A characterization of sets of finite perimeter via their covariogram

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Abstract

The covariogram of a measurable set A is the function g_A which to each $y \in \mathbb{R}^d$ associates the Lebesgue measure of $A \cap y + A$. This paper proves two formulas. The first equates the directional derivatives at the origin of g_A to the directional variations of A. The second equates the average directional derivative at the origin of g_A to the perimeter of A. These formulas, previously known with restrictions, are proved for any measurable set. They therefore constitute a characterization of sets of finite perimeter and of sets of finite directional variation. As a consequence we prove that the covariogram of a set A is Lipschitz if and only if A has finite perimeter, the Lipschitz constant being half the maximal directional variation. The formulas also permit to compute the expected perimeter per unit volume of any stationary random closed set. As an illustration, the expected perimeter per unit volume of homogeneous Boolean models having any grain distribution is computed.

Keywords: covariogram, set covariance, sets of finite perimeter, directional variation, Lipschitz constant, strict positive-definiteness, random closed sets, specific variation, variation intensity, specific directional variation, directional variation intensity, Boolean model.

1 Introduction

The object of study of this paper is the *covariogram* g_A of a measurable set $A \subset \mathbb{R}^d$ defined for all $y \in \mathbb{R}^d$ by $g_A(y) = \mathcal{L}^d(A \cap (y + A))$, where $\mathcal{L}^d$ denotes the Lebesgue measure. Note that some authors prefer the terms *set covariance* or *covariance function* [6, 5, 22].

Results By definition, the directional variation in the direction $u \in S^{d-1}$ of A is [3, Section 3.11]

$$V_u(A) = \sup\left\{ \int_{\mathbb{R}^d} \mathbb{1}_A(x)\langle \nabla\varphi(x), u\rangle dx : \ \varphi \in \mathcal{C}_c^1\left(\mathbb{R}^d, \mathbb{R}\right), \ \|\varphi\|_\infty \leq 1 \right\}.$$

We prove that for every measurable set A with finite Lebesgue measure,

$$\lim_{r\to 0} \frac{g_A(0) - g_A(ru)}{|r|} = \frac{1}{2} V_u(A), \ u \in S^{d-1}. \tag{1}$$

In addition, noting $\left(g_A^u\right)'(0) := \lim_{r\to 0^+} \frac{g_A(ru)-g_A(0)}{r}$ the directional derivatives at the origin of the covariogram, it is shown that

$$\operatorname{Per}(A) = -\frac{1}{\omega_{d-1}} \int_{S^{d-1}} \left(g_A^u\right)'(0)\mathcal{H}^{d-1}(du), \tag{2}$$

where $\operatorname{Per}(A)$ denotes the perimeter of A [8, 3] and ω_{d-1} the Lebesgue measure of the unit ball in $\mathbb{R}^{d-1}$. Consequently, a measurable set A has finite perimeter if and only if its covariogram is Lipschitz, and

$$\operatorname{Lip}(g_A) = \frac{1}{2} \sup_{u\in S^{d-1}} V_u(A).$$

Previous works Formula (1) has already been proved for certain classes of sets. It was well-known by the mathematical morphology school [16, 10, 19, 20] that the directional derivative at the origin of the covariogram g_A of a convex set equals minus the length of the orthogonal projection of the set A. The convexity assumption was withdrawn in [22] where Rataj extends the result to compact sets in $\mathcal{U}_{PR}$ satisfying a condition of full-dimensionality, $\mathcal{U}_{PR}$ being the family of locally finite unions of sets with positive reach such that all their finite intersections also have positive reach[1]. In this more general framework, the length of the orthogonal projection is replaced by the total projection $TP_u(A)$. One can easily verify that $V_u(A) = 2TP_u(A)$ by using a recent result due to Ambrosio, Colesanti and Villa [2]: a full-dimensional compact set with positive reach A satisfies $\operatorname{Per}(A) = 2\Phi_{d-1}(A)$ [2, Theorem 9], where $\Phi_{d-1}(A)$ denotes the $(d-1)$-total curvature of A [9]. Since Formula (1) is valid for any measurable set A such that $\mathcal{L}^d(A) < +\infty$, one can argue that the directional variation is the relevant general concept when it comes to the derivative at the origin of the covariogram.

Formula (2) has been widely stated in the mathematical morphology literature [10, 19, 26, 15]. We rigorously establish that it is valid for any measurable set A having finite Lebesgue measure, provided the perimeter $\operatorname{Per}(A)$ is understood as the variation of A.

The Lipschitzness of the covariogram seems to have received less attention in the literature. It is stated in [20] that the covariogram of a compact convex set is Lipschitz. The given upper bound of the Lipschitz constant is the double of the actual Lipschitz constant.

Applications The covariogram is of particular importance in stochastic geometry when dealing with random closed sets (RACS) [19, 27, 21, 25]. In this context, one defines the mean covariogram of a RACS X as the function $\gamma_X(y) = \mathbb{E}\left(\mathcal{L}^d\left(X \cap y + X\right)\right)$. The mean covariogram of a RACS X is related to the probability that two given points belong to X according to the following relation

$$\gamma_X(y) = \int_{\mathbb{R}^d} \mathbb{P}\left(x \in X \text{ and } x + y \in X\right) dx.$$

[1] We refer to [9] and [23] for definitions and results regarding sets with positive reach and $\mathcal{U}_{PR}$-sets respectively

As a consequence the mean covariogram is systematically involved in second order statistics of classic germ-grain models, such as the Boolean model [19, 27, 25], the shot noise model [24, 11], or the dead leaves model [18, 13, 15, 4].

All the established properties of covariograms of deterministic sets extend to the case of mean covariograms of random closed sets. In particular, the stochastic equivalent of (1) and (2) show that the expectations of the variations of a RACS X are proportional to the directional derivatives of its mean covariogram γ_X.

