank-k solutions of the form (1.2).

The proof of this proposition has been explained in detail in [4]. Note that these symmetries are not symmetries of the original system, but they can be used to construct solutions of the overdetermined system composed of (1.1) and (1.5).

In general, the overdetermined system composed of (1.6 i) and (1.6 ii) is nonlinear and cannot always be solved in a closed form. Nevertheless, particular rank-k solutions for many physically interesting systems of PDEs are well worth pursuing. These particular solutions of (1.6 i) and (1.6 ii) can be obtained by assuming that the function f is in the form of a rational function, which may also be interpreted as a truncated Laurent series in the variables r^A. This method can work only for equations having the Painlevé property [2]. Consequently, these equations can be very often integrated in terms of known functions.

2 Invariant rank-k solutions with time-dependent sound velocity

Let us now consider the isentropic flow of an ideal and compressible fluid in the case when the sound velocity depends on the time t only. The system (1.1) in $(k+1)$ dimensions becomes

$$\begin{aligned}&\vec{u}_t + (\vec{u}\cdot\nabla)\vec{u} = 0,\\ &a_t + \kappa^{-1} a \,\mathrm{div}\vec{u} = 0, \quad a_{x^j} = 0, \quad a > 0, \quad \kappa = 2(\gamma-1)^{-1}, \quad j = 1,\dots,k.\end{aligned} \tag{2.1}$$

It was shown [4] that our approach enables us to construct general rank-k solutions of (2.1). The change of coordinates on $\mathbb{R}^{k+1}\times\mathbb{R}^{k+1}$

$$\bar{t} = t, \quad \bar{x}^1 = x^1 - u^1 t, \dots \bar{x}^k = x^k - u^k t, \quad \bar{a} = a, \quad \bar{u} = u \in \mathbb{R}^k.$$

transforms (2.1) into the system

$$\frac{\partial \bar{u}}{\partial \bar{t}} = 0, \quad \frac{\partial}{\partial \bar{t}} \ln \bar{a} + \kappa^{-1} \mathrm{tr}\left((\mathcal{I}_k + \bar{t} D\bar{u}(\bar{x}))^{-1} D\bar{u}(\bar{x})\right) = 0, \quad \frac{\partial \bar{a}}{\partial \bar{x}} = 0,$$

where $D\bar{u}(\bar{x}) \in \mathbb{R}^{k\times k}$ is the Jacobian matrix and $\bar{x} = (\bar{x}^1,\dots,\bar{x}^k) \in \mathbb{R}^k$.

It is easy to demonstrate that, in the original coordinates $(x,u)\in\mathbb{R}^p\times\mathbb{R}^q$, the general rank-$k$ solution of (2.1) takes the form

$$u = f(x^1 - u^1 t, \dots, x^k - u^k t), \quad a(t) = A_1(1 + p_1 t + \dots + p_k t^k)^{-1/\kappa},$$

where it is required that the coefficients p_n, $n = 1,\dots,k$, of the characteristic polynomial

$$Df(\bar{x})^k - \sum_{n=1}^{k} p_n(Df(\bar{x}))(Df(\bar{x}))^{k-n} = 0$$

of the Jacobian matrix $Df(\bar{x})$ be constant.

For the particular case when $k = 2$, the general rank-2 solution of (2.1) invariant under the vector field

$$X = \frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y}$$

is implicitly defined by

$$\begin{aligned}&u(t,x,y) = C_1(x - ut) + \frac{\partial h}{\partial r^2}(x-ut, y-vt), \quad v(t,x,y) = C_1(y - vt) - \frac{\partial h}{\partial r^1}(x-ut, y-vt),\\ &a(t) = A_1((1 + C_1 t)^2 + B_1 t^2)^{-1/\kappa}, \quad A_1 \in \mathbb{R}^+.\end{aligned}$$

Here the function h depends on two variables, $r^1 = x - ut$ and $r^2 = y - vt$, and satisfies the inhomogeneous Monge-Ampère (MA) equation

$$h_{r^1r^1} h_{r^2r^2} - h^2_{r^1r^2} = b. \tag{2.2}$$

where $b = B_1 - C_1^2$ can be normalized to $0, \pm 1$. Note that the Monge-Ampère (MA) equation has a geometrical meaning in projective geometry in $\mathbb{R}^3$. Namely, if $r^3 = h(r^1, r^2)$ is the equation of an improper affine sphere with affine normals parallel to the r^3-axis, then h has to satisfy the MA equation (2.2).

