Elliptic general analytic solutions*

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Abstract

In order to find analytically the travelling waves of partially integrable autonomous nonlinear partial differential equations, many methods have been proposed over the ages: "projective Riccati method", "tanh-method", "exponential method", "Jacobi expansion method", "new ...", etc. The common default to all these "truncation methods" is to only provide some solutions, not all of them. By implementing three classical results of Briot, Bouquet and Poincaré, we present an algorithm able to provide in closed form *all* those travelling waves which are elliptic or degenerate elliptic, i.e. rational in one exponential or rational. Our examples here include the Kuramoto-Sivashinsky equation and the cubic and quintic complex Ginzburg-Landau equations.

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1 Introduction

In physical language, given an autonomous nonlinear partial differential equation (PDE) $E(v, v_x, v_t, \dots, v_{nx,mt}) = 0$, one calls **travelling wave** any solution of the reduction $(v, x, t) \to \xi = x - ct, v(x,t) = u(\xi)$, with c constant, and **solitary wave** a travelling wave which obeys the decay condition that $u(\xi)$ should have constant limits, possibly different, as $\xi \to \pm\infty$. The problem addressed here is to find all the travelling waves of a given PDE, in closed form, or more generally all the solutions of similarity reductions of a given PDE to some autonomous ODE. Two cases naturally arise. If the PDE is "integrable", for instance in the sense of the inverse spectral transform [1], there exist powerful methods to solve this problem completely; we will discard this case. If the PDE is partially integrable (i.e. fails at least one of the integrability criteria), there exist several nonperturbative methods (mostly "truncations" [28]) able to find *some* travelling waves but generically unable to find *all* of them.

In a more mathematical language, let us denote N the differential order of the autonomous ordinary differential equation (ODE)

$$E(u, u', \dots, u^{(N)}) = 0, \ ' = \frac{\mathrm{d}}{\mathrm{d}\xi}, \tag{1}$$

which is the travelling wave reduction of the considered PDE. If the ODE has the Painlevé property [11] (i.e. no "bad" singularities), the problem is to find its general solution (depending on N arbitrary constants) in closed form; we discard this case. If the ODE fails the Painlevé test ("partially integrable" case), by definition it admits some movable critical singularities (movable: which depends on the initial conditions; critical: the solution is multi-valued around it). Let us define its **general analytic solution** as the M-parameter particular solution without movable critical singularities, with M maximal and of course $M < N$. The problem is then to find this general analytic solution in closed form. This is equivalent to find the largest order subequation of the given ODE with the Painlevé property (subequation: the ODE is a differential consequence of it).

The present article, which is self-contained, gives a synthetic presentation of a method [23, 11] able to find this general analytic solution (i.e. **all** the single-valued travelling waves) when it is elliptic or degenerate of elliptic. This includes the physically relevant solitary waves.

There are many equivalent definitions for elliptic functions. From the point of view of nonlinear differential equations adopted here, the suitable definition for an **elliptic function** $f(\xi)$ is any rational function of $\wp(\xi), \wp'(\xi)$,

$$f(\xi) = R(\wp(\xi), \wp'(\xi)), \tag{2}$$

in which $\wp$ is the function of Weierstrass, defined by the first order ODE

$$\wp'^2 = 4\wp^3 - g_2\wp - g_3 = 4(\wp(\xi) - e_1)(\wp(\xi) - e_2)(\wp(\xi) - e_3),\ \wp'' = 6\wp^2 - \frac{g_2}{2}. \tag{3}$$

Elliptic functions are doubly periodic and have two successive degeneracies,

- when one root e_j is double ($g_2^3 - 27g_3^2 = 0$), degeneracy to simply periodic functions (i.e. rational functions of one exponential e^{kx}) according to

$$\forall x, d:\ \wp(x, 3d^2, -d^3) = 2d - \frac{3d}{2}\coth^2\sqrt{\frac{3d}{2}}x, \tag{4}$$

- when the root e_j is triple ($g_2 = g_3 = 0$), degeneracy to rational functions of ξ.

In section 2, we handle a textbook example of an integrable equation in order to make several points. In section 3, we make precise the notion of *general analytic solution* of a partially integrable ODE. In section 4, we shortly review the existing methods and point out their weaknesses. In section 5, we present the algorithmic method, then we process a few selected examples. Finally, section 7 examines possible extensions of the method.

2 Tutorial integrable example

Consider the Korteweg-de Vries (KdV) equation

$$bv_t + v_{xxx} - \frac{6}{a}vv_x = 0,\ (a,b) \text{ constant}, \tag{5}$$

and its travelling wave reduction

$$v = -\frac{ab}{6}c + u,\ \xi = x - ct,\ u''' - \frac{6}{a}uu' = 0. \tag{6}$$

2.1 Mathematical part

If one *a priori* knows nothing on (6), some skill is needed to find the successive first integrals g_2 (arising from the conservative form)

$$u'' - \frac{3}{a}u^2 + ag_2 = 0, \tag{7}$$

and g_3 (integrating factor u')

$$\frac{1}{2}u'^2 - \frac{1}{a}u^3 + ag_2 u + 2a^2 g_3 = 0. \tag{8}$$

The general solution is then elliptic (doubly periodic)

$$u(\xi) = 2a\wp(\xi - \xi_0, g_2, g_3), \tag{9}$$

in which the three constants of integration ξ_0, g_2, g_3 are allowed to be complex.