For a stationary RACS, the mean covariogram is not defined. Nevertheless (1) and (2) also permit to study the mean variation of stationary RACS. Define the *specific directional variation* $\theta_{V_u}(X)$ of X as the mean amount of directional variation of X per unit volume. For any stationary RACS X, it is shown using (1) that

$$\theta_{V_u}(X) = 2 \lim_{r \to 0} \frac{1}{|r|} \mathbb{P}\left(ru \in X, 0 \notin X\right).$$

Again, integrating over all directions u, one obtains an expression of the *variation intensity* $\theta_V(X)$ of X (*i.e.* the mean amount of variation of X per unit volume)

$$\theta_V(X) = \frac{1}{\omega_{d-1}} \int_{S^{d-1}} \lim_{r \to 0} \frac{1}{|r|} \mathbb{P}\left(ru \in X, 0 \notin X\right) \mathcal{H}^{d-1}(du).$$

As for (2), the above formula has been stated in the early works of Matheron [17, p. 30] [15, p. 26]. It should be emphasized that the specific variation is well-defined for any stationary RACS, and that it can easily be computed as soon as one knows the probabilities $\mathbb{P}\left(ru \in X, 0 \notin X\right)$. As an illustration, the specific directional variations and the specific variation of homogeneous Boolean models are computed in this paper. The obtained expressions generalize known statistics of Boolean models with convex grains [25]. Because it is well-defined for any stationary RACS and easily computable, we claim that the specific variation is an interesting alternative to the usual specific surface area [25] when one deals with non negligible RACS.

Plan In Section 2 the covariogram g_A of a Lebesgue measurable set A is defined and several properties of g_A are recalled and established. In particular it is shown that as soon as A is non negligible its covariogram g_A is a strictly positive-definite function. Section 3 gathers several known results from the theory of function of bounded variation. In Section 4, the main results relating both the derivative at the origin and the Lipschitzness of the covariogram of a set to its directional variations and its perimeter are stated. Finally, applications of these results to the theory of random closed sets are discussed and illustrated in Section 5.

2 Covariogram of a measurable set

Definition 1. *(covariogram of a measurable set)*
Let $A \subset \mathbb{R}^d$ be a $\mathcal{L}^d$-measurable set of finite Lebesgue measure. The covariogram

of A *is the function* $g_A : \mathbb{R}^d \to [0, +\infty[$ *defined for all* $y \in \mathbb{R}^d$ *by*

$$g_A(y) = \mathcal{L}^d (A \cap (y + A)) = \int_{\mathbb{R}^d} \mathbb{1}_A(x)\mathbb{1}_A(x + y)dx.$$

As initially noted by Matheron [16], the covariogram of A can be expressed as the convolution between the indicator functions of A and its symmetric $\check{A} = \{-x \mid x \in A\}$:

$$g_A = \mathbb{1}_A * \mathbb{1}_{\check{A}}.$$

As illustrated in the following proposition, this point of view is useful to establish some analytic properties of g_A.

Proposition 2. *Let* $A \subset \mathbb{R}^d$ *be a* $\mathcal{L}^d$*-measurable set of finite Lebesgue measure and* g_A *be its covariogram. Then*

1. *For all* $y \in \mathbb{R}^d$, $0 \leq g_A(y) \leq g_A(0) = \mathcal{L}^d(A)$.
2. g_A *is even: for all* $y \in \mathbb{R}^d$, $g_A(-y) = g_A(y)$.
3. $\displaystyle\int_{\mathbb{R}^d} g_A(y)dy = \mathcal{L}^d(A)^2$.
4. g_A *is uniformly continuous over* $\mathbb{R}^d$ *and* $\displaystyle\lim_{|y|\to+\infty} g_A(y) = 0$.

Proof. The three first points are elementary proved. The fourth point is obtained in applying the L^p-$L^{p'}$-convolution theorem to $g_A = \mathbb{1}_A * \mathbb{1}_{\check{A}}$ (see [1, Chapter 2] for example). □

It is well-known that the covariogram is a positive-definite function [16, p. 22], [15, p. 23]. The next proposition improves slightly this result. In particular, it shows that for all $x \neq 0$, $g_A(x) < g_A(0)$.

Proposition 3 (strict positive-definiteness of the covariogram)**.** *Let* A *be a* $\mathcal{L}^d$*-measurable set such that* $0 < \mathcal{L}^d(A) < +\infty$*. Then its covariogram* g_A *is a strictly positive-definite function, that is, for all* $p \in \mathbb{N}^*$*, for all* p*-tuple* $(y_1, \dots, y_p)$ *of distinct vectors of* $\mathbb{R}^d$*, and for all* $(w_1, \dots, w_p) \in \mathbb{R}^p \setminus \{0\}$ *we have*

$$\sum_{j,k=1}^{p} w_j w_k g_A(y_k - y_j) > 0.$$

Proof. By Lemma 4 below, the function $x \mapsto \displaystyle\sum_{j=1}^{p} w_j \mathbb{1}_A(x + y_j)$ is not a.e. equal to zero. Hence

$$\begin{aligned}\sum_{j,k=1}^{p} w_j w_k g_A(y_k - y_j) &= \sum_{j,k=1}^{p} w_j w_k \int_{\mathbb{R}^d} \mathbb{1}_A(x)\mathbb{1}_A(x + y_k - y_j)dx \\ &= \sum_{j,k=1}^{p} w_j w_k \int_{\mathbb{R}^d} \mathbb{1}_A(x + y_j)\mathbb{1}_A(x + y_k)dx \\ &= \int_{\mathbb{R}^d} \left(\sum_{j=1}^{p} w_j \mathbb{1}_A(x + y_j)\right)^2 dx > 0.\end{aligned}$$