According to E. Goursat [3], the MA equation (2.2) can be transformed into the linear Laplace-Beltrami equation

$$\tilde{h}_{r^2r^2} + b\tilde{h}_{zz} = 0, \quad b = 0, \pm 1, \tag{2.3}$$

by using the half-Legendre transformation $(r^1, r^2, h) \to (s, r^2, \tilde{h})$ given by

$$\tilde{h}(z, r^2) = h(s, r^2) - s h_s(s, r^2), \quad z = h_s(s, r^2), \quad h_{ss} \neq 0.$$

Using an explicit form of solution of the linear equation (2.3), it is possible to find the general solution of the MA equation (2.2). In general, for the elliptic ($b > 0$), hyperbolic ($b < 0$) and parabolic ($b = 0$) cases all global real solutions of the MA equation (2.2) are known. Using this Goursat approach, for each solution $\tilde{h}(z, r^2)$ of the equation (2.3) we can associate a solution $h(r^1, r^2)$ of the Monge-Ampère equation (2.2). Depending on the selection of particular solutions of the MA equation (2.2) we obtain Riemann double waves or other types of rank-2 solutions of (2.1). Invariant solutions of system (2.1) associated with particular solutions of MA equation (2.2) are summarized in Table I.

3 Rank-2 and rank-3 solutions

The construction approach outlined in Section 1 has been applied to the isentropic flow equations (1.1) in order to obtain rank-2 and rank-3 solutions. The results of our analysis are summarized in Tables II and III. Several of them possess a certain amount of freedom. They depend on one or two arbitrary functions of one or two Riemann invariants, depending on the case. The range of the types of solutions obtained depends on different combinations of the vector fields X_a. For convenience, we denote by E_iE_j,E_iS_j,S_iS_j,$E_iE_jE_k$, etc, $i, j, k = 1, 2, 3$, the solutions which are the result of nonlinear superpositions of rank-1 solutions associated with different types of wave vectors (1.4 i) and (1.4 ii). By r^1, r^2 and r^3 we denote the Riemann invariants which coincide with the group invariants of the differential operators X_a of the solution under consideration.

The arbitrary functions appearing in the solutions listed in Tables II and III allow us to change the geometrical properties of the governed fluid flow in such a way as to exclude the presence of singularities. This fact is of special significance here since, as is well known, in most cases, even for arbitrary smooth and sufficiently small initial data at $t = t_0$, the magnitude of the first derivatives of Riemann invariants becomes unbounded in some finite time T. Thus, solutions expressible in terms of Riemann invariants usually admit a gradient catastrophe. Nevertheless, we have been able to show that it is still possible in these cases to construct bounded solutions of soliton-type expressed in terms of elliptic functions, through the proper selection of the arbitrary functions appearing in the general solution. For this purpose it is useful to select DCs corresponding to a certain class of the nonlinear Klein-Gordon equation which is known to possess rich families of soliton-like solutions [1]. Thus, we submit the arbitrary function(s) appearing in the general solutions listed in Tables II and III, say v, to the differential constraint in the form of the Klein-Gordon v^6-field equation in three independent variables r^1, r^2 and r^3

$$\Box_{(r^1,r^2,r^3)} v = cv^5, \quad c \in \mathbb{R}. \tag{3.1}$$

It is well known that equation (3.1) is invariant with respect to the similitude Lie algebra involving translations, rotations, Lorentz boosts and a dilation. A systematic use of the subgroup structure [7] of the invariance group of (3.1) allows us to generate all symmetry variables ξ in terms of Riemann invariants r^1, r^2, r^3. We concentrate here only on the case when symmetry variables are invariants of the assumed subgroups $G_i \subset G$ having generic orbits of codimension one. The set of symmetry variables ξ enables us to reduce, after some transformation, equation (3.1) to many possible ODEs. The application of the symmetry reduction to equation (3.1) leads to solutions of the form

$$v(r) = \alpha(r)F(\xi), \quad r = (r^1, r^2, r^3), \tag{3.2}$$

where the multiplier $\alpha(r)$ and the symmetry variables $\xi(r)$ are given explicitly by group theoretical considerations and $F(\xi)$ satisfies an ODE obtained by substituting (3.2) into equation (3.1). The results are presented in Table IV which includes the reduction obtained by subgroups $G_i \subset G$. The detailed procedure for obtaining such ODEs can be found in [7].

Under a transformation $(F, \xi) \to (U, \zeta)$ of the equations listed in the Table IV which preserves the Painlevé property,

$$F(\xi) = \left(\frac{U(\zeta)}{g(\xi)}\right)^{1/2}, \ \left(\frac{d\zeta}{d\xi}\right)^2 = \frac{1}{Gg^2},$$

in which the four sets of functions G, g and constants e_0, c_4 obey the respective conditions

$$
\begin{aligned}
&1.\quad G=-\frac{3c_4}{4},\ g^2=\frac{4e_0}{c_4},\\
&2.\quad G=-\frac{3c_4}{4}\xi(\xi+1),\ g^2=-\frac{64e_0}{c_4}\xi,\\
&3.\quad G=-\frac{3c_4}{4},\ (a,e_0,g^2)=(0,0,k_1),\ (4/3,0,k_1\xi^{4/3}),\ (2,e_0,-\frac{16e_0}{c_4}\xi^2),\\
&4.\quad G=-\frac{3c_4}{4}(\xi^2+1),\ g=k_1(1+\xi^2)^{1/3},\ e_0=0,\quad k_1\neq 0,
\end{aligned}
$$