2.2 Physical part

One then prescribes some behaviour at $\xi \to \pm\infty$, e.g.

$$\lim_{\xi \to -\infty} u = B_-, \ \lim_{\xi \to +\infty} u = B_+, \ B_\pm \text{ real}, \tag{10}$$

which for $B_- = B_+ = B$ makes the doubly periodic function degenerate to simply periodic (trigonometric)

$$\left\{ \begin{array}{l} g_2 = 3((B-\kappa)/a)^2, \ g_3 = -((B-\kappa)/a)^3, \ \kappa = -abc/6. \\ u = -\dfrac{ab}{6}c + 2a\left(\dfrac{4}{3}k^2 - k^2 \coth^2 k(\xi - \xi_0)\right), \ k^2 = \dfrac{3(B-\kappa)}{2a}, \end{array} \right. \tag{11}$$

in which the constants of integration are k^2 (real) and ξ_0 (complex).

Finally one requires this solution u to be bounded on the real axis, by moving all singularities outside $\mathcal{R}$

$$\xi_0 = \xi_1 + i\pi/(2k), \ \xi_1 \in \mathcal{R}, \ \coth k(\xi - \xi_0) = \tanh k(\xi - \xi_1), \tag{12}$$

to obtain the physical, bell-shaped, solitary wave on the background B,

$$u = B - 2ak^2 \operatorname{sech}^2 \left(kx - 4k^3 t/b + 6k(B/a)t/b - k\xi_1\right). \tag{13}$$

2.3 Lessons from this tutorial example

Although it is completely straightforward, this example nevertheless teaches us several lessons.

- Skill was needed to find the first integrals, we want to avoid that.
- If possible, it is better to first find the mathematical solution, because all the physical solutions follow. In the case of the nonlinear Schrödinger equation for instance, which admits two (one bright, one dark) solitary waves depending on the focusing or defocusing situation, the bright wave is found as above, and the dark one arises from a factorisation of the first order subequation analogous to (8).
- Elliptic functions seem privileged. We will come back to that later.

3 Partially integrable case

By definition, partially integrable PDEs fail the Painlevé test. Our specific examples will be the following reductions:

- the similarity reduction of the one-dimensional cubic complex Ginzburg-Landau equation (CGL3)

$$iA_t + pA_{xx} + q|A|^2 A - i\gamma A = 0,\ pq\gamma \ne 0,\ (A,p,q) \in \mathcal{C},\ \gamma \in \mathcal{R} \tag{14}$$

$$A(x,t) = \sqrt{M(\xi)} e^{i(-\omega t + \varphi(\xi))},\ \xi = x - ct, \tag{15}$$

  (a generic equation for slowly varying amplitudes, see the review [25]);

- the travelling wave reduction of the Kuramoto-Sivashinsky equation (KS)

$$v_t + \nu v_{xxxx} + b v_{xxx} + \mu v_{xx} + v v_x = 0,\ \nu \ne 0, \tag{16}$$

$$v(x,t) = c + u(\xi),\ \xi = x - ct,\ \nu u''' + b u'' + \mu u' + \frac{u^2}{2} + A = 0,\ \nu \ne 0, \tag{17}$$

  in which A is an integration constant (flame on a vertical wall, phase equation of CGL3, see [22] for some physical background);

- the similarity reduction of the one-dimensional quintic complex Ginzburg-Landau equation (CGL5) [25],

$$iA_t + pA_{xx} + q|A|^2 A + r|A|^4 A - i\gamma A = 0,\ pr \ne 0,\ \Im(p/r) \ne 0, \tag{18}$$
$$(A,p,q,r) \in \mathcal{C},\ \gamma \in \mathcal{R}.$$

$$A(x,t) = \sqrt{M(\xi)} e^{i(-\omega t + \varphi(\xi))},\ \xi = x - ct. \tag{19}$$

3.1 General analytic solution of a partially integrable ODE

Given an N-th order ODE (1) which is partially integrable and therefore displays movable multi-valuedness, one first needs to count the number of integration constants which correspond to this multi-valuedness. Let us take an example.

3.2 Local representation of the general analytic solution

The KS PDE (16) admits the travelling wave reduction (17). It has a chaotic behavior [22], and it depends on two dimensionless parameters, $b^2/(\mu\nu)$ and $\nu A/\mu^3$. The assumption of a singular algebraic behaviour

$$\xi \to \xi_0 :\ u \sim u_0 \chi^p,\ \chi = \xi - \xi_0,\ u_0 \ne 0,\ p \notin \mathcal{N}, \tag{20}$$

generates for (p, u_0) the system

$$p - 3 = 2p,\ \nu p(p-1)(p-2)u_0 + (1/2)u_0^2 = 0,\ u_0 \ne 0, \tag{21}$$

which admits the unique solution $p = -3, u_0 = 120\nu$, with the common value of the two powers: $q = p - 3 = 2p = -6$. Near this triple pole, the linearized equation has the indicial equation

$$\lim_{\chi \to 0} \chi^{-j-q} (\nu \partial_\xi^3 + u_0 \chi^p) \chi^{j+p} \tag{22}$$

$$= \nu(j-3)(j-4)(j-5) + 120\nu = \nu(j+1)(j^2 - 13j + 60) \tag{23}$$

$$= \nu(j+1)\left(j - \frac{13 + i\sqrt{71}}{2}\right)\left(j - \frac{13 - i\sqrt{71}}{2}\right) = 0, \tag{24}$$

i.e. two of the three Fuchs indices are irrational complex numbers. Computing the next terms after the dominant one yields the Laurent series

$$u^{(0)} = 120\nu\chi^{-3} - 15b\chi^{-2} + \frac{15(16\mu\nu - b^2)}{4 \times 19\nu}\chi^{-1} + \frac{13(4\mu\nu - b^2)b}{32 \times 19\nu^2} + O(\chi), \tag{25}$$