□

Lemma 4 (the translations of an integrable function are linearly independent). *Let f be a non null function of $L^1\left(\mathbb{R}^d\right)$ and let $y_1, \dots, y_p$ be p distinct vectors of $\mathbb{R}^d$. Then the functions $x \mapsto f(x+y_j)$, $j = 1, \dots, p$, are linearly independent in $L^1\left(\mathbb{R}^d\right)$.*

Proof. Let $(w_1, \dots, w_p) \in \mathbb{R}^p$ be such that $\sum_{j=1}^{p} w_j f(x+y_j) = 0$ for a.e. $x \in \mathbb{R}^d$. Applying the Fourier transform we have

$$\left(\sum_{j=1}^{p} w_j e^{i\langle \xi, y_j \rangle}\right) \hat{f}(\xi) = 0 \text{ for all } \xi \in \mathbb{R}^d.$$

Since f is non null and integrable, $\hat{f}$ is non null and continuous. Hence there exists $\xi_0 \in \mathbb{R}^d$ and $r > 0$ such that for all $\xi \in B(\xi_0, r)$, $\hat{f}(\xi) \neq 0$, and thus $\forall \xi \in B(\xi_0, r)$, $S(\xi) := \sum_{j=1}^{p} w_j e^{i\langle \xi, y_j \rangle} = 0$. One easily shows that the sum $S(\xi)$ is null for all $\xi \in \mathbb{R}^d$ in considering the one-dimensional restriction of S on the line containing ξ and ξ_0: by the identity theorem, this one-dimensional function is null since it is analytic and null over an open interval. Applying the inverse generalized Fourier transform to $S = 0$ shows that $\sum_{j=1}^{p} w_j \delta_{y_j} = 0$. This implies that $w_1 = \dots = w_p = 0$, since by hypothesis the vectors y_j are distinct. □

Proposition 5. *Let $A \subset \mathbb{R}^d$ be a $\mathcal{L}^d$-measurable set of finite Lebesgue measure and let g_A be its covariogram. Then for all $y, z \in \mathbb{R}^d$*

$$|g_A(y) - g_A(z)| \leq g_A(0) - g_A(y-z).$$

Proof. First let us show that for all measurable sets A_1, A_2, and A_3

$$\mathcal{L}^d(A_1 \cap A_2) - \mathcal{L}^d(A_1 \cap A_3) \leq \mathcal{L}^d(A_2 \setminus A_3) = \mathcal{L}^d(A_2) - \mathcal{L}^d(A_3 \cap A_2). \tag{3}$$

We have

$$\begin{aligned} \mathcal{L}^d(A_1 \cap A_2) - \mathcal{L}^d(A_1 \cap A_3) &\leq \mathcal{L}^d(A_1 \cap A_2) - \mathcal{L}^d(A_1 \cap A_2 \cap A_3) \\ &\leq \mathcal{L}^d((A_1 \cap A_2) \setminus (A_1 \cap A_2 \cap A_3)). \end{aligned}$$

Now using that $(A_1 \cap A_2) \setminus (A_1 \cap A_2 \cap A_3)$ is included in the set $A_2 \setminus A_3$, (3) is proved. Applying (3) to the sets $A_1 = A$, $A_2 = y + A$ and $A_3 = z + A$ we get

$$\begin{aligned} g_A(y) - g_A(z) &= \mathcal{L}^d\left(A \cap (y+A)\right) - \mathcal{L}^d\left(A \cap (z+A)\right) \\ &\leq \mathcal{L}^d(y+A) - \mathcal{L}^d\left((y+A) \cap (z+A)\right) \\ &\leq \mathcal{L}^d(A) - \mathcal{L}^d\left(A \cap ((z-y)+A)\right) \\ &\leq g_A(0) - g_A(z-y). \end{aligned}$$

□

Some remarks:

- The weaker inequality

$$|g_A(y) - g_A(z)| \leq 2(g_A(0) - g_A(y-z))$$

was established by Matheron [20, p. 1].

- The inequality of Proposition 5 shows that the Lipschitzness of the covariogram only depends on the behavior of the function in 0.

3 Facts from the theory of functions of bounded directional variation

Definition 6. *(variation and directional variation) [3]*
Let G be an open subset of $\mathbb{R}^d$ and let $f : G \longrightarrow \mathbb{R}$, $f \in L^1(G)$. The (total) variation of f in G is

$$V(f,G) = \sup\left\{\int_G f(x)\operatorname{div}\varphi(x)dx : \ \varphi \in \mathcal{C}_c^1\left(G,\mathbb{R}^d\right), \ \|\varphi\|_\infty \leq 1\right\}.$$

The directional variation of f in G in the direction $u \in S^{d-1}$ is

$$V_u(f,G) = \sup\left\{\int_G f(x)\langle\nabla\varphi(x), u\rangle dx : \ \varphi \in \mathcal{C}_c^1(G,\mathbb{R}), \ \|\varphi\|_\infty \leq 1\right\}.$$

If $A \subset \mathbb{R}^d$ is a $\mathcal{L}^d$-measurable set, the perimeter of A in G is $\operatorname{Per}(A,G) := V(\mathbb{1}_A, G)$ *and one writes $V_u(A,G) := V_u(\mathbb{1}_A, G)$ for the directional variation of A in G.*

In the case $G = \mathbb{R}^d$, one simply writes $V(f) = V\left(f,\mathbb{R}^d\right)$ and $V_u(f) = V_u\left(f,\mathbb{R}^d\right)$, and similarly for the variations of a set. One denotes by $BV(G)$ and $BV_u(G)$ the set of functions of bounded variation in G and the set of functions of bounded variation in G in the direction u respectively.