we obtain the first integral

$$U'^2-c_4U^4-12e_0U^2-4K'U=0,\ c_4\neq 0.$$

When $K'=0$, U^{-1} is either a sin, cos, sinh or a cosh function, depending on the signs of the constants, then bounded solutions are easily characterized. When $K'\neq 0$, it is convenient to first integrate this elliptic equation in terms of the Weierstrass $\wp$ function,

$$U(\zeta)=\frac{K'}{\wp(\zeta)-e_0},\ g_2=12e_0^2,\quad g_3=-8e_0^3-c_4K'^2,\quad \wp'^2=4\wp^3-g_2\wp-g_3,$$

then to use the classical relations between $\wp$ and the various bounded Jacobi functions [5]. We give here the explicit solutions of the equations listed in Table IV in terms of the Weierstrass $\wp$-function, leaving the conversion to Jacobi's notation to the reader. These solutions are obtained by convenient choices of the normalization constants e_0, c_4, K', k_1.

With the normalization $e_0=-1/3,\ c_4=-4/3,\ K'=C$, the general solution of equation no 1 is

$$F^2(\xi)=\frac{C}{\wp(\xi)+1/3},\ \zeta=\xi,\ g_2=\frac{4}{3},\ g_3=\frac{8}{27}+\frac{4}{3}C^2,\ C\in\mathbb{R}. \tag{3.3}$$

With the normalization $e_0=k_0^{-2}/48,\ c_4=-(4/3)k_0^{-2},\ K'=C$, the solution of equation no 2 is

$$F^2(\xi)=\frac{C\xi^{-1/2}}{\wp(\zeta)-\frac{1}{48k_0^2}},\ \zeta=-2k_0\,\mathrm{Argth}\sqrt{\xi+1},\ g_2=\frac{1}{192k_0^4},\ g_3=-\frac{1}{13824k_0^6}+\frac{4C^2}{3k_0^2}, \tag{3.4}$$

with $k_0, C\in\mathbb{R}$.
The three cases for equation no 3 yield the respective solutions

$$
\begin{aligned}
&q=-k/3: F^2(\xi)=\frac{C}{\wp(\xi)},\ \zeta=\xi,\ g_2=0,\ g_3=\frac{4C^2}{3},\\
&q=4-3k: F^2(\xi)=\frac{C\xi^{-2/3}}{\wp(\zeta)},\ \zeta=3k_0\xi^{1/3},\ g_2=0,\ g_3=\frac{4C^2}{3k_0^2},\\
&q=k-2: F^2(\xi)=\frac{C\xi^{-1}}{\wp(\zeta)-\frac{1}{12k_0^2}},\ \zeta=k_0\log\xi,\ g_2=\frac{1}{12k_0^4},\ g_3=-\frac{1}{216k_0^6}+\frac{4C^2}{3k_0^2}.
\end{aligned}
\tag{3.5}
$$

Finally, equation no 4 integrates as (equation no 4 in Table IV)

$$F^2(\xi)=\frac{C(\xi^2+1)^{-1/3}}{\wp(\zeta)},\zeta=\xi\ {}_2F_1\left(\frac{1}{2},\frac{5}{6};\frac{3}{2};-\xi^2\right),\ g_2=0,\ g_3=\frac{4C^2}{3k_0^2}, \tag{3.6}$$

where ${}_2F_1$ denotes the hypergeometric function.

Using these results, we construct bounded rank-3 solutions of the equations (1.1). For this purpose, for each general solution appearing in Tables II and III, we introduce the arbitrary functions into the Klein-Gordon equation (3.1) and select only the solutions expressed in terms of the Weierstrass $\wp$-function.

As an illustration, let us consider the rank-3 solution for the case $E_1E_2E_3$. This solution exists if the angles ϕ_{ij} between the entropic wave vectors $\lambda^1, \lambda^2, \lambda^3$ satisfy the following condition [6]

$$\cos\phi_{ij}=-\frac{1}{\kappa},\quad i\neq j=1,2,3.$$

Requiring that each of the functions $a_i(r^i)$, $i = 1, 2, 3$, obeys the equation no 1 listed in Table IV, the rank-3 solution of the type $E_1E_2E_3$ becomes, according to equation (3.3),

$$a = \sum_{i=1}^{3} \frac{C_i}{\left(\wp(r^i, \frac{4}{3}, \frac{8}{27} + \frac{4}{3}C_i^4) + \frac{1}{3}\right)^{1/2}}, \quad \vec{u} = \kappa \sum_{i=1}^{3} \frac{C_i\vec{\lambda}^i}{\left(\wp(r^i, \frac{4}{3}, \frac{8}{27} + \frac{4}{3}C_i^4) + \frac{1}{3}\right)^{1/2}},$$
$$r^i = -(1+\kappa)\frac{C_i}{\left(\wp(r^i, \frac{4}{3}, \frac{8}{27} + \frac{4}{3}C_i^4) + \frac{1}{3}\right)^{1/2}} t + \vec{\lambda}^i \cdot \vec{x}, \quad i = 1, 2, 3.$$

This solution is physically interesting since it remains bounded for every value of the Riemann invariants r^i. Thus, it represents a bounded solution with periodic flow velocities. Similarly, it is possible to substitute the arbitrary functions into the differential equations no $2, 3, 4$ listed in Table IV to obtain other types of bounded solutions. In Table V, we present resulting types of solutions with their corresponding Riemann invariants. They are all bounded solutions of periodic, bump or kink type. Note that these solutions admit the gradient catastrophe at some finite time. Hence, some discontinuities can occur like shock waves. Note also that the solutions remain bounded even when the first derivatives of r^i tend to infinity after some finite time T. However, after time T, the solution cannot be represented in parametric form by the Riemann invariants and ceases to exist.

