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FAST TRACK COMMUNICATION

Explicit solutions of the four-wave mixing model

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Abstract

Dynamical degenerate four-wave mixing is studied analytically in detail. By removing the unessential freedom, we first characterize this system by a lower-dimensional closed subsystem of a deformed Maxwell–Bloch type, involving only three physical variables: the intensity pattern, the dynamical grating amplitude, the relative net gain. We then classify by the Painlevé test all the cases when single-valued solutions may exist, according to the two essential parameters of the system: the real relaxation time τ, and the complex response constant γ. In addition to the stationary case, the only two integrable cases occur for a purely nonlocal response ($\mathrm{Re}(\gamma) = 0$), these are the complex unpumped Maxwell–Bloch system and another one, which is explicitly integrated with elliptic functions. For a generic response ($\mathrm{Re}(\gamma) \neq 0$), we display strong similarities with the cubic complex Ginzburg–Landau equation.

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1. Introduction

The wave self-action by the degenerate mixing in a nonlinear medium involves three simultaneous processes: the interference of waves, the recording of the dynamical grating by an interference pattern, and the wave diffraction by the grating. This process is now the basic technique of some important practical applications in real-time holography, including optical phase conjugation, holographic interferometry, novelty filters, all-optical signal processing, etc [15, 17, 22].

During the wave mixing, the self-diffraction of waves is governed by a self-consistent set of five equations for five complex amplitudes A_j, $j = 1, 2, 3, 4$ and $\mathcal{E}$, see e.g. [22]

$$\partial_z A_1 = -\mathrm{i}\mathcal{E} A_2, \quad \partial_z \bar{A}_2 = \mathrm{i}\mathcal{E} \bar{A}_1, \quad \partial_z \bar{A}_3 = -\mathrm{i}\mathcal{E} \bar{A}_4, \quad \partial_z A_4 = \mathrm{i}\mathcal{E} A_3, \tag{1}$$

$$\partial_t \mathcal{E} = \gamma I_\mathrm{m} - \frac{\mathcal{E}}{\tau}, \tag{2}$$

$$I_\mathrm{m} = A_1 \bar{A}_2 + \bar{A}_3 A_4, \tag{3}$$

where (1) is the coupled wave system for slow variable amplitudes $A_j(z,t)$ [26], (2) is the evolution equation of the grating amplitude $\mathcal{E}$ with a rhs including the grating gain and the grating relaxation, (3) is the relevant interference pattern of the interacting waves. In our notation a bar denotes complex conjugation, ∂ denotes partial derivation, τ is a real constant.

It must be emphasized that the response constant

$$\gamma = |\gamma| \, \mathrm{e}^{\mathrm{i}g} \tag{4}$$

is complex. We will use the terms 'local' and 'nonlocal' response to describe the phase shift between the index grating $\mathcal{E}$ and the interference pattern I_m. In the case of a purely nonlocal response (γ purely imaginary), an energy transfer occurs between the interacting waves, whereas a local response (γ real) is characterized by an exchange of the phases of the waves [22]. In particular, the complex value of the coupling coefficient $\mathcal{E}$ is an essential feature for the existence of solitonlike solutions.

Apart from t and τ, all variables are assumed dimensionless, after normalizing the physical variables A'_j, z',

$$A_j = \frac{A'_j}{\sqrt{I_0}}, \qquad z = \frac{k_0^2}{2k_z} z', \tag{5}$$

where k_0 is the amplitude of the wave vector in the free space, I_0 is the total input intensity

$$I_0 = \sum_{j=1}^{4} I_j = \text{constant}, \qquad I_j = |A'_j|^2. \tag{6}$$

We restrict ourselves here to the so-called degenerate four-wave mixing (the four frequencies are identical), in the transmission geometry and in two space dimensions,

$$\vec{k}_j = k_{j,x} \vec{e}_x + k_{j,z} \vec{e}_z, \qquad j = 1, 2, 3, 4, \tag{7}$$

$$\vec{k}_1 - \vec{k}_2 = \vec{k}_4 - \vec{k}_3 = \vec{K} \tag{8}$$

($\vec{e}_x$ and $\vec{e}_z$ are unit vectors, $\vec{K}$ is the grating vector).