Proposition 7 (variation and directional variation)**.** *Let G be an open subset of $\mathbb{R}^d$ and let $f \in L^1(G)$. Then the variation $V(f,G)$ is finite if and only if for every direction $u \in S^{d-1}$ the directional variation $V_u(f,G)$ is finite and*

$$V(f,G) = \frac{1}{2\omega_{d-1}}\int_{S^{d-1}} V_u(f,G)\mathcal{H}^{d-1}(du), \tag{4}$$

where ω_{d-1} denotes the Lebesgue measure of the unit ball in $\mathbb{R}^{d-1}$.

References for the proof. From the definitions one easily shows that

$$\frac{1}{d}V(f,G) \leq \sup_{u\in S^{d-1}} V_u(f,G) \leq V(f,G).$$

The integral geometric formula (4) is elementary proved in [7, Lemma 3.8]. See also [14, Theorem 3.2]. □

The next proposition recalls fundamental properties related to the approximation of functions with bounded directional variation. For simplicity we restrict ourselves to the case $G = \mathbb{R}^d$. See [3, Section 3.11] for the proofs.

Proposition 8 (directional variation and approximation)**.**

- Variation of smooth functions: *If* $f \in \mathcal{C}^1\left(\mathbb{R}^d\right) \cap L^1\left(\mathbb{R}^d\right)$ *then*
$$V_u(f) = \int_{\mathbb{R}^d} \left| \frac{\partial f}{\partial u}(x) \right| dx,$$
where $\frac{\partial f}{\partial u}(x) := \langle \nabla f(x), u \rangle$.

- Lower semi-continuity with respect to the L^1-convergence: *If* f_n *converges towards* f *in* $L^1\left(\mathbb{R}^d\right)$ *then* $V_u(f) \leq \liminf_{n \to +\infty} V_u(f_n)$.

- Approximation by smooth functions: *for every function* $f \in BV_u\left(\mathbb{R}^d\right)$, *there exists a sequence of smooth functions* $f_n \in \mathcal{C}^\infty\left(\mathbb{R}^d\right) \cap BV_u\left(\mathbb{R}^d\right)$ *such that* f_n *converges towards* f *in* $L^1\left(\mathbb{R}^d\right)$ *and* $\lim_{n \to +\infty} V_u(f_n) = V_u(f)$.

Proposition 9 (directional variation and difference quotient)**.** *Let* $u \in S^{d-1}$ *and let* $f \in L^1\left(\mathbb{R}^d\right)$ *be any integrable function. Then for all* $r \neq 0$,
$$\int_{\mathbb{R}^d} \frac{|f(x+ru) - f(x)|}{|r|} dx \leq V_u(f)$$
and
$$\lim_{r \to 0} \int_{\mathbb{R}^d} \frac{|f(x+ru) - f(x)|}{|r|} dx = V_u(f).$$

Proof. To prove the inequality we can suppose that $f \in BV_u\left(\mathbb{R}^d\right)$. First suppose that $f \in \mathcal{C}^1\left(\mathbb{R}^d\right) \cap BV_u\left(\mathbb{R}^d\right)$. Then
$$|f(x+ru) - f(x)| = \left| \int_0^1 r \frac{\partial f}{\partial u}(x + tru) dt \right| \leq \int_0^1 |r| \left| \frac{\partial f}{\partial u}(x+tru) \right| dt.$$

Hence, using Fubini's theorem and the first point of Proposition 8,
$$\int_{\mathbb{R}^d} \frac{|f(x+ru) - f(x)|}{|r|} dx \leq \int_0^1 \left(\int_{\mathbb{R}^d} \left| \frac{\partial f}{\partial u}(x+tru) \right| dx \right) dt = V_u(f).$$

This inequality is shown to be valid for any $f \in BV_u\left(\mathbb{R}^d\right)$ by using approximation by smooth functions (see Proposition 8).
Let us now turn to the second part of the statement. Let $f \in L^1\left(\mathbb{R}^d\right)$. Using the above inequality it is enough to show that
$$V_u(f) \leq \liminf_{r \to 0} \int_{\mathbb{R}^d} \frac{|f(x+ru) - f(x)|}{|r|} dx.$$

Let us consider a family of mollifiers $(\rho_\varepsilon)_{\varepsilon > 0}$ and define $f_\varepsilon = f * \rho_\varepsilon$. Then $f_\varepsilon \in \mathcal{C}^\infty\left(\mathbb{R}^d\right) \cap BV_u\left(\mathbb{R}^d\right)$ and f_ε converges towards f in $L^1\left(\mathbb{R}^d\right)$ as ε tends to 0. By Fatou's lemma we have
$$V_u(f_\varepsilon) = \int_{\mathbb{R}^d} \left| \frac{\partial f_\varepsilon}{\partial u}(x) \right| dx \leq \liminf_{r \to 0} \int_{\mathbb{R}^d} \frac{|f_\varepsilon(x+ru) - f_\varepsilon(x)|}{|r|} dx.$$

Since $\|f_\varepsilon(.+ru)-f_\varepsilon\|_1 = \|(f(.+ru)-f)*\rho_\varepsilon\|_1 \le \|f(.+ru)-f\|_1$ we deduce that for all $\varepsilon > 0$

$$V_u(f_\varepsilon) \le \liminf_{r\to 0} \int_{\mathbb{R}^d} \frac{|f(x+ru)-f(x)|}{|r|} dx.$$

Using the lower semi-continuity of the directional variation with respect to the L^1-convergence we get the result. □

4 Characterization of sets of finite perimeter via their covariogram

Lemma 10 ([20]). *Let A be a $\mathcal{L}^d$-measurable set having finite Lebesgue measure and let g_A be its covariogram. Then for all $y \in \mathbb{R}^d$*

$$g_A(0) - g_A(y) = \frac{1}{2}\int_{\mathbb{R}^d} |\mathbb{1}_A(x+y) - \mathbb{1}_A(x)|\, dx.$$

Proof.

$$\int_{\mathbb{R}^d} |\mathbb{1}_A(x+y) - \mathbb{1}_A(x)|\, dx = \int_{\mathbb{R}^d} (\mathbb{1}_A(x+y) - \mathbb{1}_A(x))^2\, dx = 2\left(g_A(0) - g_A(x)\right).$$

□

Theorem 11 (characterization of sets of finite directional variation). *Let A be a $\mathcal{L}^d$-measurable set having finite Lebesgue measure, let g_A be its covariogram, and let $u \in S^{d-1}$. The following assertions are equivalent:*

(i) A has finite directional variation $V_u(A)$.