4 Concluding remarks

In this paper we have shown how to construct rank-2 and 3 periodic solutions expressed in terms of the Weierstrass $\wp$-function. They represent bumps, kinks and multiple wave solutions which are parametrized by Riemann invariants. These solutions remain bounded even when the invariants admit the gradient catastrophe. We have also found rank-2 solutions of the ideal isentropic fluid flow when the sound velocity depends on the time only. These solutions were obtained through the use of the conditional symmetry method and the Legendre transformation applied to the Monge-Ampère equations. We are currently looking at the stability property of the obtained solutions. Solutions which possess the property of stability should be observable physically and such analysis could be the starting point for perturbative computations. This task will be undertaken in a future work.

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Table I. Isentropic solutions of the system (1.1) associated with MA equation (2.2). The Riemann invariants r^1, r^2 are obtained by introducing the expressions for u and v into $r^1 = x - ut$, $r^2 = y - vt$.

No	b	Solutions	Comments
1.i	-1	$u = -\frac{-C_1\, r^1 \sqrt{-6\, r^1 - 36\, (r^2)^2} + 12\, r^2\, r^1 + 72\, (r^2)^3}{\sqrt{-6\, r^1 - 36\, (r^2)^2}}$ $v = \frac{C_1\, r^2 \sqrt{-6\, r^1 - 36\, (r^2)^2} + r^1 + 6\, (r^2)^2}{\sqrt{-6\, r^1 - 36\, (r^2)^2}}$ $a = A_1(1 + 2C_1 t + (2{C_1}^2 - 1)t^2)^{-1/k}$	$C_1 \in \mathbb{R}$
1.ii	-1	$u = r^1 \left(C_1 - \frac{(A_2 e^{2r^2} + B_2)^2 + 2A_2 r^1 e^{2r^2}}{(A_2 e^{2r^2} + B_2)(r^1 + (r^2 - 1)(A_2 e^{2r^2} + B_2))} \right)$ $v = C_1 r^2 + \frac{r^1}{r^1 + (r^2 - 1)(A_2 e^{2r^2} + B_2)}$ $a = A_1(1 + 2C_1 t + (2{C_1}^2 - 1)t^2)^{-1/k}$	$A_1, A_2, B_2, C_1 \in \mathbb{R}$
2.i	1	$u = \frac{C_1 x + ({C_1}^2 + 1)tx - 4y}{(C_1 + 1)t^2 + 2C_1 t + 1}$ $v = \frac{1}{4} \frac{4({C_1}^2 + 1)ty + 4C_1 y + x}{(C_1 + 1)t^2 + 2C_1 t + 1}$ $a = A_1(1 + 2C_1 t + (2{C_1}^2 + 1)t^2)^{-1/k}$	$C_1 \in \mathbb{R}$
2.ii	1	$u = C_1 r^1 + \frac{12 r^2 (r^1 - 6(r^2)^2)}{\sqrt{36(r^2)^2 - 6r^1}}$ $v = C_1 r^2 + \frac{r^1 - 6(r^2)^2}{\sqrt{36(r^2)^2 - 6r^1}}$ $a = A_1(1 + 2C_1 t + (2{C_1}^2 + 1)t^2)^{-1/k}$	$A_1, C_1 \in \mathbb{R}$
2.iii	1	$u = (C_1 + 2\alpha_{12})r^1 + 2\alpha_{22} r^2 + \beta_2$ $v = (C_1 - 2\alpha 12)r^2 - 2\alpha_{11} r^1 - b_1$ $a = A_1(1 + 2C_1 t + (2{C_1}^2 + 1)t^2)^{-1/k}$	$\alpha_{ij}, \beta_i \in \mathbb{R}, i, j = 1, 2$
3.i	0	$u = C_1 r^1 + \frac{1}{2}(r^1 + r^2)$ $v = C_1 r^2 - \frac{1}{2}(r^1 + r^2)$ $a = A_1(1 + 2C_1 t(1 + C_1 t))^{-1/k}$	$A_1, C_1 \in \mathbb{R}$
3.ii	0	$u = C_1 r^1 + \sqrt{1 - \left(\frac{r^1}{1 + r^2}\right)^2}$ $v = C_1 r^2 - \arcsin\left(\frac{r^1}{1 + r^2}\right)$ $a = A_1(1 + 2C_1 t(1 + C_1 t))^{-1/k}$	$A_1, C_1 \in \mathbb{R}$
3.iii	0	$u = C_1 r^1 + \alpha_2 \varphi' + \beta_2$ $v = C_1 r^2 - \alpha_1 \varphi' - \beta_1$ $a = A_1(1 + 2C_1 t(1 + C_1 t))^{-1/k}$	$C_1, \alpha_i, \in \mathbb{R}, i = 1, 2$

Table II : Rank-2 solutions with the freedom of one,two or three arbitrary functions of one or two variables. Unassigned unknown functions $a(\cdot), u(\cdot), \dots$ are arbitrary functions of their respective arguments.

