# A connection between HH 3 and KdV with one source* 

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#### Abstract

In the system made of Korteweg-de Vries with one source, we first show by applying the Painlevé test that the two components of the source must have the same potential. We then explain the natural introduction of an additional term in the potential of the source equations while preserving the existence of a Lax pair. This allows us to prove the identity between the travelling wave reduction and one of the three integrable cases of the cubic Hénon-Heiles Hamiltonian system.


Keywords: KdV with source, cubic Hénon-Heiles, Lax pair, Painlevé test.
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## 1 Introduction

In several soliton equations, it is possible to add a so-called "source term" without destroying the soliton property. For instance, the Korteweg-de Vries (KdV) equation

$$
\begin{equation*}
b u_{t}+\left(u_{x x}-\frac{3}{a} u^{2}\right)_{x}=0, a, b=\text { constants }, \tag{1}
\end{equation*}
$$

[^0]retains its solitonic property when extended to [21, 26]
\[

\left\{$$
\begin{array}{l}
E_{1} \equiv b u_{t}+\left(u_{x x}-\frac{3}{a} u^{2}-d v w\right)_{x}=0, \quad a, b, d=\text { constants }  \tag{2}\\
E_{2} \equiv v_{x x}+\left(-\frac{u}{a}+\mu_{1}\right) v=0 \\
E_{3} \equiv w_{x x}+\left(-\frac{u}{a}+\mu_{2}\right) w=0
\end{array}
$$\right.
\]

provided $\mu_{1}=\mu_{2}=\mu$. The parameter $\mu$ is then inessential and it could be removed by the Galilean transformation

$$
\begin{equation*}
(u, x, t) \rightarrow\left(u+a \mu, x+6 \frac{\mu}{b} t, t\right) . \tag{3}
\end{equation*}
$$

If one denotes $W=v w, Z=v / w$, the transformed system for $(u, W, Z)$,

$$
\left\{\begin{array}{l}
b u_{t}+\left(u_{x x}-\frac{3}{a} u^{2}-d W\right)_{x}=0  \tag{4}\\
\left(\partial_{x}^{3}-4\left(\frac{u}{a}-\mu\right) \partial_{x}-2 \frac{u_{x}}{a}\right) W=0, \\
\left(W(\log Z)_{x}\right)_{x}=0
\end{array}\right.
$$

allows the easy elimination of $W$ and $Z$, resulting in the KdV6 equation [17, 18],

$$
\begin{equation*}
\left(\partial_{x}^{3}-\frac{4}{a} U_{x} \partial_{x}-\frac{2}{a} U_{x x}\right)\left(b U_{t}+U_{x x x}-\frac{3}{a} U_{x}^{2}\right)=0, U_{x}=u-\alpha \mu . \tag{5}
\end{equation*}
$$

Solutions of system (2) with $\mu_{1}=\mu_{2}$ have been investigated by means of the inverse scattering method [22] and the Darboux transformation [27, 19]. In the more general situation $\mu_{1} \neq \mu_{2}$, it has been noticed [20] that one can also build a variety of solutions.

The purpose of this paper is to present some new results concerning the KdV with one source system (2). In section 2, we first examine the system (2) and prove that a necessary condition to pass the Painlevé test is $\mu_{1}=\mu_{2}$. In section 3, we introduce an extension of this system admitting a Lax pair, and we show that this generalized KdV with one source admits a reduction which can be identified to one of the three integrable sets of equations of motion of the cubic Hénon-Heiles Hamiltonian system.