(ii) $\lim_{r\to 0} \frac{g_A(0) - g_A(ru)}{|r|}$ exists and is finite.

(iii) The one-dimensional restriction of the covariogram $g_A^u : r \mapsto g_A(ru)$ is Lipschitz.

In addition,

$$\mathrm{Lip}\left(g_A^u\right) = \lim_{r\to 0} \frac{g_A(0) - g_A(ru)}{|r|} = \frac{1}{2} V_u\left(A\right),$$

the second equality being valid both in the finite and infinite case.

Proof. Since from Lemma 10,

$$\frac{g_A(0) - g_A(ru)}{|r|} = \frac{1}{2}\int_{\mathbb{R}^d} \frac{|\mathbb{1}_A(x+ru) - \mathbb{1}_A(x)|}{|r|} dx,$$

by applying Proposition 9 with $f = \mathbb{1}_A$ one obtains the equivalence of *(i)* and *(ii)* as well as the formula $\lim_{r\to 0} \frac{g_A(0) - g_A(ru)}{|r|} = \frac{1}{2} V_u\left(A\right)$.
Let us show that *(i)* implies *(iii)*. By Proposition 5, for all r and $s \in \mathbb{R}$

$$|g_A(ru) - g_A(su)| \le g_A(0) - g_A((r-s)u) = \frac{1}{2}\int_{\mathbb{R}^d} |\mathbb{1}_A(x+(r-s)u) - \mathbb{1}_A(x)|\, dx.$$

Applying the inequality of Proposition 9 with $f = \mathbb{1}_A$,

$$|g_A(ru) - g_A(su)| \leq \frac{1}{2}|r-s| \int_{\mathbb{R}^d} \frac{|\mathbb{1}_A(x+(r-s)u) - \mathbb{1}_A(x)|}{|r-s|} dx \leq \frac{1}{2} V_u(A)|r-s|.$$

Hence g_A^u is Lipschitz and $\mathrm{Lip}\,(g_A^u) \leq \frac{1}{2} V_u(A)$.
Let us now show that *(iii)* implies *(i)*. For all $r \neq 0$ we have

$$\mathrm{Lip}\,(g_A^u) \geq \frac{g_A(0) - g_A(ru)}{|r|} = \frac{1}{2} \int_{\mathbb{R}^d} \frac{|\mathbb{1}_A(x+ru) - \mathbb{1}_A(x)|}{|r|} dx.$$

By Proposition 9 the right-hand side tends towards $\frac{1}{2} V_u(A)$ as r tends to 0. Hence A has finite directional variation in the direction u and $\mathrm{Lip}\,(g_A^u) \geq \frac{1}{2} V_u(A)$. All in all we have shown that *(i)* and *(iii)* are equivalent and that $\mathrm{Lip}\,(g_A^u) = \frac{1}{2} V_u(A)$. □

Considering all the possible directions $u \in S^{d-1}$, the results of the previous theorem yield to a characterization of sets of finite perimeter.

Theorem 12 (characterization of sets of finite perimeter)**.** *Let A be a $\mathcal{L}^d$-measurable set having finite Lebesgue measure, and let g_A be its covariogram. The following assertions are equivalent:*

(i) A has finite perimeter $\mathrm{Per}(A)$.

(ii) For all $u \in S^{d-1}$, $(g_A^u)'(0) := \lim_{r \to 0^+} \frac{g_A(ru) - g_A(0)}{r}$ exists and is finite.

(iii) The covariogram g_A is Lipschitz.

In addition the following relations hold:

$$\mathrm{Lip}\,(g_A) = \frac{1}{2} \sup_{u \in S^{d-1}} V_u(A) \leq \frac{1}{2} \mathrm{Per}(A)$$

and

$$\mathrm{Per}(A) = -\frac{1}{\omega_{d-1}} \int_{S^{d-1}} (g_A^u)'(0) \mathcal{H}^{d-1}(du), \tag{5}$$

this last formula being valid both in the finite and infinite case.

Proof. The equivalence of *(i)* and *(ii)* as well as the integral geometric formula (5) derive from Proposition 7 and the identity

$$(g_A^u)'(0) = \lim_{r \to 0^+} \frac{g_A(ru) - g_A(0)}{r} = -\frac{1}{2} V_u(A).$$

Let us now show that *(i)* implies *(iii)*. Let $y, z \in \mathbb{R}^d$. Denote by u the direction of S^{d-1} such that $y - z = |y-z|u$. By Proposition 5 and Theorem 11

$$|g_A(y) - g_A(z)| \leq g_A(0) - g_A(y-z) \leq \frac{1}{2} V_u(A)|y-z| \leq \left(\frac{1}{2} \sup_{u \in S^{d-1}} V_u(A) \right) |y-z|.$$

Hence g_A is Lipschitz and $\mathrm{Lip}\,(g_A) \leq \frac{1}{2} \sup_u V_u(A)$. As for the converse implication and inequality, for all $u \in S^{d-1}$,

$$\mathrm{Lip}\,(g_A) \geq \lim_{r \to 0} \frac{g_A(0) - g_A(ru)}{|r|} = \frac{1}{2} V_u(A).$$

Hence for all $u \in S^{d-1}$, $V_u(A) < +\infty$ and $\mathrm{Lip}\,(g_A) \geq \frac{1}{2} \sup_u V_u(A)$. This concludes the proof. □

One natural question is whether Formula (5) extends to the case of functions. The answer to this question is negative. Indeed, if one considers a smooth function $f \in \mathcal{C}_c^1\left(\mathbb{R}^d\right)$, then its covariogram $g_f(y) = \int f(x+y)f(x)dx$ is well-defined and is differentiable in 0. But since g_f is even, its derivative at the origin equals zero, and thus the variation of f is not equal to the integral of the directional derivatives of the covariogram g_f.