So far, there exist two main analytic results:

- for γ purely imaginary (purely nonlocal response) and in the stationary regime, a sech profile grating amplitude [18];
- when the phases of each A_j are independent of z, a parametric representation of the five amplitudes also restricted to a purely imaginary γ [4, 5, 18],

$$\mathrm{Re}(\gamma) = 0: \begin{cases} \mathcal{E} = (\partial_z u) \, \mathrm{e}^{\mathrm{i}\varphi_e}, \quad \gamma = \mathrm{i}\gamma_\mathrm{NL}, \quad \gamma_\mathrm{NL} \text{ real}, \\ A_1 = f_{12} \sin(s_{12}(u - C_{12})) \, \mathrm{e}^{\mathrm{i}\varphi_1}, \quad A_2 = f_{12} \cos(s_{12}(u - C_{12})) \, \mathrm{e}^{i\varphi_2}, \\ A_4 = -f_{43} \sin(s_{43}(u + C_{43})) \, \mathrm{e}^{\mathrm{i}\varphi_4}, \quad A_3 = f_{43} \cos(s_{43}(u + C_{43})) \, \mathrm{e}^{\mathrm{i}\varphi_3}, \\ \varphi_1 - \varphi_2 - \varphi_e + \dfrac{\pi}{2} = n_{12}\pi, \quad s_{12} = (-1)^{n_{12}}, \\ \varphi_4 - \varphi_3 - \varphi_e + \dfrac{\pi}{2} = n_{43}\pi, \quad s_{43} = (-1)^{n_{43}}, \\ I_\mathrm{m} = \dfrac{1}{2} \mathrm{e}^{\mathrm{i}(\varphi_e - \pi/2)} \left(f_{12}^2 \sin 2(u - C_{12}) - f_{43}^2 \sin 2(u + C_{43}) \right), \\ n_{12}, n_{43} \in \mathcal{Z}, \end{cases} \tag{9}$$

in terms of the real solution u of a damped sine-Gordon equation [4, 5, 18],

$$u_{zt} + \frac{1}{\tau} u_z - K \sin(2u + \alpha) = 0, \qquad K\, \mathrm{e}^{\mathrm{i}\alpha} = \frac{\gamma_{\mathrm{NL}}}{2} \left(f_{12}^2\, \mathrm{e}^{-2\mathrm{i}C_{12}} - f_{43}^2\, \mathrm{e}^{2\mathrm{i}C_{43}} \right). \tag{10}$$

The representation (9) displays the invariance $(1, 2, 3, 4, \partial_z, u) \to (4, 3, 2, 1, -\partial_z, -u)$ and depends on six arbitrary real functions of t ($f_{12}, f_{43}, C_{12}, C_{43}$ and the values of $\varphi_1 + \varphi_2$ and $\varphi_4 + \varphi_3$) and one arbitrary real constant (the phase φ_e). The stationary sech solution [18] is then represented by [5] (see equation (23) below),

$$\operatorname{tg} u = \mathrm{e}^{2k(z-z_0)}. \tag{11}$$

In the present paper we classify all cases when the system admits solutions with a single-valued dependence on the initial conditions, and, with one major exception, we integrate all these cases. This major exception, left for future work, presents analogous difficulties to the search, in the complex cubic Ginzburg–Landau equation (CGL3),

$$\mathrm{i}A_t + pA_{xx} + q|A|^2 A - \mathrm{i}\gamma A = 0, \qquad pq\gamma \neq 0, \qquad (A, p, q) \in \mathcal{C}, \qquad \gamma \in \mathcal{R}, \tag{12}$$

for source [3], pulse [23] or front [21] solutions.

2. The intrinsic four-wave mixing, a deformed Maxwell–Bloch system

The ten-dimensional system (1)–(3) is invariant under any time-dependent rotation in the space $\{A_1, \overline{A}_2, A_4, \overline{A}_3\}$ which preserves the interference pattern (3). In order to remove this five-parameter unessential freedom, let us apply repeatedly the derivation operator ∂_z, starting from the interference pattern (3), until a closed system has been obtained. This process ends after two steps and results in the intrinsic system

$$\partial_z I_{\mathrm{m}} = -\mathrm{i}\mathcal{E} I_{\mathrm{d}}, \qquad \partial_z I_{\mathrm{d}} = -2\mathrm{i}\bar{\mathcal{E}} I_{\mathrm{m}} + 2\mathrm{i}\mathcal{E}\overline{I_{\mathrm{m}}}, \qquad \partial_t \mathcal{E} = \gamma I_{\mathrm{m}} - \frac{\mathcal{E}}{\tau}, \tag{13}$$

admitting the first integral

$$4|I_{\mathrm{m}}|^2 + I_{\mathrm{d}}^2 = K(t), \qquad K \text{ arbitrary}. \tag{14}$$

The real field I_{d} which is thus introduced,

$$I_{\mathrm{d}} = -|A_1|^2 + |A_2|^2 - |A_3|^2 + |A_4|^2, \tag{15}$$

has a natural interpretation: this is the relative *net gain*, therefore the four-wave mixing is characterized by three intrinsic variables: the intensity pattern I_{m}, the grating amplitude $\mathcal{E}$ and the relative net gain I_{d}.