which only depends on one integration constant. Laurent series such as (25) are convergent, as proven by Chazy [3]. By perturbing this solution [3, pages 338, 384] [7], one obtains the three-parameter solution

$$\begin{aligned} u(\xi - \xi_0, \varepsilon c_+, \varepsilon c_-) \ = \ & u^{(0)} \\ & +\varepsilon\chi^{-3}[c_{-1}\chi^{-1}\mathrm{Regular}(\chi) \\ & + \quad c_+\chi^{(13+i\sqrt{71})/2}\mathrm{Regular}(\chi) \\ & + \quad c_-\chi^{(13-i\sqrt{71})/2}\mathrm{Regular}(\chi)] + \mathcal{O}(\varepsilon^2)\}, \end{aligned} \tag{26}$$

which is a local representation valid for $\xi \to \xi_0$ and $\varepsilon \to 0$, depending on the three arbitrary constants $\xi_0, \varepsilon c_+, \varepsilon c_-$ (indeed, as shown by Poincaré[1] and Darboux [13], the coefficient of εc_{-1} in (26) is the derivative of $u^{(0)}$ with respect to its arbitrary constant ξ_0, therefore c_{-1} only represents a perturbation of ξ_0 and it can be set to 0). The only way for (26) to become single-valued is to cancel the contribution of the two irrational Fuchs indices, thus restricting (26) to (25).

The general analytic solution as defined in section 1 depends on $3 - 2 = 1$ integration constant, it is locally represented by the Laurent series (25) and the problem addressed is to obtain a closed form expression for $u^{(0)}$.

4 Existing methods (sufficient)

Given a nonlinear ODE such as (1), all the methods able to yield single-valued solutions in closed form necessarily make use of the structure of movable singularities. Many "new", "extended", etc, methods claim not to do so and to be original, but this is not true: no method can avoid the analytic structure of singularities.

Moreover, in order to be nonperturbative, i.e. to only require a finite number of steps, the integration methods have better use an elementary "unit of integration" which has singularities (such as tanh or sech) rather than an entire function (such as exp).

When nothing is known about the ODE (1), most methods follow the pioneering work of Weiss, Tabor and Carnevale [28]. The restriction of their method to ODEs was certainly known to the classical authors involved with explicit integration (Painlevé, Gambier, Chazy, Bureau), although we could not find an explicit, convincing reference.

All these methods proceed as follows:

1. Assume a given class of expressions for the general analytic solution,
2. Check whether there are indeed solutions in that class.

Various names are used to describe these methods: truncation method, tanh method, Jacobi expansion method, ...

Typical classes of assumed expressions are:

- polynomials in tanh (Weiss *et al.* [28] and followers [5]),
- polynomials in $\wp, \wp'$ [14, 20],
- polynomials in tanh and sech [17, 9, 24].

We will call all these methods **sufficient** because, by construction, they cannot find any solution outside the given class of expressions.

[1] With his "équation aux variations", see Chazy [3, page 338 footnote 1].

4.1 Examples of sufficient methods

The KS ODE (17) has one family of movable triple poles, and the Weierstrass function $\wp$ one family of double poles. Therefore it is consistent to assume

$$\begin{aligned} & u = c_0\wp' + c_1\wp + c_2,\ c_0 \neq 0, \\ & E(u) \equiv \nu u''' + bu'' + \mu u' + \frac{u^2}{2} + A = 0,\ \nu \neq 0, \\ & \wp'' = 6\wp^2 - \frac{g_2}{2},\ \wp'^2 = 4\wp^3 - g_2\wp - g_3. \end{aligned} \tag{27}$$

The elimination of $\wp''$ and $\wp'^2$ transforms the lhs of (27) into

$$E(u) = \sum_{j=0}^{k}\sum_{k=0}^{1} E_{j,k} u^j u'^k,$$

and one has to solve the six determining equations

$$\begin{cases} E_{3,0} \equiv c_0(120\nu + 2c_0) = 0, \\ E_{1,1} \equiv 12bc_0 + c_0c_1 + 12\nu c_1 = 0, \\ E_{2,0} \equiv 6\mu c_0 + 6bc_1 + \dfrac{c_1^2}{2} = 0, \\ E_{0,1} \equiv c_0c_2 + \mu c_1 = 0, \\ E_{1,0} \equiv c_1c_2 - \dfrac{1}{2}g_2c_0^2 - 18\nu g_2 c_0 = 0, \\ E_{0,0} \equiv A - 12\nu g_3 c_0 + \dfrac{1}{2}\left(c_2^2 - bg_2c_1 - \mu g_2 c_0 - g_3 c_0^2\right) = 0. \end{cases}$$

As always in these truncations [28], these equations *must* be solved by decreasing value of the singularity degree $3j + 4k$, for the successive unknowns c_0, c_1, c_2, g_2, g_3. The result is [14, 20]

$$\begin{cases} u = -60\nu\wp' - 15b\wp - \dfrac{b\mu}{4\nu}, \\ g_2 = \dfrac{\mu^2}{12\nu^2},\ g_3 = \dfrac{13\mu^3 + \nu A}{1080\nu^3}, b^2 = 16\mu\nu, \end{cases} \tag{28}$$

and this would be the general analytic solution if $b^2 - 16\mu\nu$ were unconstrained. Removing this constraint is still an open problem.