## 2 The Painlevé test

Among the possible leading behaviours

$$
\begin{equation*}
u \sim u_{0} \chi^{p_{1}}, v \sim v_{0} \chi^{p_{2}}, w \sim w_{0} \chi^{p_{3}}, u_{0} v_{0} w_{0} \neq 0 \tag{6}
\end{equation*}
$$

in which $\chi$ denotes the expansion variable near a movable singularity $\varphi=\varphi(x, t)[6]$

$$
\begin{equation*}
\chi=\left(\frac{\varphi_{x}}{\varphi}-\frac{\varphi_{x x}}{2 \varphi_{x}}\right)^{-1} \tag{7}
\end{equation*}
$$

there exist at least two families in which all $p_{i}$ are negative integers, these are

$$
\left\{\begin{array}{l}
\text { F1: } \mathbf{p}=(-2,-1,-1), u_{0}=2 a,\left(v_{0}, w_{0}\right)=\text { arbitrary }  \tag{8}\\
\text { F2: } \mathbf{p}=(-2,-2,-2), u_{0}=6 a, v_{0} w_{0}=-\frac{72 a}{d}
\end{array}\right.
$$

with $\mathbf{p}=\left(p_{1}, p_{2}, p_{3}\right)$, and their respective Fuchs indices are

$$
\left\{\begin{array}{l}
\text { F1: }  \tag{9}\\
\text { F2: }-1,0,0,3,3,4,6, \\
-1,0,4,5,6,8 .
\end{array}\right.
$$

When one checks the existence of the Laurent series, one finds the following conditions for the absence of movable logarithms,

$$
\begin{equation*}
\text { F1 : } Q_{6} \equiv\left(\mu_{1}-\mu_{2}\right)^{2} v_{0} w_{0}=0 \tag{10}
\end{equation*}
$$

and

$$
\text { F2: }\left\{\begin{array}{l}
Q_{6} \equiv\left(\mu_{1}-\mu_{2}\right)\left(b C+3\left(\mu_{1}+\mu_{2}\right)\right)=0  \tag{11}\\
Q_{8} \equiv 2 S Q_{6}+\left(\mu_{1}-\mu_{2}\right)\left(20 S_{x x}-20 S^{2}-4 b C_{x x}+200 u_{4}+5\left(\mu_{1}-\mu_{2}\right)^{2}\right)=0
\end{array}\right.
$$

where $C=C(x, t), S=S(x, t)$ are functions given by the singular manifold [6]

$$
\begin{equation*}
S=\left(\frac{\varphi_{x x}}{\varphi_{x}}\right)_{x}-\frac{1}{2}\left(\frac{\varphi_{x x}}{\varphi_{x}}\right)^{2}, \quad C=-\frac{\varphi_{t}}{\varphi_{x}} \tag{12}
\end{equation*}
$$

and $u_{4}$ is the arbitrary coefficient introduced at Fuchs index $i=4$. Therefore a necessary condition for the system (2) to pass the Painlevé test is that $\mu_{1}=\mu_{2}$.

## 3 Link with the cubic Hénon-Heiles

The last two equations of system (4) each admit a first integral related to the Wronskian of $v$ and $w$,

$$
\begin{align*}
& 2 G(t)=W W_{x x}-\frac{1}{2} W_{x}^{2}-2\left(\frac{u}{a}-\mu\right) W^{2},  \tag{13}\\
& W(\log Z)_{x}=g(t),  \tag{14}\\
& g(t)=v_{x} w-v w_{x}, \quad G(t)=-\frac{1}{2} g^{2}(t) . \tag{15}
\end{align*}
$$

The conservation of the Wronskian is of course intrinsic to the original system (2) (with $\mu_{1}=\mu_{2}$ ), and it manifests itself by the natural introduction in the transformed system for ( $u, W, Z$ ) of one arbitrary function of time, thus ensuring the conservation of the total differential order (seven) between the original and the transformed systems.

One deduces that the field

$$
\begin{equation*}
Q=W^{1 / 2} \tag{16}
\end{equation*}
$$

obeys a nonlinear ODE of the Ermakov-Pinney type [13, 25]

$$
\begin{equation*}
\left(\frac{\partial^{2}}{\partial_{x}^{2}}-\frac{u}{a}+\mu-\frac{G(t)}{Q^{4}}\right) Q=0, \tag{17}
\end{equation*}
$$

i.e. an equation which only differs from the linear ODE for $v$ or $w$ in system (2) by the contribution of $G(t)$.