5 Application to random closed sets

5.1 Mean covariogram of a random closed set

A random closed set (RACS) X is a measurable map from a probability space $(\Omega, \mathcal{A}, \mathbb{P})$ to the space $\mathcal{F}\left(\mathbb{R}^d\right)$ of closed subsets of $\mathbb{R}^d$ endowed with the σ-algebra generated by the sets $\left\{\left\{F \in \mathcal{F}\left(\mathbb{R}^d\right),\ F \cap K = \emptyset\right\},\ K \text{ compact}\right\}$ [19, 21, 27].

Definition 13 (mean covariogram of a random closed set)**.** *Let X be a random closed set (RACS) of $\mathbb{R}^d$ with finite mean Lebesgue measure,* i.e. $\mathbb{E}\left(\mathcal{L}^d\left(X\right)\right) < +\infty$. *The mean covariogram γ_X of X is the expectation of the covariogram of X with respect to its distribution, that is $\gamma_X : \mathbb{R}^d \to [0, \infty[$ is the function defined by*

$$\gamma_X(y) = \mathbb{E}\left(g_X(y)\right) = \mathbb{E}\left(\mathcal{L}^d\left(X \cap y + X\right)\right) = \int_{\mathcal{F}(\mathbb{R}^d)} \mathcal{L}^d\left(A \cap y + A\right)\mathbb{P}_X(dA).$$

All the results established in the deterministic case can be adapted for mean covariograms of RACS.

Proposition 14 (properties of mean covariograms)**.** *Let X be a RACS of $\mathbb{R}^d$ satisfying $\mathbb{E}\left(\mathcal{L}^d(X)\right) < +\infty$ and let γ_X be its mean covariogram. Then*

1. *For all $y \in \mathbb{R}^d$, $0 \le \gamma_X(y) \le \gamma_X(0) = \mathbb{E}\left(\mathcal{L}^d(X)\right)$.*
2. *γ_X is even.*
3. *$\gamma_X(y) = \displaystyle\int_{\mathbb{R}^d} \mathbb{P}\left(x \in X \text{ and } x + y \in X\right) dx$.*
4. *$\int_{\mathbb{R}^d} \gamma_X(y)dy = \mathbb{E}\left(\mathcal{L}^d(X)^2\right) \in [0, +\infty]$.*
5. *If $\mathbb{E}\left(\mathcal{L}^d(X)\right) > 0$, then γ_X is a strictly positive-definite function.*
6. *For all $y, z \in \mathbb{R}^d$, $|\gamma_X(y) - \gamma_X(z)| \le \gamma_X(0) - \gamma_X(y - z)$.*
7. *γ_X is uniformly continuous over $\mathbb{R}^d$ and $\displaystyle\lim_{|y| \to +\infty} \gamma_X(y) = 0$.*
8. *We have*
$$\lim_{r \to 0} \frac{\gamma_X(0) - \gamma_X(ru)}{|r|} = \frac{1}{2}\mathbb{E}\left(V_u(X)\right).$$
and, noting $\left(\gamma_X^u\right)'(0) = \displaystyle\lim_{r \to 0^+} \frac{\gamma_X(ru) - \gamma_X(0)}{r}$,
$$-\frac{1}{\omega_{d-1}} \int_{S^{d-1}} \left(\gamma_X^u\right)'(0)\mathcal{H}^{d-1}(du) = \mathbb{E}\left(\mathrm{Per}(X)\right).$$

The proofs are omitted since they mostly consist in integrating the results of the previous sections with respect to the distribution of the RACS X. The convergence results follow easily from the bounded convergence theorem.

5.2 Specific variation of a stationary RACS

A RACS X is said to be *stationary* if for all $y \in \mathbb{R}^d$, the translated RACS $y+X$ has the same distribution as X. If a RACS X is stationary, one defines its *variogram* ν_X as the function $\nu_X(y) = \mathbb{P}(y \in X, 0 \notin X)$ (see e.g. [15] for more details on variograms).

Given a stationary RACS X, the map $G \mapsto \mathbb{E}(\mathrm{Per}(X,G))$, $G \subset \mathbb{R}^d$ open, defines a measure which is translation invariant. Hence there exists a real number $\theta_V(X) \in [0,\infty]$ such that

$$\mathbb{E}(\mathrm{Per}(X,G)) = \theta_V(X)\mathcal{L}^d(G).$$

We choose to call this constant $\theta_V(X)$ the *specific variation* of X or the *variation intensity* of X (see the discussion below). Similarly, for all $u \in S^{d-1}$ there exists a real $\theta_{V_u}(X) \in [0,\infty]$ such that $\mathbb{E}(V_u(X,G)) = \theta_{V_u}(X)\mathcal{L}^d(G)$. $\theta_{V_u}(X)$ is called the *specific directional variation* of X in the direction u (or also the *directional variation intensity*). In this context the integral-geometric formula (4) gives

$$\theta_V(X) = \frac{1}{2\omega_{d-1}} \int_{S^{d-1}} \theta_{V_u}(X)\mathcal{H}^{d-1}(du).$$