In previous integration methods [14] for the four-wave mixing, one would mainly look for the wave amplitudes A_j from some nonlinear system. Thanks to the existence of the above intrinsic system, the integration, whether analytic or numerical, now becomes systematic and involves two steps,

(i) integration of the *nonlinear* intrinsic system (13);
(ii) knowing the grating $\mathcal{E}$, integration of the two-dimensional *linear* system

$$\partial_z X = -\mathrm{i}\mathcal{E} Y, \qquad \partial_z Y = -\mathrm{i}\bar{\mathcal{E}} X; \tag{16}$$

indeed, given two linearly independent solutions $(X, Y) = (X_1, Y_1), (X_2, Y_2)$, the general solution of (1) is defined in a matrix form by

$$\begin{pmatrix} A_1 \\ A_2 \end{pmatrix} = a_{12} \begin{pmatrix} X_1 \\ Y_1 \end{pmatrix} + b_{12} \begin{pmatrix} X_2 \\ Y_2 \end{pmatrix}, \qquad \begin{pmatrix} \bar{A}_3 \\ \bar{A}_4 \end{pmatrix} = a_{34} \begin{pmatrix} X_1 \\ Y_1 \end{pmatrix} + b_{34} \begin{pmatrix} X_2 \\ Y_2 \end{pmatrix}, \tag{17}$$

in which the eight integration constants a_{ij}, b_{ij}, constrained by the three relations (3) and (15), depend on five arbitrary parameters according to relation (A.8) in the appendix.

The above system (13) is very similar to another classical system of nonlinear optics, the pumped Maxwell–Bloch system, which is an integrable system defined in complex form as [8]

$$\partial_X \rho = N e, \quad \partial_X \overline{\rho} = N \overline{e}, \quad \partial_X N = -(\rho \overline{e} + \overline{\rho} e)/2 + 4s = 0, \quad \partial_T e = \rho, \quad \partial_T \overline{e} = \overline{\rho}, \tag{18}$$

with s being a real constant (the system is 'pumped' when s is nonzero).

In fact, there is only one situation when the intrinsic four-wave mixing system (13) and the pumped Maxwell–Bloch system (18) can be identified. This occurs when, at the same time, the four-wave mixing model is undamped ($\tau = +\infty$) and has a purely nonlocal response ($\mathrm{Re}(\gamma) = 0$), while the Maxwell–Bloch system is unpumped ($s = 0$). After this identification,

$$\frac{1}{\tau} = 0, \quad \mathrm{Re}(\gamma) = 0, \quad s = 0 : \frac{z}{X} = \frac{t}{T} = \frac{2|\gamma| I_{\mathrm{m}}}{\rho} = \frac{2|\gamma| \overline{I_{\mathrm{m}}}}{\overline{\rho}} = \frac{|\gamma| I_{\mathrm{d}}}{N} = \frac{-2\mathrm{i}\mathcal{E}}{e} = \frac{2\mathrm{i}\bar{\mathcal{E}}}{\overline{e}}, \tag{19}$$

the undamped, purely nonlocal response four-wave mixing model admits all the solutions of the unpumped complex Maxwell–Bloch system.

The undamped case (relaxation time $\tau = +\infty$) physically means the recording of a permanent grating. In optics that can be, for example, the permanent holographic memory realized in nonlinear media.

For practical computations, it may be advisable to eliminate I_{m} from the grating evolution (2) and to equivalently consider the three-dimensional fifth-order closed system,

$$\begin{cases} |\gamma|^2 \partial_z I_{\mathrm{d}} - 2\mathrm{i}\gamma \mathcal{E}(\partial_t \bar{\mathcal{E}} + \bar{\mathcal{E}}/\tau) + 2\mathrm{i}\bar{\gamma}\bar{\mathcal{E}}(\partial_t \mathcal{E} + \mathcal{E}/\tau) = 0, \\ \left(\partial_z \partial_t + \dfrac{1}{\tau}\partial_z\right)\mathcal{E} + \mathrm{i}\gamma \mathcal{E} I_{\mathrm{d}} = 0, \\ 4|\gamma|^{-2}|\partial_t \mathcal{E} + \mathcal{E}/\tau|^2 + I_{\mathrm{d}}^2 = K(t), \quad K \text{ arbitrary.} \end{cases} \tag{20}$$

The following will display the crucial role of the third intrinsic variable (the relative net gain I_{d}) to perform the explicit analytic integration whenever it is possible.