No	Type	Vector Fields	Riemann Invariants	Solutions
1	E_1S_1	$X_1 = \frac{\partial}{\partial x^2} - \frac{\sigma_2}{\beta_1}\frac{\partial}{\partial t} - \frac{\beta_2}{\beta_1}\frac{\partial}{\partial x^1}$ $X_2 = \frac{\partial}{\partial x^3} - \frac{\sigma_3}{\beta_1}\frac{\partial}{\partial t} - \frac{\beta_3}{\beta_1}\frac{\partial}{\partial x^1}$ $\beta_i = -(\vec{e}^2 \times \vec{m}^2)_i(a + \vec{e}^1 \cdot \vec{u}) + e_i^1[\vec{u}, \vec{e}^2, \vec{m}^2]$ $\sigma_j = -e_1^1(\vec{e}^2 \times \vec{m}^2)_j + e_j^1(\vec{e}^2 \times \vec{m}^2)_1, j = 2, 3$	$r^1 = ((1+k)\bar{a}_1(r^1) + C_2)t - \vec{e}^1 \cdot \vec{x}$ $r^2 = Ct - [\vec{x}, \vec{e}^2, \vec{m}^2], \quad [\vec{e}^1, \vec{e}^2, \vec{m}^2] = 0$ $C_2 = (C_1 e_1^1 - e_3^1)^{-1}$	$\bar{a} = \bar{a}_1(r^1) + a_0, \quad [\vec{u}_2, \vec{e}^2, \vec{m}^2] = C$ $\vec{u} = k\bar{a}_1(r^1) + \vec{u}_2(r^2), \quad \bar{u}_2^3(r^2) = C_1\bar{u}_2^1(r^2)$ $a_0, C, C_1, C_2 \in \mathbb{R}$
$2a$	S_1S_2	$X_1 = \frac{\partial}{\partial t} + u^1\frac{\partial}{\partial x^1} + u^2\frac{\partial}{\partial x^2}$ $X_2 = \frac{\partial}{\partial x^3}$	$r^1 = x^1 - u^1 t$ $r^2 = x^2 - u^2 t$	$\bar{a} = a_0, \quad \bar{u}^1 = -\phi_{r^2}, \quad \bar{u}^2 = \phi_{r^1},$ $\phi = \varphi(\alpha_1 r^1 + \alpha_2 r^2) + \beta_1 r^1 + \beta_2 r^2 + \gamma,$ $\bar{u}^3 = \bar{u}^3(r^1, r^2), \quad a_0, \alpha_i, \beta_i, \gamma \in \mathbb{R}, i = 1, 2,$
$2b$	S_1S_2	$X_1 = \frac{\partial}{\partial t} + u^1\frac{\partial}{\partial x^1} + u^2\frac{\partial}{\partial x^2}$ $X_2 = \frac{\partial}{\partial x^3}$	$r^1 = x^1 - u^1 t$ $r^2 = x^2 - u^2 t$	$\bar{a} = a_0, \quad \bar{u}^2 = \bar{u}^3 = g(x^1 - x^2), \quad a_0 \in \mathbb{R},$ $\bar{u}^1 = b(x^1 - tg(x^1 - x^2), x^2 - tg(x^1 - x^2))$
$2c$	S_1S_2	$X_2 = \frac{\partial}{\partial x^2} - \frac{\sigma_2}{\beta_1}\frac{\partial}{\partial t} - \frac{\beta_2}{\beta_1}\frac{\partial}{\partial x^1}$ $X_3 = \frac{\partial}{\partial x^3} - \frac{\sigma_3}{\beta_1}\frac{\partial}{\partial t} - \frac{\beta_3}{\beta_1}\frac{\partial}{\partial x^1}$ $\beta_j = \lambda_j^2[\vec{u}, \vec{e}^1, \vec{m}^1] - \lambda_j^1[\vec{u}, \vec{e}^2, \vec{m}^2]$ $\sigma_i = \lambda_1^1\lambda_i^2 - \lambda_i^1\lambda_1^2$	$r^1 = \left(C_1 + \frac{\lambda_1^1}{\lambda_1^2}C_2\right)t - \vec{\lambda}^1 \cdot \vec{x}$ $r^2 = \left(C_2 + \frac{\lambda_1^2}{\lambda_1^1}C_1 + G(r^1)\right)t - \vec{\lambda}^2 \cdot \vec{x}$ $\lambda_i^j = -(\vec{e}^j \times \vec{m}^j)_i$ $G(r^1) = \frac{1}{\lambda_1^1}\big((\lambda_1^1\lambda_2^2 - \lambda_2^1\lambda_1^2)\bar{u}_1^2(r^1)$ $+ (\lambda_1^1\lambda_3^2 - \lambda_1^2\lambda_3^1)\bar{u}_1^3(r^1)\big)$	$\bar{a} = a_0, \quad a_0, C_1, C_2 \in \mathbb{R}$ $\bar{u}^1 = \frac{1}{\lambda_1^1}(C_1 - \lambda_2^1\bar{u}_1^2(r^1) - \lambda_3^1\bar{u}_1^3(r^1))$ $- \left(\frac{\lambda_3^2}{\lambda_1^2}\eta + \frac{\lambda_2^2}{\lambda_1^2}\right)\bar{u}_2^2(r^2) + \frac{C_2}{\lambda_1^2}$ $\bar{u}^2 = \bar{u}_1^2(r^1) + \bar{u}_2^2(r^2)$ $\bar{u}^3 = \bar{u}_1^3(r^1) + \eta\bar{u}_2^2(r^2), \quad \eta = \frac{\lambda_1^2\lambda_2^1 - \lambda_1^1\lambda_2^2}{\lambda_1^1\lambda_3^2 - \lambda_3^1\lambda_1^2}$
3	$E_1E_2S_1$	$X = \frac{\partial}{\partial x^3} - \frac{\sigma_1}{\beta_{12}}\frac{\partial}{\partial t} + \frac{\beta_{23}}{\beta_{12}}\frac{\partial}{\partial x^1} + \frac{\beta_{31}}{\beta_{12}}\frac{\partial}{\partial x^2}$ $\sigma_1 = \epsilon_{ijk}\, e_i^1 e_j^2(\vec{e}^3 \times \vec{m})_k$ $\beta_{ij} = (e_j^1 e_i^2 - e_i^1 e_j^2)[\vec{u}, \vec{e}^3, \vec{m}^3]$ $+(e_j^2(\vec{e}^3 \times \vec{m}^3)_i - e_i^2(\vec{e}^3 \times \vec{m}^3)_j)(a + \vec{e}^1 \cdot \vec{u})$ $+(e_i^1(\vec{e}^3 \times \vec{m}^3)_j - e_j^1(\vec{e}^3 \times \vec{m}^3)_i)(a + \vec{e}^2 \cdot \vec{u})$	$r^1 = \frac{\beta\bar{u}_3^1(r^3)t - e_1^1x^1 - e_2^1x^2}{1 - \alpha(1+\kappa)t}$ $r^2 = \frac{-\beta\bar{u}_3^1(r^3)t - e_1^2x^1 - e_2^2x^2}{1 - \alpha(1+\kappa)t}$ $r^3 = x^3 - u_0^3 t$ $\beta = (1 + \kappa^{-1})/(e_1^1 - e_1^2)$	$\bar{a} = \frac{\alpha((e_1^1 + e_1^2)x^1 + (e_2^1 + e_2^2)x^2)}{1 - \alpha(1+\kappa)t}, \quad \bar{u}^3 = u_0^3$ $\bar{u}^1 = \frac{-\kappa\alpha\left(((e_1^1)^2 + (e_1^2)^2)x^1 + (e_1^1e_2^1 + e_1^2e_2^2)x^2\right) - \bar{u}_3^1(r^3)}{1 - \alpha(1+\kappa)t}$ $\bar{u}^2 = \kappa\alpha\Bigg(\frac{e_2^1\left(\beta\bar{u}_3^1(r^3)t - e_1^1x^1 - e_2^1x^2\right)}{1 - \alpha(1+\kappa)t}$ $+ \frac{e_2^2\left(-\beta\bar{u}_3^1(r^3)t - e_1^2x^1 - e_2^2x^2\right)}{1 - \alpha(1+\kappa)t}\Bigg) + \frac{e_2^2 - e_2^1}{e_1^2 - e_1^1}\bar{u}_3^1(r^3)$ $\alpha, u_0^3 \in \mathbb{R}$