Theorem 15 (specific variations and variogram)**.** *Let X be a stationary RACS, let ν_X be its variogram, and for all $u \in S^{d-1}$ denote $(\nu_X^u)'(0) := \lim_{r\to 0} \frac{1}{|r|}\nu_X(ru)$. Then for all $u \in S^{d-1}$ the specific directional variation $\theta_{V_u}(X)$ is given by*

$$\theta_{V_u}(X) = 2(\nu_X^u)'(0) = 2\lim_{r\to 0}\frac{1}{|r|}\mathbb{P}(ru \in X, 0 \notin X).$$

In other words, the specific directional variation is twice the directional derivative of the variogram at the origin. Integrating over all directions, one obtains the specific variation of X:

$$\theta_V(X) = \frac{1}{\omega_{d-1}} \int_{S^{d-1}} (\nu_X^u)'(0)\mathcal{H}^{d-1}(du). \tag{6}$$

Before proving this theorem let us discuss the terminology *specific variation* of X for the constant $\theta_V(X)$. Equation (6) is exactly the formula given in [15, p. 26] and which originates from Matheron [17, p. 30]. In these references, the constant corresponding to the variation intensity $\theta_V(X)$ is called the *specific $(d-1)$-volume* of X (*specific perimeter* if $d=2$, *specific surface area* if $d=3$). However, in the later works of Matheron [19] as well as on recent reference textbooks [27, 25], the *specific surface measure* refers to the surface measure that derives from Steiner's formula. This measure has different names, depending on its normalization and the degree of generalization: intrinsic volume of index $d-1$ and Minkowski's content of index 1 for convex sets [25], total curvature of index $d-1$ for sets with positive reach and $\mathcal{U}_{PR}$-sets [9, 23], or also in a more general setting outer Minkowski content [2, 28]; see also [12]. Even though the (variational) perimeter of a set and this notion of surface measure agree for convex sets [2], the distinction is important. Indeed their extensions to non convex sets have different behaviors. For example, the outer Minkowski content counts twice the isolated fine parts of a set having a bounded and $(d-1)$-rectifiable topological boundary, whereas these fine parts have no influence on the perimeter [28, Proposition 4.1] (here "isolated fine parts" denotes the part of the boundary which has Lebesgue density 0). In order to make a clear distinction between the (variational) perimeter and the surface measure from Steiner's formula, the constant $\theta_V(X)$ is named the *specific variation* of X and not its "specific perimeter".

As mentioned in the introduction, one should notice that, contrary to the specific surface area [25], the specific variation $\theta_V(X)$ is well-defined for any stationary RACS. Besides, Theorem 15 shows that the specific directional variations $\theta_{V_u}(X)$ and the specific variation $\theta_V(X)$ are easily computed as soon as one knows the variogram of X. This will be illustrated in the next section where the specific variations of homogeneous Boolean models are computed.

Let us now turn to the proof of Theorem 15 which uses the following intuitive lemma.

Lemma 16. *Let A be a $\mathcal{L}^d$-measurable set and B be an open ball. Then for all $u \in S^{d-1}$,*

$$V_u(A, B) \leq V_u\left(A \cap B, \mathbb{R}^d\right) \leq V_u(A, B) + V_u(B, \mathbb{R}^d).$$

References for the proof. The first inequality is immediate from the definition of the directional variation on an open set [3]. The second inequality is easily proved using the interpretation of the directional variation as an oriented Hausdorff measure of the essential boundary [7]. □

Proof of Theorem 15. First remark that

$$\begin{aligned}\mathbb{P}(ru \in X, 0 \notin X) &= \mathbb{P}(0 \in X, -ru \notin X)\\ &= \mathbb{P}(0 \in X) - \mathbb{P}(0 \in X \text{ and } -ru \in X)\\ &= \mathbb{P}(0 \in X) - \mathbb{P}(0 \in X \cap (ru + X)).\end{aligned}$$

Let B be any open ball. Since X is a stationary RACS

$$\mathbb{P}(0 \in X) = \frac{\mathbb{E}\left(\mathcal{L}^d(X \cap B)\right)}{\mathcal{L}^d(B)}.$$

As $X \cap (ru + X)$ is also a stationary RACS, we have

$$\mathbb{P}(0 \in X \cap (ru + X)) = \frac{\mathbb{E}\left(\mathcal{L}^d(X \cap (ru + X) \cap B)\right)}{\mathcal{L}^d(B)}.$$

In order to introduce the mean covariogram of the set $X \cap B$, let us denote $E_r = (X \cap B) \cap (ru + (X \cap B))$. Clearly we have the following inclusions

$$E_r \subset X \cap (ru + X) \cap B \quad \text{and} \quad [X \cap (ru + X) \cap B] \setminus E_r \subset B \setminus (B \cap (ru + B)).$$

Noting that $\mathcal{L}^d\left(B \setminus (B \cap (ru + B))\right) = g_B(0) - g_B(ru)$, we obtain

$$\frac{\gamma_{X \cap B}(ru)}{\mathcal{L}^d(B)} \leq \frac{\mathbb{E}\left(\mathcal{L}^d(X \cap (ru + X) \cap B)\right)}{\mathcal{L}^d(B)} \leq \frac{\gamma_{X \cap B}(ru)}{\mathcal{L}^d(B)} + \frac{g_B(0) - g_B(ru)}{\mathcal{L}^d(B)}.$$

This yields both an upper and a lower bound of $\mathbb{P}(ru \in X, 0 \notin X)$. We have

$$\mathbb{P}(ru \in X, 0 \notin X) \leq \frac{\gamma_{X\cap B}(0) - \gamma_{X\cap B}(ru)}{\mathcal{L}^d(B)}.$$