3. The stationary case: general solution

When the amplitudes are independent of the time t, the integration can be performed completely. The intrinsic system (13)–(14) for I_{m}, I_{d}, $\mathcal{E}$ reduces to

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}z} I_{\mathrm{m}} = -\mathrm{i}\mathcal{E} I_{\mathrm{d}}, \quad \dfrac{\mathrm{d}}{\mathrm{d}z} I_{\mathrm{d}} = -4|\gamma|\tau(\sin g)|I_{\mathrm{m}}|^2, \quad \mathcal{E} = \gamma\tau I_{\mathrm{m}}, \\ 4|I_{\mathrm{m}}|^2 + I_{\mathrm{d}}^2 = K, \end{cases} \tag{21}$$

in which the first integral K is independent of t, therefore I_{d} obeys a first-order ordinary differential equation (ODE) of the Riccati type,

$$\frac{\mathrm{d}}{\mathrm{d}z} I_{\mathrm{d}} = |\gamma|\tau(\sin g)\left(I_{\mathrm{d}}^2 - K\right). \tag{22}$$

The case γ real is uninteresting for it involves no energy exchange and the intensities $|\mathcal{E}|^2$, $|I_{\mathrm{m}}|^2$, I_{d} are all constant.

For γ nonreal, the nonlinear intrinsic system (21) admits the general solution

$$\gamma \notin \mathcal{R}: \begin{cases} I_{\rm d} = -\dfrac{k \tanh kz}{|\gamma|\tau \sin g}, \quad \mathcal{E} = \gamma\tau I_{\rm m} = \dfrac{{\rm e}^{2{\rm i}\varphi_0}}{2\sin g}\,(k\,{\rm sech}\,kz)^{1-{\rm i}\,{\rm cotg}\,g}, \\ K = \left(\dfrac{k}{|\gamma|\tau \sin g}\right)^2, \end{cases} \tag{23}$$

in which k, z_0, φ_0 are constants of integration, with $z - z_0$ written for shortness as z.

These bright profiles for $|\mathcal{E}|^2$ and $|I_{\rm m}|^2$ extrapolate the solution of [5] which was restricted to γ purely imaginary.

The amplitudes are found by noticing that each variable $A_j\mathcal{E}^{-1/2}, j = 1, 4$ and $A_j\bar{\mathcal{E}}^{-1/2}, j = 2, 3$ obeys a second-order linear ODE with constant coefficients. The result is

$$\gamma \notin \mathcal{R}: \begin{cases} A_1 = (k\,{\rm sech}\,kz)^{(1-{\rm i}\,{\rm cotg}\,g)/2}\,{\rm e}^{+{\rm i}\varphi_0-{\rm i}g/2}(a_{12}{\rm c}_- + b_{12}{\rm s}_-), \\ A_2 = (k\,{\rm sech}\,kz)^{(1+{\rm i}\,{\rm cotg}\,g)/2}\,{\rm e}^{-{\rm i}\varphi_0+{\rm i}g/2}(-a_{12}{\rm s}_+ + b_{12}{\rm c}_+), \\ A_3 = (k\,{\rm sech}\,kz)^{(1+{\rm i}\,{\rm cotg}\,g)/2}\,{\rm e}^{-{\rm i}\varphi_0+{\rm i}g/2}(B_{34}{\rm c}_+ + A_{34}{\rm s}_+), \\ A_4 = (k\,{\rm sech}\,kz)^{(1-{\rm i}\,{\rm cotg}\,g)/2}\,{\rm e}^{+{\rm i}\varphi_0-{\rm i}g/2}(-B_{34}{\rm s}_- + A_{34}{\rm c}_-), \\ {\rm c}_\pm = \cosh(1 \pm {\rm i}\,{\rm cotg}\,g)\dfrac{kz}{2}, \quad {\rm s}_\pm = \sinh(1 \pm {\rm i}\,{\rm cotg}\,g)\dfrac{kz}{2}, \end{cases} \tag{24}$$

in which the conditions that $\bar{A}_j$ be complex conjugate of A_j requires the four complex constants $a_{12}, b_{12}, A_{34}, B_{34}$ to be represented as

$$\begin{cases} a_{12} = R\cos\lambda\,{\rm e}^{{\rm i}\alpha_{12}}, \quad b_{12} = R\cos\mu\,{\rm e}^{{\rm i}\beta_{12}}, \quad A_{34} = R\sin\lambda\,{\rm e}^{-{\rm i}\alpha_{34}}, \quad B_{34} = R\sin\mu\,{\rm e}^{-{\rm i}\beta_{34}}, \\ 2R^2 = \dfrac{k}{|\gamma|\tau\sin g}, \quad \dfrac{\sin(\alpha_{34}-\beta_{34})}{\sin(\alpha_{12}-\beta_{12})} = -\tan\lambda\tan\mu. \end{cases} \tag{25}$$

The five additional constants of integration, chosen to be all real, are $\lambda, \mu, \alpha_{12}+\beta_{12}, \alpha_{34}+\beta_{34}$, and for instance $\alpha_{12} - \beta_{12} + \alpha_{34} - \beta_{34}$.