Instead of considering the Weierstrass equation, one may consider the Riccati equation with constant coefficients

$$\tau' + \tau^2 + \frac{S}{2} = 0,\ S = -\frac{k^2}{2} = \text{ constant} \in \mathcal{C}, \tag{29}$$

whose singularities are one family of movable simple poles, and make the similar consistent assumption [4, 8]

$$u = \sum_{j=0}^{-p} c_j \tau^{-j-p},\ c_0 \neq 0,\ p = -3, \tag{30}$$

with c_j constants to be determined. The elimination of τ' from (29) transforms the lhs of (27) into a similar polynomial

$$E(u) = \sum_{j=0}^{-q} E_j \tau^{-j-q} = 0,\ q = -6. \tag{31}$$

This set of determining equations $E_j = 0$ admits six solutions, listed in Table 1, all represented by

$$\begin{aligned} u &= 120\nu\tau^3 - 15b\tau^2 + \left(\frac{60}{19}\mu - 30\nu k^2 - \frac{15b^2}{4\times 19\nu}\right)\tau \\ &+\frac{5}{2}bk^2 - \frac{13b^3}{32\times 19\nu^2} + \frac{7\mu b}{4\times 19\nu},\ \tau = \frac{k}{2}\tanh\frac{k}{2}(\xi-\xi_0). \end{aligned} \tag{32}$$

The solitary waves $b = 0$ were found by Kuramoto and Tsuzuki [21], and the three other values of $b^2/(\mu\nu)$ were added by Kudryashov [19].

Currently, no other solution is known to (27).

Table 1: The six trigonometric solutions of KS, Eq. (17), with the notation $k^2 = -2S$. They all have the form (32). The last line is a degeneracy of the elliptic solution (28).

$b^2/(\mu\nu)$	$\nu A/\mu^3$	$\nu k^2/\mu$
0	$-4950/19^3,\ 450/19^3$	$11/19,\ -1/19$
144/47	$-1800/47^3$	$1/47$
256/73	$-4050/73^3$	$1/73$
16	$-18,\ -8$	$1,\ -1$

It is easy to build an example making truncation methods fail. Knowing that rational functions are usually not assumed classes, such an example is for instance

$$u = \frac{\tanh(\xi-\xi_0)}{2+\tanh^2(\xi-\xi_0)} = \text{outside usual classes}, \tag{33}$$

$$2u'^2 + (24u^2-3)u' + 72u^4 - 17u^2 + 1 = 0. \tag{34}$$

5 Subequation method (necessary)

5.1 The class of elliptic functions

All solutions found in previous examples are elliptic or degenerate. This quite frequent result for solutions of autonomous nonlinear ODEs has a simple explanation, dating back to Lazarus Fuchs. Indeed, this is a classical result (L. Fuchs, Poincaré, Painlevé) that the only autonomous first order algebraic ODEs having a single-valued general solution are algebraic transforms of either the Weierstrass elliptic equation, see (2), or the Riccati equation (29). These first order ODEs are therefore the elementary building blocks with which to build solutions to autonomous ODEs of higher order.

Conversely, given an N-th order autonomous algebraic ODE, is it possible to find **all** solutions in this class of elliptic or degenerate elliptic functions? The present section brings a positive answer.

5.2 Two results of Briot and Bouquet

Given an elliptic function, its **elliptic order** is defined as the number of poles in a period parallelogram, counting multiplicity of course. It is equal to the number of zeros.

We will need two classical theorems.

Theorem [2, theorem XVII p. 277]. Given two elliptic functions u, v with the same periods of respective elliptic orders m, n, they are linked by an algebraic equation

$$F(u,v) \equiv \sum_{k=0}^{m} \sum_{j=0}^{n} a_{j,k} u^j v^k = 0, \tag{35}$$

with $\deg(F,u) = \mathrm{order}(v)$, $\deg(F,v) = \mathrm{order}(u)$. If in particular v is the derivative of u, the first order ODE obeyed by u takes the precise form

$$F(u,u') \equiv \sum_{k=0}^{m} \sum_{j=0}^{2m-2k} a_{j,k} u^j u'^k = 0,\ a_{0,m} \neq 0. \tag{36}$$

Theorem (Briot and Bouquet, Poincaré, Painlevé). If a first order m-th degree autonomous ODE has the Painlevé property,

- it must have the form (36),
- its general solution is either elliptic (two periods) or rational in one exponential e^{kx} (one period) or rational in x (no period) (successive degeneracies $g_2^3 - 27g_3^2 = 0$, then $g_2 = 0$ in $\wp'^2 = 4\wp^3 - g_2\wp - g_3$).

Remark. Equation (36) is invariant under an arbitrary homographic transformation having constant coefficients, this is another useful feature of elliptic equations.

5.3 General method to find all elliptic solutions

Consider an N-th order autonomous algebraic ODE (1) admitting at least one Laurent series

$$u = \chi^p \sum_{j=0}^{+\infty} u_j \chi^j,\ \chi = \xi - \xi_0. \tag{37}$$

If its general analytic solution, as defined section 1, is elliptic or degenerate elliptic, there exists an algorithm to find it in closed form, which we now present. It only requires four ingredients: the two theorems of Briot and Bouquet stated section 5.2, the existence of at least one Laurent series, an algorithm of Poincaré to be presented soon. Its input and output are as follows.

Input: an N-th order ($N \geq 2$) any degree autonomous algebraic ODE admitting a Laurent series.

Output: all its elliptic or degenerate elliptic solutions in closed form.

The successive steps are [23, 11]:

1. Find the analytic structure of singularities (e.g., 4 families of simple poles, 2 of double poles). Deduce the elliptic orders m, n of u, u'.
2. Compute slightly more than $(m+1)^2$ terms in the Laurent series.
3. Define the first order m-th degree subequation $F(u,u') = 0$ (it contains at most $(m+1)^2$ coefficients $a_{j,k}$),

$$F(u,u') \equiv \sum_{k=0}^{m} \sum_{j=0}^{2m-2k} a_{j,k} u^j u'^k = 0,\ a_{0,m} \neq 0. \tag{38}$$

4. Require each Laurent series (37) to obey $F(u, u') = 0$,

$$F \equiv \chi^{m(p-1)} \left(\sum_{j=0}^{J} F_j \chi^j + \mathcal{O}(\chi^{J+1}) \right), \ \forall j \ : \ F_j = 0. \tag{39}$$

   and solve this **linear overdetermined** system for $a_{j,k}$.