The main point is that the KdV with one source system (2) is incomplete, in the sense that the term $G(t) / W^{2}$ can be added to the potential of the linear equation for $v$ and
$w$ while retaining the Painlevé property. Indeed, the extrapolation of (2) to an arbitrary value of $G(t)$

$$
\left\{\begin{array}{l}
F_{1} \equiv b u_{t}+\left(u_{x x}-\frac{3}{a} u^{2}-d v w\right)_{x}=0  \tag{18}\\
F_{2} \equiv v_{x x}+\left(-\frac{u}{a}+\mu-\frac{G(t)}{v^{2} w^{2}}\right) v=0 \\
F_{3} \equiv w_{x x}+\left(-\frac{u}{a}+\mu-\frac{G(t)}{v^{2} w^{2}}\right) w=0
\end{array}\right.
$$

admits the second order matrix Lax pair

$$
\begin{align*}
& \Psi_{x}=L \Psi, \Psi_{t}=M \Psi, \Psi=\binom{\psi_{1}}{\psi_{2}},  \tag{19}\\
& L=\left(\begin{array}{cc}
0 & \frac{u}{a}-\mu+\lambda \\
\hline
\end{array}\right), b M=\left(-\frac{u_{x}}{a}-\frac{d(v w)_{x}}{4 a \lambda}\right)\left(\begin{array}{cc}
1 & 0 \\
0 & -1
\end{array}\right)+\left(\begin{array}{cc}
0 & a_{12} \\
a_{21} & 0
\end{array}\right), \\
& a_{12}=\frac{2 u}{a}-4(\lambda-\mu)+\frac{d v w}{2 a \lambda}, \\
& a_{21}=\left(\frac{u}{a}+\lambda-\mu\right)\left(2 \frac{u}{a}-4(\lambda-\mu)\right)-\frac{u_{x x}}{a}-\frac{d G(t)}{2 a \lambda v w}+\frac{d v w}{2 a}-\frac{d v_{x} w_{x}}{2 a \lambda}=0,
\end{align*}
$$

in which $\lambda$ is the spectral parameter. The zero-curvature vanishing condition

$$
b\left(L_{t}-M_{x}+[L, M]\right) \equiv \frac{d\left(w F_{2}+v F_{3}\right)}{4 a \lambda}\left(\begin{array}{cc}
1 & 0  \tag{20}\\
0 & -1
\end{array}\right)+\left(\frac{F_{1}}{a}+\frac{d\left(w_{x} F_{2}+v_{x} F_{3}\right)}{2 a \lambda} \quad 00\right)
$$

is indeed equivalent to the condition that the $\operatorname{lhs} F_{1}, F_{2}, F_{3}$ of the system (18) simultaneously vanish.

Let us now prove that the system (18) admits for $G(t)$ arbitrary a noncharacteristic reduction to an ODE system which is complete in the Painlevé sense. The transformed system of (18) for ( $u, W=v w, Z=v / w$ ) reads

$$
\left\{\begin{array}{l}
b u_{t}+\left(u_{x x}-\frac{3}{a} u^{2}-d W\right)_{x}=0  \tag{21}\\
\left(\partial_{x}^{3}-4\left(\frac{u}{a}-\mu-\frac{G(t)}{W^{2}}\right)^{2} \partial_{x}-2\left(\frac{u}{a}-\frac{G(t)}{W^{2}}\right)_{x}\right) W=0 \\
W(\log Z)_{x}=g(t)
\end{array}\right.
$$

the second equation is independent of $G(t)$ and therefore admits the first integral $G(t)$ defined in (13), and the first two equations define a closed susbsystem for $u, W$. The resulting system for ( $u, Q=W^{1 / 2}$ ) is

$$
\left\{\begin{array}{l}
b u_{t}+\left(u_{x x}-\frac{3}{a} u^{2}-d Q^{2}\right)_{x}=0,  \tag{22}\\
\left(\frac{\partial^{2}}{\partial_{x}^{2}}-\frac{u}{a}+\mu-\frac{G(t)}{Q^{4}}\right) Q=0 .
\end{array}\right.
$$