4. Determination of the cases of singlevaluedness

In the nonstationary case, the only existing analytic result, valid for a purely nonlocal response ($\text{Re}(\gamma) = 0$) and recalled in the introduction, is the parametric representation of the five complex amplitudes in terms of the solution u of the damped sine-Gordon equation (10). Rather than looking for solutions of this damped sine-Gordon equation, which would only concern the case $\text{Re}(\gamma) = 0$, let us investigate the question of finding single-valued closed form solutions, by applying the Painlevé test [13] in order to detect all obstacles to singlevaluedness.

4.1. The Painlevé test

For the basic notation (singular manifold variable φ, expansion variable χ, auxiliary functions S, C), we refer to detailed lecture notes [10].

Near a noncharacteristic (i.e., $\partial_z\partial_t \ne 0$) movable singular manifold, as shown in our preliminary article [11], the amplitudes have the leading order,

$$\begin{cases} A_k \sim a_k\chi^{-1+{\rm i}b}, & \bar{A}_k \sim b_k\chi^{-1-{\rm i}b}, & k = 1, 4, \\ A_k \sim a_k\chi^{-1-{\rm i}b}, & \bar{A}_k \sim b_k\chi^{-1+{\rm i}b}, & k = 2, 3, \\ \mathcal{E} \sim q_0\chi^{-1+2{\rm i}b}, & I_{\rm m} \sim I_{{\rm m},0}\chi^{-2+2{\rm i}b}, & I_{\rm d} \sim I_{{\rm d},0}\chi^{-2}, \\ \bar{\mathcal{E}} \sim r_0\chi^{-1-2{\rm i}b}, & \overline{I_{\rm m}} \sim \overline{I_{{\rm m},0}}\chi^{-2-2{\rm i}b}, & \end{cases} \tag{26}$$

in which b is any one of the two real constants defined in terms of γ by

$$(2b^2 - 1)\cos g + 3b\sin g = 0, \qquad g = \arg\gamma. \tag{27}$$

The leading coefficients depend on the nonzero auxiliary function $C(z,t)$ and four arbitrary complex functions $\lambda, \mu, p_{12}, p_{43}$ of (z,t),

$$\begin{cases} a_1 = N\lambda p_{12}\cosh\mu, \quad b_2 = -N\lambda p_{12}^{-1}\cosh\mu, \\ a_4 = N\lambda p_{43}\sinh\mu, \quad b_3 = N\lambda p_{43}^{-1}\sinh\mu, \\ a_2 = N\lambda^{-1}p_{12}\cosh\mu, \quad b_1 = N\lambda^{-1}p_{12}^{-1}\cosh\mu, \\ a_3 = -N\lambda^{-1}p_{43}\sinh\mu, \quad b_4 = N\lambda^{-1}p_{43}^{-1}\sinh\mu, \\ q_0 = -\mathrm{i}(1-\mathrm{i}b)\lambda^2, \quad r_0 = -\mathrm{i}(1+\mathrm{i}b)\lambda^{-2}, \\ I_{\mathrm{m},0} = -N^2\lambda^2, \quad \overline{I_{\mathrm{m},0}} = N^2\lambda^{-2}, \quad I_{\mathrm{d},0} = -2N^2, \\ N^2 = \dfrac{C}{|\gamma|}((1-2b^2)\sin g + 3b\cos g), \quad C \neq 0. \end{cases} \tag{28}$$

The Fuchs indices of the linearized system only depend on the value of b; for the ten-dimensional system (1)–(3), these are [11]

$$j = -1, 0, 0, 0, 0, 2, 2, 2, \frac{5 \pm \sqrt{1-48b^2}}{2}. \tag{29}$$

For the intrinsic five-dimensional system (13), the indices are

$$j = -1, 0, \frac{5 \pm \sqrt{1-48b^2}}{2}, 4, \tag{30}$$

then the linear system (16) admits the Fuchs indices

$$j = 0, 2. \tag{31}$$

The diophantine condition that all Fuchs indices be integer therefore only admits the solution $b = 0$, $\mathrm{Re}(\gamma) = 0$ corresponding to a purely nonlocal response of the medium.