5. Integrate each resulting ODE $F(u, u') = 0$.

Several remarks are in order.

1. The fourth step generates a *linear*, infinitely overdetermined, system of equations $F_j = 0$ for the unknown coefficients $a_{j,k}$. It is quite an easy task to solve such a system, and this is the key advantage of the present algorithm.

2. In the fifth step, two cases arise, depending on the genus of the algebraic curve $F = 0$. If the genus is one (nondegenerate elliptic), there exists an algorithm of Poincaré which builds explicitly the rational function R, Eq. (2). This algorithm has been implemented by Mark van Hoeij [15] as the package "algcurves" of the computer algebra language Maple. If the genus is zero (degenerate elliptic), finding u as a rational function of one exponential $e^{k(\xi-\xi_0)}$ or as a rational function of $\xi - \xi_0$ is classical, and also implemented in "algcurves".

3. In the third and fourth steps, the requirement can be weakened, at the price of finding less that the general analytic solution. Consider for instance the artificial ODE (which actually occurs in the travelling wave of the modified KdV equation),

$$E(u) \equiv \text{some differential consequence of } a^2 u'^2 - (u^2 + b)^2 + c = 0, \tag{40}$$

   which admits two Laurent series

$$u = \pm a \chi^{-1} + \dots. \tag{41}$$

   With the assumption $m = 2$ in step 3 and "Require each Laurent series" in step 4, one will find the Jacobi solution. But, with the weaker assumption $m = 1$ in step 3 and "Require one [not two] Laurent series" in step 4, one will find the constraint $c = 0$ in $E(u)$. The interested reader can practice on the ODE admitting, for instance, the rational solution

$$u = c_1(x - x_1)^{-1} + c_2(x - x_2)^{-2} + (x - x_3)^{-3}, \ c_j \text{ fixed}, \ x_j \text{ movable}, \tag{42}$$

   for which the assumptions can be $m = 6, 5, 4, 3$.

6 Examples

6.1 Tutorial integrable example: KdV

The ODE (6) presents one movable double pole $u \sim 2a\chi^{-2}$, hence the respective elliptic orders 2 and 3 for u and u'. Nine terms (in fact five because of parity) will prove sufficient in the computation of the unique Laurent series,

$$u = 2a\chi^{-2} + U_4\chi^2 + U_6\chi^4 + \frac{U_4^2}{6a}\chi^6 + \dots, \tag{43}$$

in which U_4 and U_6 are arbitrary constants. Assuming in step 3

$$F \equiv {u'}^2 + a_{0,1}u' + a_{1,1}uu' + a_{0,0} + a_{1,0}u + a_{2,0}u^2 + a_{3,0}u^3, a_{0,2} = 1, \tag{44}$$

one generates in step 4 the linear overdetermined system (39),

$$\begin{cases} F_0 \equiv 16a^2 a_{0,2} + 8a^3 a_{3,0} = 0, \\ F_1 \equiv -8a^2 a_{1,1} = 0, \\ F_2 \equiv 4a^2 a_{2,0} = 0, \\ F_3 \equiv -4aa_{0,1} = 0, \\ F_4 \equiv 2aa_{1,0} - 16aa_{0,2}U_4 + 12a^2 a_{3,0}U_4 = 0, \\ F_5 \equiv 0, \\ F_6 \equiv a_{0,0} + 4aa_{2,0}U_4 - 32aa_{0,2}U_6 + 12a^2 a_{3,0}U_6 = 0, \\ \dots \end{cases} \tag{45}$$

whose unique solution is

$${u'}^2 - (2/a)u^3 + 20U_4 u + 56aU_6 = 0. \tag{46}$$

The two arbitrary constants U_4, U_6 correspond to the two first integrals of (6), but no skill is required to find them, the process is systematic.

In step 5, the Maple commands would look like

```
with(algcurves);                                    load the package
genus(eq46,u,uprime);                               will answer "1"
Weierstrassform(eq46,u,uprime,wp,wpprime);
```

the last command yielding the four formulae of the birational transformation between (u, u') and $(\wp, \wp')$, one of them being precisely (9).

6.2 Partially integrable example, KS

The Laurent series of (17) is (25).

In the first step, the unique family of movable triple poles yields elliptic orders 3 and 4 for respectively u and u'. With the normalization $a_{0,3} = 1$, the subequation $F(u, u') = 0$ contains ten coefficients. In step 4, these ten coefficients are first determined by solving the Cramer system of ten equations $F_j = 0, j = 0 : 6, 8, 9, 12$. The remaining infinitely overdetermined nonlinear system for (ν, b, μ, A) contains as greatest common divisor (gcd) $b^2 - 16\mu\nu$ which defines a first solution

$$\begin{aligned} & \frac{b^2}{\mu\nu} = 16,\ u_s = u + \frac{3b^3}{32\nu^2}, \\ & \left(u' + \frac{b}{2\nu}u_s\right)^2 \left(u' - \frac{b}{4\nu}u_s\right) + \frac{9}{40\nu}\left(u_s^2 + \frac{15b^6}{1024\nu^4} + \frac{10A}{3}\right)^2 = 0. \end{aligned} \tag{47}$$