Table III : Rank-3 solutions. Unassigned unknown functions $a(\cdot), u(\cdot), \ldots$ are arbitrary functions of their respective arguments.

No	Type	Vector Fields	Riemann Invariants	Solutions
1	$E_1E_2E_3$	$X_1 = \frac{\partial}{\partial x^3} + \frac{\sigma_1}{\beta_3}\frac{\partial}{\partial t} + \frac{\beta_1}{\beta_3}\frac{\partial}{\partial x^1} + \frac{\beta_2}{\beta_3}\frac{\partial}{\partial x^2}$ $\sigma_1 = -[\vec{e}^1, \vec{e}^2, \vec{e}^3]$ $\beta_i = (\vec{e}^2 \times \vec{e}^3)_i(a + \vec{e}^1 \cdot \vec{u})$ $+(\vec{e}^1 \times \vec{e}^3)_i(a + \vec{e}^2 \cdot \vec{u})$ $+(\vec{e}^1 \times \vec{e}^2)_i(a + \vec{e}^3 \cdot \vec{u})$	$r^i = (1+\kappa)a_i(r^i)t - \vec{e}^i \cdot \vec{x},\ i = 1,2,3$ $\vec{e}^i \cdot \vec{e}^j = -1/\kappa,\ i \neq j = 1,2,3$	$\bar{a} = \bar{a}_1(r^1) + \bar{a}_2(r^2) + \bar{a}_3(r^3)$ $\vec{u} = \kappa(\vec{e}^1\bar{a}_1(r^1) + \vec{e}^2\bar{a}_2(r^2) + \vec{e}^3\bar{a}_3(r^3))$
$2a$	$E_1S_1S_2$	$X = e_1^2\frac{\partial}{\partial x^1} + e_2^2\frac{\partial}{\partial x^2}$	$r^1 = ((1+k^{-1})f(r^1) + a_0 + u_0^3)t - x^3$ $r^2 = t - x^1\sin g(r^2,r^3) + x^2\cos g(r^2,r^3)$ $\frac{\partial r^3}{\partial t} + (f(r^1) + u_0^3)\frac{\partial r^3}{\partial x^3} = 0$	$\bar{a} = k^{-1}f(r^1) + a_0, \quad \bar{u}^1 = \sin g(r^2,r^3)$ $\bar{u}^2 = -\cos g(r^2,r^3), \quad \bar{u}^3 = f(r^1) + u_0^3$ $a_0, u_0^3 \in \mathbb{R}$
$2b$	$E_1S_1S_2$	$X = e_1^2\frac{\partial}{\partial x^1} + e_2^2\frac{\partial}{\partial x^2}$	$r^1 = \frac{((1+k^{-1})B + a_0 + u_0^3)t - x^3}{1-(1+k^{-1})At}$ $r^2 = t - x^1\sin g(r^2,r^3) + x^2\cos g(r^2,r^3)$ $r^3 = \Psi\left[\frac{1}{A}(A(ka_0 - u_0^3)t + x^3 - ka_0 - B)((1+k)At - k)^{-k/k+1}\right]$	$\bar{a} = k^{-1}(Ar^1 + B) + a_0,$ $\bar{u}^1 = \sin g(r^2,r^3), \bar{u}^2 = -\cos g(r^2,r^3)$ $\bar{u}^3 = Ar^1 + B + u_0^3, \quad a_0, u_0^3 \in \mathbb{R}$
$2c$	$E_1S_1S_2$	$X = \frac{\partial}{\partial x^3}$	$r^1 = (k^{-1}f(r^1) + a_0)t - x^1\cos f(r^1) - x^2\sin f(r^1)$ $r^2 = -t\cos f(r^1) - x^2$ $r^3 = -t\sin f(r^1) + x^1$	$\bar{a} = k^{-1}f(r^1) + a_0, \quad \bar{u}^1 = \sin f(r^1)$ $\bar{u}^2 = -\cos f(r^1), \quad a_0 \in \mathbb{R}$ $\bar{u}^3 = g(r^2\cos f(r^1) + r^3\sin f(r^1))$

Table IV. Reduction of equation (3.1) in Minkowski space $M(1,2)$ to a second order ODE; the solution is of the form $v = \alpha F(\xi)$; we use the following basis for the Lie algebra $sim(1,2)$: dilation $D = r^i - \frac{1}{2}v\partial_v$, translations $P_i = \partial_{r^i}$, rotations $L_{ab} = r^a\partial_{r^b} - r^b\partial_{r^a}$, Lorentz boosts $K_{1a} = -r^1\partial_{r^a} - r^a\partial_{r^1}$.

No	Algebra	α	ξ	ODE
1	D, P_1	$\{4c[(r^2)^2 + (r^3)^2]\}^{-1/4}$	$\frac{1}{2}\arctan\frac{r^3}{r^2}$	$F'' + F + F^5 = 0$
2	D, L_{31}	$\{-c(r^1)^2/4\}^{-1/4}$	$\frac{(r^2)^2 + (r^3)^2}{(r^1)^2}$	$\xi(1+\xi)F'' + \left(2\xi + \frac{3}{2}\right)F' + \frac{3}{16}F + F^5 = 0$
3	$D + \frac{1+q}{q}K_{12}, L_{23}$	$\{-\frac{(2q+1)}{c}\}^{1/4}(r^1 + r^2)^{q/2}$	$[(r^1)^2 - (r^2)^2 - (r^3)^2](r^1 + r^2)^q$	$F'' + \frac{3q+k}{2q+1}\frac{1}{\xi}F' + F^5 = 0, \quad q = -k/3, k-2, 4-3k$
4	$D + \frac{1}{2}K_{12}, L_1 - K_{13}$	$(9/4C)^{1/4}\{r^2 - (r^1 + r^3)^2/4\}^{-1/2}$	$\frac{6(r^3 - r^1) + 6r^2(r^1 + r^3) - (r^1 + r^3)^3}{8(r^2 - (r^1 + r^3)^2/4)^{3/2}}$	$(1+\xi^2)F'' + \frac{7}{3}F' + \frac{1}{3}F + F^5 = 0$