In order to compute the necessary conditions for the absence of movable logarithms arising from the integer Fuchs indices, one can handle equivalently either the ten-dimensional nonlinear system (1)–(3), or the five-dimensional nonlinear system (13) followed by the two-dimensional linear system (16). One must distinguish $b = 0$ from $b \neq 0$, and it is useless to test the quadruple index 0 (because the leading order already introduces four arbitrary functions) and the index 4 (because of the existence of the single-valued first integral $K(t)$, equation (14). In the generic situation $b \neq 0$ no movable logarithm is detected at the triple index 2. In the nongeneric situation $b = 0$, for instance with the five-dimensional system (13), two such necessary conditions $Q_j = 0$ are generated, at the Fuchs indices $j = 2$ and 3,

$$\mathrm{Re}(\gamma) = 0: \begin{cases} Q_2 \equiv \tau^{-1}(C_t + CC_z - (2/\tau)C) = 0, \quad C \neq 0, \\ Q_3 \equiv \tau^{-1}\left(-\Lambda_{tt} + (2/\tau)\Lambda_t - 2C\Lambda_{zt} - C^2\Lambda_{zz}\right) = 0, \quad \lambda = \mathrm{e}^{\mathrm{i}\Lambda}, \end{cases} \tag{32}$$

and no additional condition arises from the Fuchs index 2 of the linear system (16).

Remark 1. The analysis of the damped sine-Gordon equation (10) only generates the condition $Q_2 = 0$ [11], since the condition $Q_3 = 0$ which involves the phases of the complex amplitudes is then identically satisfied.

A first solution to the conditions (32) is $1/\tau = 0$, which identifies the unpumped complex Maxwell–Bloch system as the purely nonlocal response, undamped limit ($\mathrm{Re}(\gamma) = 0$, $1/\tau = 0$) of the four-wave mixing model.

Table 1. Possible single-valued solutions, according to time dependence, response (γ) and damping (τ). The reduced variable is $\xi = \sqrt{2z}\,\mathrm{e}^{-t/\tau}$.

∂_t	Re(γ)	$1/\tau$	Dependence	Solution	Section
$=0$			$f(z)$	8-param	3
$\neq 0$	$=0$	$=0$	$f(z,t)$	Maxwell–Bloch	5
$\neq 0$	$=0$	$\neq 0$	$f(\xi)$	10-param	6.1
$\neq 0$	$\neq 0$	0	$f(z,t)$		
$\neq 0$	$\neq 0$	$\neq 0$	$f(z,t)$		

The second solution $1/\tau \neq 0$ puts restrictions on the functions C and Λ. The condition on C, whose general solution is [11]

$$2z/\tau = C + F(\mathrm{e}^{-2t/\tau} C),\ F \text{ arbitrary function,} \tag{33}$$

restricts the expansion variable χ to only depend on the reduced variable $\xi = \sqrt{2z}\,\mathrm{e}^{-t/\tau}$ (the $\sqrt{2}$ is pure convenience) and therefore defines a reduction $(z,t) \to \xi$ of the PDE system to an ODE system written and studied in section 6. As to the restriction on Λ, which only makes sense for this ξ reduction, it will be further examined in section 6.

4.2. *Conclusion of the test*

The result of the test provides the guidelines to be followed in order to obtain explicitly single-valued solutions of the four-wave mixing model. These detailed guidelines, summarized in table 1, are the following.

- In the stationary case $\partial_t = 0$, the test (not performed here) succeeds, therefore an eight-parameter single-valued solution may exist. It has already been obtained in section 3.
- In the nonstationary, purely nonlocal response, undamped case ($\partial_t \neq 0, \mathrm{Re}(\gamma) = 0, 1/\tau = 0$), the system is equivalent to the unpumped complex Maxwell–Bloch system (18), integrable in the sense of the inverse spectral transform [1], i.e. it admits N-soliton solutions, see section 5.
- In the nonstationary, purely nonlocal response, damped case, no single-valued solution exists unless the dependence on (z,t) is through the reduced variable $\xi = \sqrt{2z}\,\mathrm{e}^{-t/\tau}$. Then, a single-valued solution may exist which depends on ten arbitrary parameters, we obtain it explicitly in section 6.1.
- In the nonstationary, arbitrary response case, whether damped or undamped, which includes the generic situation of the four-wave mixing, the structure of singularities is quite similar to that of the cubic complex Ginzburg–Landau equation (12) (total differential order four, two irrational Fuchs indices, no movable logarithm [9]). Single-valued solutions are locally represented by two Laurent series depending on eight (instead of ten as in the two previous cases) arbitrary functions, and the question of finding closed form solutions *a priori* presents the same difficulty as for the CGL3 equation.