As to the cubic Hénon-Heiles Hamiltonian, it is defined as $[16,14]$

$$
\begin{equation*}
H=\frac{1}{2}\left(p_{1}^{2}+p_{2}^{2}+c_{1} q_{1}^{2}+c_{2} q_{2}^{2}\right)+\alpha q_{1} q_{2}^{2}-\frac{1}{3} \beta q_{1}^{3}+\frac{c_{3}}{2 q_{2}^{2}}, \alpha \neq 0 \tag{23}
\end{equation*}
$$

in which $p_{i}=p_{i}(\xi), q_{i}=q_{i}(\xi)(i=1,2)$ and $\alpha, \beta, c_{1}, c_{2}, c_{3}$ are constants. The corresponding equations of motion,

$$
\left\{\begin{array}{l}
\frac{\mathrm{d}^{2} q_{1}}{\mathrm{~d} \xi^{2}}+c_{1} q_{1}-\beta q_{1}^{2}+\alpha q_{2}^{2}=0  \tag{24}\\
\frac{\mathrm{~d}^{2} q_{2}}{\mathrm{~d} \xi^{2}}+c_{2} q_{2}+2 \alpha q_{1} q_{2}-\frac{c_{3}}{q_{2}^{3}}=0
\end{array}\right.
$$

pass the Painlevé test (in the dependent variables $q_{1}, q_{2}^{2}$ ) for only three sets of values of $\left(\beta / \alpha, c_{1}, c_{2}\right)[4,5,15]:\left(\beta / \alpha=-1, c_{1}=c_{2}\right),(\beta / \alpha=-6),\left(\beta / \alpha=-16, c_{1}=16 c_{2}\right)$. These three cases are equivalent to the stationary reduction of three fifth order soliton equations, respectively known as the Sawada-Kotera (SK), fifth order Korteweg-de-Vries $\left(\mathrm{KdV}_{5}\right)$ and Kaup-Kupershmidt (KK) equations, which belong respectively to the BKP, KP and CKP hierarchies.

The link between the extended KdV with one source system (18) now becomes obvious. Under the traveling wave reduction

$$
\begin{equation*}
G(t)=\text { constant }, \quad \xi=x-c t \tag{25}
\end{equation*}
$$

the two-component system is readily identified to the Hamilton equations (24), with

$$
\begin{align*}
& u=q_{1}, Q=W^{1 / 2}=(v w)^{1 / 2}=q_{2} \\
& c_{1}=-b c, \beta=\frac{3}{a}, \alpha=-d, c_{2}=\mu, \alpha=-\frac{1}{2 a}, c_{3}=G(t), \tag{26}
\end{align*}
$$

which imply

$$
\begin{equation*}
\frac{\beta}{\alpha}=-6 . \tag{27}
\end{equation*}
$$

Therefore the traveling-wave reduction of the extended KdV with one source system (18) is the cubic HH system corresponding to $\mathrm{KdV}_{5}$.

Since one cannot include additional terms in the system (24) without destroying its Painlevé property $[11,14,10]$ (completeness property), this proves that the initial system (2) (with $\mu_{1}=\mu_{2}$ ) was incomplete. This is why the extended KdV with one source system (18) deserves to be qualified as "complete".

Remark. As proven in Ref. [12], the number of degrees of freedom in (23) can be extended arbitrarily.

## 4 Conclusion

It is known that the cubic and quartic Hénon-Heiles Hamiltonians pass the Painlevé test only for seven sets of coefficients. The seven Hénon-Heiles Hamiltonians all have the Painlevé property and have been extensively studied $[3,8,9,2]$. However, the explicit integration of three of the quartic cases, namely $1: 6: 1,1: 6: 8,1: 12: 16$, is not yet optimal, in the sense that the expressions are quite intricate. The reason is that one could not yet associate each of these three cases to an optimal PDE system [1]. The link established in this paper between a PDE with source system and one of the cubic Hénon-Heiles Hamiltonians strongly suggests that there could exist three privileged systems of the type PDE+source, whose reduction $x-c t$ would be identical to the Hamilton equations of the quartic cases $1: 6: 1,1: 6: 8,1: 12: 16$. A preliminary step in this direction has been made in [7, Eq. (A.24)] for the 1:6:1 and 1:6:8 cases, whose integration with the autonomous F-VI equation [10] needs to be improved.

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