After division by this factor, the remaining nonlinear system for (ν, b, μ, A) with $b^2 - 16\mu\nu \neq 0$ admits exactly four solutions (stopping the series at $j = 16$ is enough to obtain the result), identical to those listed in Table 1 Section 4.1, each solution defining a first order third degree subequation,

$$\begin{aligned} & b = 0, \\ & \left(u' + \frac{180\mu^2}{19^2\nu}\right)^2 \left(u' - \frac{360\mu^2}{19^2\nu}\right) + \frac{9}{40\nu}\left(u^2 + \frac{30\mu}{19}u' - \frac{30^2\mu^3}{19^2\nu}\right)^2 = 0, \end{aligned} \tag{48}$$

$$b=0,\ u'^3+\frac{9}{40\nu}\left(u^2+\frac{30\mu}{19}u'+\frac{30^2\mu^3}{19^3\nu}\right)^2=0, \tag{49}$$

$$\frac{b^2}{\mu\nu}=\frac{144}{47},\ u_s=u-\frac{5b^3}{144\nu^2},\ \left(u'+\frac{b}{4\nu}u_s\right)^3+\frac{9}{40\nu}u_s^4=0, \tag{50}$$

$$\frac{b^2}{\mu\nu}=\frac{256}{73},\ u_s=u-\frac{45b^3}{2048\nu^2},$$

$$\left(u'+\frac{b}{8\nu}u_s\right)^2\left(u'+\frac{b}{2\nu}u_s\right)+\frac{9}{40\nu}\left(u_s^2+\frac{5b^3}{1024\nu^2}u_s+\frac{5b^2}{128\nu}u'\right)^2=0. \tag{51}$$

In order to integrate the two sets of subequations (47), (48)–(51), one must first compute their genus, which is one for (47), and zero for (48)–(51). Therefore (47) has the elliptic general solution (28).

As to the general solution of the four others (48)–(51), this is the third degree polynomial (32) in $\tanh k(\xi-\xi_0)/2$ already found by the one-family truncation method.

Remark. Canceling the gcd $b^2-16\mu\nu$ is equivalent to cancel the residue in the Laurent series (25), therefore KS admits no elliptic solution other than (28) [16]. Indeed, this is a property of elliptic functions that, inside a period parallelogram, the sum of the residues is necessarily zero. Conversely, by implementing the condition that the sum of the residues of any differential polynomial of u vanishes, Hone [16] has found an elegant method to isolate all the nondegenerate elliptic (genus one) solutions, proving in particular that CGL3 has no travelling wave which is nondegenerate elliptic. This method cannot however also yield the degenerate elliptic solutions.

6.3 Partially integrable example, CGL5

As example of a new solution found by the method of section 5.3, one can quote the unique elliptic (nondegenerate, i.e. genus one) travelling wave (19) of the CGL5 equation (18) [27],

$$\left\{\begin{array}{l} M=c_0\dfrac{4\wp^2(\xi)-g_2}{4\wp^2(\xi)+g_2},\ \varphi'=\dfrac{cs_r}{2}+\left(-\dfrac{3g_2}{4\wp(2\xi)}\right)^{1/2},\\ \text{notation } s_r-is_i\equiv p^{-1},\ g_r\equiv s_r\gamma+s_i\omega,\ e_i\equiv\Im(r/p),\\ \text{constraints } q=0,\ \Re(r/p)=0,\ cs_i=0,\ s_r\omega-s_i\gamma+(cs_r/2)^2=0,\\ \text{values of } c_0,g_2,g_3:\ c_0^2=\dfrac{4g_r}{3e_i},\ g_2=-\dfrac{g_r^2}{27},\ g_3=0. \end{array}\right. \tag{52}$$

This solution, obtained by combining the present method with the conditions on residues [16], exists at the price of five real constraints among the coefficients p,q,r,γ,ω,c of the differential system for (M,φ').

In this example, care should be taken that the two elliptic subequations,

$$\left\{\begin{array}{l} 3^4e_iM'^4-M^2(3e_iM^2-4g_r)^3=0,\\ 9\psi'^2-12\psi^4-g_r^2=0,\ \psi=\varphi'-\dfrac{cs_r}{2}, \end{array}\right. \tag{53}$$

respectively of a canonical Briot-Bouquet type and of a Jacobi type, involve elliptic functions with *different* first arguments, see $\wp(2\xi)$ term in (52).

6.4 Results for various partially integrable PDEs

For the various PDEs considered in the text, and recalled below for convenience,

$$\text{KdV}\ :\ u'''-(6/a)uu'=0,$$

$$\begin{aligned}
&\text{KS}\ :\ \nu u''' + bu'' + \mu u' + u^2/2 + A = 0,\ \nu \neq 0,\\
&\text{CGL3}\ :\ iA_t + pA_{xx} + q|A|^2 A - i\gamma A = 0,\ pq \neq 0,\ \Im(q/p) \neq 0,\\
&\text{CGL5}\ :\ iA_t + pA_{xx} + q|A|^2 A + r|A|^4 A - i\gamma A = 0,\ pr \neq 0,\ \Im(p/r) \neq 0,\\
&\text{2 CGL3}\ :\ \partial_t A_\pm = rA_\pm \mp v\partial_x A_\pm + (1+i\alpha)\partial_x^2 A_\pm - (1+i\beta)(|A_\pm|^2 + \gamma|A_\mp|^2)A_\pm,\\
&\qquad\qquad r, v, \alpha, \beta, \gamma \text{ real parameters,}
\end{aligned}$$

Table 2 collects the results produced by the method for the various PDEs in the text (we have added in last line an interesting system, not yet processed). In this table, nPq is short for "n families of q-th order poles", $(m+1)^2-1$ is the maximal number of coefficients in the subequation (36), and the column "$a_{j,k}$" shows the true number of such coefficients. The column 'u_k" displays the minimal number of terms to be computed in the Laurent series (37).