5. The unpumped Maxwell–Bloch system limit

Since the pumped complex Maxwell–Bloch system (18) admits the Lax pair [19]

$$\partial_X \Psi = L\Psi, \qquad \partial_T \Psi = M\Psi,$$

$$L = \frac{1}{2}\begin{pmatrix} 0 & e \\ -\bar{e} & 0 \end{pmatrix} + f\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad M = \frac{1}{4f}\begin{pmatrix} N & -\rho \\ -\bar{\rho} & -N \end{pmatrix}, \quad f^2 = 2sT + \lambda^2, \tag{34}$$

in which λ is an arbitrary complex constant (the spectral parameter), the undamped four-wave mixing model with a purely nonlocal response then admits N-soliton solutions, etc, as well as solutions in terms of the third Painlevé function [12, 20, 25].

6. The dynamical case, reduction $\xi = (2z)^{1/2}\,\mathrm{e}^{-t/\tau}$

The reduction $(z,t) \to \xi = (2z)^{1/2}\,\mathrm{e}^{-t/\tau}$ (with an arbitrary origin for z and t) isolated by the Painlevé test also exists for any value of γ and we define it so as to preserve the definitions (3) and (15),

$$\frac{1}{\tau} \neq 0, \quad \gamma \text{ arbitrary: } \begin{cases} I_{\mathrm{m}}(z,t) = \mathrm{e}^{-2t/\tau-\mathrm{i}\omega t} I_{\mathrm{m,r}}(\xi), \quad I_{\mathrm{d}}(z,t) = \mathrm{e}^{-2t/\tau} I_{\mathrm{d,r}}(\xi), \\ \mathcal{E}(z,t) = (1/2)\,\mathrm{e}^{-t/\tau-\mathrm{i}\omega t}(2z)^{-1/2}\mathcal{E}_{\mathrm{r}}(\xi), \\ A_j(z,t) = \mathrm{e}^{-t/\tau-\mathrm{i}\omega t/2} A_{j,\mathrm{r}}(\xi), \quad j=1,4, \\ A_j(z,t) = \mathrm{e}^{-t/\tau+\mathrm{i}\omega t/2} A_{j,\mathrm{r}}(\xi), \quad j=2,3. \end{cases} \tag{35}$$

It introduces one arbitrary real parameter ω.

The intrinsic system (13)–(14) for I_{m}, I_{d}, $\mathcal{E}$ and the linear system for the amplitudes A_j reduce to

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}\xi} I_{\mathrm{m,r}} = -\mathrm{i}\mathcal{E}_{\mathrm{r}} I_{\mathrm{d,r}}, \quad \dfrac{\mathrm{d}}{\mathrm{d}\xi} I_{\mathrm{d,r}} = 2i(\mathcal{E}_{\mathrm{r}}\overline{I_{\mathrm{m,r}}} - \bar{\mathcal{E}}_{\mathrm{r}} I_{\mathrm{m,r}}), \quad \dfrac{\mathrm{d}}{\mathrm{d}\xi}\mathcal{E}_{\mathrm{r}} = -\gamma\tau I_{\mathrm{m,r}} - \dfrac{\mathrm{i}\omega\tau}{\xi}\mathcal{E}_{\mathrm{r}}, \\ \dfrac{\mathrm{d}}{\mathrm{d}\xi} A_{1,\mathrm{r}} = -\mathrm{i}\mathcal{E}_{\mathrm{r}} A_{2,\mathrm{r}}, \quad \dfrac{\mathrm{d}}{\mathrm{d}\xi}\bar{A}_{2,\mathrm{r}} = \mathrm{i}\mathcal{E}_{\mathrm{r}}\bar{A}_{1,\mathrm{r}}, \quad \dfrac{\mathrm{d}}{\mathrm{d}\xi}\bar{A}_{3,\mathrm{r}} = -\mathrm{i}\mathcal{E}_{\mathrm{r}}\bar{A}_{4,\mathrm{r}}, \quad \dfrac{\mathrm{d}}{\mathrm{d}\xi} A_{4,\mathrm{r}} = \mathrm{i}\mathcal{E}_{\mathrm{r}} A_{3,\mathrm{r}}, \\ K_0 = \mathrm{e}^{4t/\tau} K(t) = I_{\mathrm{d,r}}^2 + 4|I_{\mathrm{m,r}}|^2. \end{cases} \tag{36}$$