Table V : Bounded real solutions for the nonscattering solution $E_1E_2E_3$ obtained by submitting the arbitrary functions to the various reductions (3.3)-(3.6) of the Klein-Gordon equation (3.1).

no	Riemann invariants	Solution	Type and comments
1	$r^i = -(1+\kappa)\frac{C_i}{\left(\wp\left(r^i, \frac{4}{3}, \frac{8}{27} + \frac{4}{3}C_i^4\right) + \frac{1}{3}\right)^{1/2}} t + \vec{\lambda}^i \cdot \vec{x}$	$a = \sum_{i=1}^{3} \frac{C_i}{\left(\wp\left(r^i, \frac{4}{3}, \frac{8}{27} + \frac{4}{3}C_i^4\right) + \frac{1}{3}\right)^{1/2}}$	Periodic solution
		$\vec{u} = \kappa \sum_{i=1}^{3} \frac{C_i \vec{\lambda}^i}{\left(\wp\left(r^i, \frac{4}{3}, \frac{8}{27} + \frac{4}{3}C_i^4\right) + \frac{1}{3}\right)^{1/2}}$	$C_i \in \mathbb{R}$
2a	$r^i = -(1+\kappa)\left(\frac{C_i}{\wp\left(r^i, 0, \frac{4C_i^2}{3}\right)}\right)^{1/2} t + \vec{\lambda}^i \cdot \vec{x}$	$a = \sum_{i=1}^{3}\left(\frac{C_i}{\wp\left(r^i, 0, \frac{4C_i^2}{3}\right)}\right)^{1/2}, \ \vec{u} = \kappa \sum_{i=1}^{3}\left(\frac{C_i}{\wp\left(r^i, 0, \frac{4C_i^2}{3}\right)}\right)^{1/2} \vec{\lambda}^i$	Periodic Solution
			$C_i > 0$
2b	$r^i = -(1+\kappa)\left(\frac{C_i (r^i)^{-2/3}}{\wp(\zeta_i, 0, \frac{4C_i^2}{3k_0^2})}\right)^{1/2} t + \vec{\lambda}^i \cdot \vec{x}$	$a = \sum_{i=1}^{3}\left(\frac{C_i (r^i)^{-2/3}}{\wp(\zeta_i, 0, \frac{4C_i^2}{3k_0^2})}\right)^{1/2}, \ \vec{u} = \kappa \sum_{i=1}^{3}\left(\frac{C_i (r^i)^{-2/3}}{\wp(\zeta_i, 0, \frac{4C_i^2}{3k_0^2})}\right)^{1/2} \vec{\lambda}^i$	Bump
	$\zeta_i = 3k_0 (r^i)^{1/3}$		$k_0 \in \mathbb{R}, C_i > 0$
2c	$r^i = -(1+\kappa)\left(\frac{C_i (r^i)^{-1}}{(\wp(\zeta_i, 12e_0^2, -8e_0^3 + 16C_i^2 e_0) - e_0)}\right)^{1/2} t + \vec{\lambda}^i \cdot \vec{x}$	$a = \sum_{i=1}^{3}\left(\frac{C_i (r^i)^{-1}}{(\wp(\zeta_i, 12e_0^2, -8e_0^3 + 16C_i^2 e_0) - e_0)}\right)^{1/2}$	Bump
	$\zeta_i = k_0 \ln r^i$	$\vec{u} = \sum_{i=1}^{3} \kappa \left(\frac{C_i (r^i)^{-1}}{(\wp(\zeta_i, 12e_0^2, -8e_0^3 + 16C_i^2 e_0) - e_0)}\right)^{1/2} \vec{\lambda}^i$	$e_0, \in \mathbb{R}, C_i > 0$
3	$r^i = -(1+\kappa)\left(\frac{C_i((r^i)^2+1)^{-1/3}}{\wp\left(\zeta_i, 0, \frac{4C_i^2}{3k_0^2}\right)}\right)^{1/2} t + \vec{\lambda}^i \cdot \vec{x}$	$a = \sum_{i=1}^{3}\left(\frac{C_i((r^i)^2+1)^{-1/3}}{\wp\left(\zeta_i, 0, \frac{4C_i^2}{3k_0^2}\right)}\right)^{1/2}$	Kink
	$\zeta_i = r^i \ {}_2F_1\left(\frac{1}{2}, \frac{5}{6}; \frac{3}{2}; -(r^i)^2\right)$	$\vec{u} = \kappa \sum_{i=1}^{3}\left(\frac{C_i((r^i)^2+1)^{-1/3}}{\wp\left(\zeta_i, 0, \frac{4C_i^2}{3k_0^2}\right)}\right)^{1/2} \vec{\lambda}^i$	$k_0, \in \mathbb{R}, C_i > 0$