Table 2: Summary of results. The solutions are labelled "ell, trig, rat" for, respectively: elliptic (genus one), rational in one exponential, rational.

PDE	poles	m	$(m+1)^2-1$	$a_{j,k}$	u_k	solutions	Ref
KdV	1P2	2	8	6	8	1 ell	
KS	1P3	3	15	10	16	1 ell + 4 trig	[23]
CGL3	2P2	4	24	18	24	6 trig + 1 rat	[23, 16]
CGL5	4P1	4	24	24		1 ell + n trig	[27]
2 CGL3	2P2	4	24	18		To be done	[10]

7 Possible extensions of the method

It should first be noted that the output of the method is all solutions which are elliptic functions of $\xi - \xi_0$ or rational in one exponential $e^{k(\xi-\xi_0)}$ or rational in $\xi - \xi_0$, in which ξ_0 is the *arbitrary* constant arising from the first order subequation.

Having this in mind, one can think of several possible extensions.

7.1 Extension to autonomous discrete equations

If the independent variable ξ is discrete ($\xi = nh$, with n integer and h a given stepsize), can the method be extended so as to yield all elliptic or degenerate elliptic solutions? The question is still open.

For instance, the discrete nonlinear equation of the Schrödinger type

$$iu_t + p\frac{u(x+h,t) + u(x-h,t) - 2u(x)}{h^2} + q\frac{|u|^2 u}{1 + \nu(qh^2/p)|u|^2} = 0,\ pq\nu \neq 0, \tag{54}$$

admits various elliptic solitary waves [18, 6] but no proof exists that these are the only ones.

Note that it is sufficient to look for rational functions of $\wp(\xi), \wp'(\xi)$ since, as shown by Briot and Bouquet [2], any elliptic function of argument $\xi + a$ can be expressed as a rational function of $\wp(\xi), \wp'(\xi)$.

7.2 Extension to nonautonomous differential equations

Consider for instance the problem (already solved, but this is just an example) of finding by the above method all the rational solutions of the second Painlevé equation P2

$$u'' = 2u^3 + xu + \alpha, \tag{55}$$

i.e. the sequence, for α integer,

$$u = \mp \frac{1}{x},\ \alpha = \pm 1, \tag{56}$$

$$u = \mp \frac{2(x^3-2)}{x(x^3+4)},\ \alpha = \pm 2, \tag{57}$$

$$u = \mp \frac{3x^2(x^6+8x^3+160)}{(x^3+4)(x^6+20x^3-80)},\ \alpha = \pm 3, \dots. \tag{58}$$

The integration constant ξ_0 of previous sections is no more arbitrary.

In order to retrieve these rational solutions by the method of section 5.3, one must

- ignore the structure of movable singularities of (55);
- consider instead the simplest differential consequence of (55) which is autonomous, i.e. (elimination of x),

$$uu''' - u'u'' - 4u^3u' - u^2 + \alpha u' = 0; \tag{59}$$

- guided by (56)–(58), manage to find the (probably α^2 in number) families of movable simple poles in (59) (it is for the moment not completely clear how this result should come out);
- apply the method to (59) with $m = \alpha^2$ and find a unique solution.

We have not yet done it but these are at least the guidelines to be followed.

Remark. The symmetries of P2 allow to considerably simplify the above by the change $(u, x) \to (U, X), u = x^2U, X = x^3$, thus lowering m to $|\alpha|$.

7.3 Extension to Painlevé solutions

After the elliptic function, the next elementary functions defined by differential equations are the six Painlevé functions, which all obey a nonautonomous nonlinear second order ODE. Hence the question: given an N-th order ODE (1), $N \ge 3$, can the method of section 5.3 be extended so as to yield all the solutions which are algebraic transforms of a given Painlevé equation? Let us take an example to illustrate the difficulties to overcome.

Consider the third order autonomous ODE built by elimination of the constant K_1 in (this example is taken from the Lorenz model)

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + 2\frac{\mathrm{d}x}{\mathrm{d}t} + \frac{x^3}{2} + \left(\frac{8}{9} - \frac{K_1}{2}e^{-2t}\right)x = 0. \tag{60}$$

This ODE happens to be integrable with a particular second Painlevé function with $\alpha = 0$ [26],

$$x = ae^{-2t/3}X,\ T = \frac{i}{2}a^{-3/2}e^{-2t/3},\ K_1 = \frac{3}{8}ia^3, \tag{61}$$

$$\frac{\mathrm{d}^2X}{\mathrm{d}T^2} = 2X^3 + TX, \tag{62}$$

The problem is to uncover the subequation (62) for $X(T)$ knowing only the third order autonomous ODE for $x(t)$ and its Laurent series. The necessity to take account of a possible change of the independent variable $t \to T$ introduces a countable[2] number of arbitrary coefficients (those of the Taylor series of $t \to T$ near the movable singularity of $x(t)$) and thus makes it difficult to use the information provided by the Laurent series.

In fact, the main privilege of the elliptic functions, already mentioned as a remark in section 5.2, is the autonomous nature of their ODE and the invariance of this ODE under homographic transformations, all features which do not exist any more for Painlevé functions.

8 Conclusion and perspectives

The main features of the method presented here are as follows:

- It provides *all* the elliptic and degenerate elliptic (i.e. rational in a single exponential or rational) solutions of a given autonomous nonlinear algebraic ODE;
- It includes all "truncation", "extended", "new", etc methods;
- It makes all these methods obsolete;
- Its main part (solving system (39)) is a linear algebraic problem.