By property 8 of Proposition 14 and Lemma 16 we obtain

$$\limsup_{r\to 0} \frac{1}{|r|}\mathbb{P}(ru \in X, 0 \notin X) \leq \frac{1}{2}\frac{\mathbb{E}(V_u(X\cap B))}{\mathcal{L}^d(B)} \leq \frac{1}{2}\theta_{V_u}(X) + \frac{1}{2}\frac{V_u(B)}{\mathcal{L}^d(B)}.$$

As for the lower bound,

$$\mathbb{P}(ru \in X, 0 \notin X) \geq \frac{\gamma_{X\cap B}(0) - \gamma_{X\cap B}(ru)}{\mathcal{L}^d(B)} - \frac{g_B(0) - g_B(ru)}{\mathcal{L}^d(B)}.$$

Again by Proposition 14 and Lemma 16 we have

$$\liminf_{r\to 0} \frac{1}{|r|}\mathbb{P}(ru \in X, 0 \notin X) \geq \frac{1}{2}\frac{\mathbb{E}(V_u(X\cap B))}{\mathcal{L}^d(B)} - \frac{1}{2}\frac{V_u(B)}{\mathcal{L}^d(B)} \geq \frac{1}{2}\theta_{V_u}(X) - \frac{1}{2}\frac{V_u(B)}{\mathcal{L}^d(B)}.$$

The two established inequalities are true for any open ball B. The enunciated formula is obtained by letting the radius of B tends to $+\infty$.

□

5.3 Computation of the specific variations of Boolean models

In this section we apply Theorem 15 to compute the specific directional variations and the specific variation of any homogeneous Boolean model. The Boolean model [27], [25] with intensity λ and grain distribution P_X is the stationary RACS Z defined by

$$Z = \bigcup_{i\in\mathbb{N}} x_i + X_i,$$

where $\{x_i, i \in \mathbb{N}\} \subset \mathbb{R}^d$ is a homogeneous Poisson point process with intensity $\lambda > 0$ and $(X_i)_{i\in\mathbb{N}}$ is a sequence of i.i.d. RACS with common distribution P_X. The avoiding functional of the Boolean model Z is well-known: for any compact $K \subset \mathbb{R}^d$ we have

$$\mathbb{P}(Z \cap K = \emptyset) = \exp\left(-\lambda\mathbb{E}\left(\mathcal{L}^d\left(X \oplus \check{K}\right)\right)\right), \tag{7}$$

where X denotes a RACS with distribution P_X and $X \oplus \check{K} = \{x - y,\ x \in X,\ y \in K\}$ (see e.g. [27, p. 65] or [15, p. 164]). Starting from the general expression (7) (which determines the distribution of Z), let us compute the variogram ν_Z of Z. For $K = \{y\}$, (7) becomes

$$q := \mathbb{P}(y \notin Z) = \exp\left(-\lambda\mathbb{E}\left(\mathcal{L}^d(X)\right)\right).$$

For $K = \{0, -ru\}$, with $r \neq 0$ and $u \in S^{d-1}$, remark that we have

$$\mathcal{L}^d\left(X \oplus \check{K}\right) = \mathcal{L}^d(X \cup ru + X) = 2\mathcal{L}^d(X) - \mathcal{L}^d(X \cap ru + X).$$

Hence in this case $\mathbb{E}\left(\mathcal{L}^d\left(X \oplus \check{K}\right)\right) = 2\mathbb{E}\left(\mathcal{L}^d\left(X\right)\right) - \gamma_X(ru)$. As a result the variogram ν_Z is equal to [27, p. 68], [15, p. 165]

$$\begin{aligned}\nu_Z(ru) = \mathbb{P}\left(-ru \in Z \text{ and } 0 \notin Z\right) &= \mathbb{P}\left(0 \notin Z\right) - \mathbb{P}\left(Z \cap \{0, -ru\} = \emptyset\right)\\ &= q - \exp\left(-\lambda\left(2\mathbb{E}\left(\mathcal{L}^d\left(X\right)\right) - \gamma_X(ru)\right)\right)\\ &= q - q\exp\left(-\lambda\left(\gamma_X(0) - \gamma_X(ru)\right)\right).\end{aligned}$$

By Theorem 15 and property 8 of Proposition 14 we deduce

$$\theta_{V_u}(Z) = 2\left(\nu_X^u\right)'(0) = 2q\lambda\frac{1}{2}\mathbb{E}\left(V_u(A)\right) = \lambda\mathbb{E}\left(V_u(A)\right)\exp\left(-\lambda\mathbb{E}\left(\mathcal{L}^d\left(X\right)\right)\right).$$

Integrating this formula over all directions u we obtain $\theta_V(Z)$. Our computation is summarized in the following statement.

Proposition 17 (specific variations of a homogeneous Boolean model)**.** *Let Z be the Boolean model with Poisson intensity λ and grain distribution P_X, and let X be a RACS with distribution P_X. Then for all $u \in S^{d-1}$,*

$$\theta_{V_u}(Z) = \lambda\mathbb{E}\left(V_u(A)\right)\exp\left(-\lambda\mathbb{E}\left(\mathcal{L}^d\left(X\right)\right)\right)$$

and

$$\theta_V(Z) = \lambda\mathbb{E}\left(\mathrm{Per}(X)\right)\exp\left(-\lambda\mathbb{E}\left(\mathcal{L}^d\left(X\right)\right)\right). \tag{8}$$

Equation (8) is valid for any grain distribution P_X and generalizes known results for Boolean models with convex grains [25, p. 386]. Similar generalizations involving intensity of surface measures deriving from Steiner's formula have recently been established [12, 29]. As already stressed out, our result is similar but not identical since the outer Minkowski content of a set differs from its (variational) perimeter [28].

A promising direction for further works is to extend the notion of specific variation for inhomogeneous RACS. In particular, following [29], one could try to derive local variation densities of certain inhomogeneous Boolean models.

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