Of course, integrating the resulting first order ODE may consume much more time.

There exists another theory in which meromorphic functions play a central role, this is Nevanlinna theory. For a short comparison between Nevanlinna theory and the present method, the interested reader can refer to [12].

Current and future work includes:

- Revisit physically interesting PDEs with insufficient solutions, such as the system of two coupled CGL3 mentioned in section 6.4.
- Find general analytic solutions which are not elliptic, for instance Lamé functions.

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References

[1] M. J. Ablowitz and P. A. Clarkson, *Solitons, nonlinear evolution equations and inverse scattering* (Cambridge University Press, Cambridge, 1991).

[2] C. Briot et J.-C. Bouquet, *Théorie des fonctions elliptiques* (Mallet-Bachelier, Paris, 1859). `http://gallica.bnf.fr/document?O=N099571`

[3] J. Chazy, Sur les équations différentielles du troisième ordre et d'ordre supérieur dont l'intégrale générale a ses points critiques fixes, Thèse, Paris (1910); Acta Math. **34** (1911) 317–385.

[2] This number can be made finite by requiring the change $(x,t) \to (X,T)$ to introduce only exponential functions as in (61), so as to make the transformed ODE polynomial in the new independent variable T, before application of the method.

[4] R. Conte, Universal invariance properties of Painlevé analysis and Bäcklund transformation in nonlinear partial differential equations, Phys. Lett. A **134** (1988) 100–104.

[5] R. Conte, Invariant Painlevé analysis of partial differential equations, Phys. Lett. A **140** (1989) 383–390.

[6] R. Conte and K.-w. Chow, Doubly periodic waves of a discrete nonlinear Schrödinger system with saturable nonlinearity, J. Nonlinear Math. Phys. **?** (2008) accepted. http://arxiv.org/abs/0812.1196

[7] R. Conte, A. P. Fordy, and A. Pickering, A perturbative Painlevé approach to nonlinear differential equations, Physica D **69** (1993) 33–58.

[8] R. Conte and M. Musette, Painlevé analysis and Bäcklund transformation in the Kuramoto-Sivashinsky equation, J. Phys. A **22** (1989) 169–177.

[9] R. Conte and M. Musette, Link between solitary waves and projective Riccati equations, J. Phys. A **25** (1992) 5609–5623.

[10] R. Conte and M. Musette, Analytic expressions of hydrothermal waves, Reports on mathematical physics **46** (2000) 77–88. nlin.SI/0009022

[11] R. Conte and M. Musette, *The Painlevé handbook*, xxiv+256 pages (Springer, Berlin, 2008). http://www.springer.com/physics/book/978-1-4020-8490-4

[12] R. Conte and Ng T-w, Meromorphic solutions of a third order nonlinear differential equation, to be submitted (2009).

[13] G. Darboux, Sur les équations aux dérivées partielles, C. R. Acad. Sc. Paris **96** (1883) 766–769.

[14] J.D. Fournier, E.A. Spiegel and O. Thual, Meromorphic integrals of two nonintegrable systems, *Nonlinear dynamics*, 366–373, ed. G. Turchetti (World Scientific, Singapore, 1989).

[15] Mark van Hoeij, package "algcurves", Maple V (1997). http://www.math.fsu.edu/~hoeij/algcurves.html

[16] A.N.W. Hone, Non-existence of elliptic travelling wave solutions of the complex Ginzburg-Landau equation, Physica D **205** (2005) 292–306.

[17] A. Jeffrey and Xu S., Travelling wave solutions to certain non-linear evolution equations, Int. J. Non-Linear Mechanics **24** (1989) 425–429.

[18] A. Khare, K.Ø. Rasmussen, M.R. Samuelsen and A. Saxena, Exact solutions of the saturable discrete nonlinear Schrödinger equation, J. Phys. A **38** (2005) 807–814.

[19] N. A. Kudryashov, Exact soliton solutions of the generalized evolution equation of wave dynamics, Prikladnaia Matematika i Mekhanika **52** (1988) 465–470 [English : Journal of applied mathematics and mechanics **52** (1988) 361–365]

[20] N. A. Kudryashov, Exact solutions of a generalized equation of Ginzburg-Landau, Matematicheskoye modelirovanie **1** (1989) 151–158. [English : none?].

[21] Y. Kuramoto and T. Tsuzuki, Persistent propagation of concentration waves in dissipative media far from thermal equilibrium, Prog. Theor. Phys. **55** (1976) 356–369.

[22] P. Manneville, *Dissipative structures and weak turbulence* (Academic Press, Boston, 1990). French adaptation: *Structures dissipatives, chaos et turbulence* (Aléa-Saclay, Gif-sur-Yvette, 1991).

[23] M. Musette and R. Conte, Analytic solitary waves of nonintegrable equations, Physica D **181** (2003) 70–79. http://arXiv.org/abs/nlin.PS/0302051

[24] A. Pickering, A new truncation in Painlevé analysis, J. Phys. A **26** (1993) 4395–4405.

[25] W. van Saarloos, Front propagation into unstable states, Physics reports **386** 29–222 (2003).

[26] H. Segur, Solitons and the inverse scattering transform, *Topics in ocean physics*, 235–277, eds. A. R. Osborne and P. Malanotte Rizzoli (North-Holland publishing co., Amsterdam, 1982).

[27] S. Yu. Vernov, Elliptic solutions of the quintic complex one-dimensional Ginzburg-Landau equation, J. Phys. A **40** 9833–9844 (2007). http://arXiv.org/abs/nlin.PS/0602060

[28] J. Weiss, M. Tabor, and G. Carnevale, The Painlevé property for partial differential equations, J. Math. Phys. **24** (1983) 522–526.