When compared to the traveling wave reduction $(z,t) \to \zeta = z - ct, c \neq 0$,

$$\begin{cases} I_{\mathrm{m}}(z,t) = \mathrm{e}^{-\mathrm{i}\omega t} I_{\mathrm{m,r}}(\zeta), \quad I_{\mathrm{d}}(z,t) = I_{\mathrm{d,r}}(\zeta), \quad \mathcal{E}(z,t) = \mathrm{e}^{-\mathrm{i}\omega t}\mathcal{E}_{\mathrm{r}}(\zeta), \\ A_j(z,t) = \mathrm{e}^{-\mathrm{i}\omega t/2} A_{j,\mathrm{r}}(\zeta), \quad j=1,4, \\ A_j(z,t) = \mathrm{e}^{+\mathrm{i}\omega t/2} A_{j,\mathrm{r}}(\zeta), \quad j=2,3. \\ \dfrac{\mathrm{d}}{\mathrm{d}\zeta} I_{\mathrm{m,r}} = -\mathrm{i}\mathcal{E}_{\mathrm{r}} I_{\mathrm{d,r}}, \quad \dfrac{\mathrm{d}}{\mathrm{d}\zeta} I_{\mathrm{d,r}} = 2\mathrm{i}(\mathcal{E}_{\mathrm{r}}\overline{I_{\mathrm{m,r}}} - \bar{\mathcal{E}}_{\mathrm{r}} I_{\mathrm{m,r}}), \quad \dfrac{\mathrm{d}}{\mathrm{d}\zeta}\mathcal{E}_{\mathrm{r}} = -\dfrac{\gamma}{c} I_{\mathrm{m,r}} - \left(\mathrm{i}\omega - \dfrac{1}{\tau}\right)\dfrac{\mathcal{E}_{\mathrm{r}}}{c}, \\ \dfrac{\mathrm{d}}{\mathrm{d}\zeta} A_{1,\mathrm{r}} = -\mathrm{i}\mathcal{E}_{\mathrm{r}} A_{2,\mathrm{r}}, \quad \dfrac{\mathrm{d}}{\mathrm{d}\zeta}\bar{A}_{2,\mathrm{r}} = \mathrm{i}\mathcal{E}_{\mathrm{r}}\bar{A}_{1,\mathrm{r}}, \quad \dfrac{\mathrm{d}}{\mathrm{d}\zeta}\bar{A}_{3,\mathrm{r}} = -\mathrm{i}\mathcal{E}_{\mathrm{r}}\bar{A}_{4,\mathrm{r}}, \quad \dfrac{\mathrm{d}}{\mathrm{d}\zeta} A_{4,\mathrm{r}} = \mathrm{i}\mathcal{E}_{\mathrm{r}} A_{3,\mathrm{r}}. \\ K_0 = K(t) = I_{\mathrm{d,r}}^2 + 4|I_{\mathrm{m,r}}|^2, \end{cases} \tag{37}$$

the two reduced systems (36) and (37) only differ by the evolution of the grating $\mathcal{E}_{\mathrm{r}}$.

6.1. Dynamical case, purely nonlocal response: general solution

A direct computation of the conditions (32) for both reduced ODE systems (36) and (37) yields

$$\mathrm{Re}(\gamma) = 0: \begin{cases} \text{reduction } (2z)^{1/2}\,\mathrm{e}^{-t/\tau}: Q_2 \equiv 0, \quad Q_3 \equiv \omega\tau\xi^{-3}, \\ \text{reduction } z - ct, \quad c \neq 0: Q_2 \equiv \dfrac{1}{c\tau^3}, \quad Q_3 \equiv \omega\left(\tau\xi^{-3} - \dfrac{1}{2}\xi^{-2} - \dfrac{1}{\tau}\xi^{-1}\right), \end{cases} \tag{38}$$

and the enforcement of $Q_j = 0$ makes both systems identical. Let us integrate the system (36) with $\mathrm{Re}(\gamma) = 0, \omega = 0$.

Thanks to the identity of the two systems (36) and (37) when the conditions $Q_j = 0$ are enforced, the first integrals of the system (36) for $(I_{\mathrm{m,r}}, I_{\mathrm{d,r}}, \mathcal{E}_{\mathrm{r}})$ can be generated systematically from the reduction $X - cT$ of the Lax pair (34) of the unpumped Maxwell–Bloch; this provides three first integrals, all real,

$$\begin{cases} K_0' = (\gamma_{\mathrm{NL}}\tau)^2\big(I_{\mathrm{d,r}}^2 + 4|I_{\mathrm{m,r}}|^2\big), \quad \gamma = \mathrm{i}\gamma_{\mathrm{NL}}, \quad \gamma_{\mathrm{NL}} \text{ real}, \\ K_1 = \gamma_{\mathrm{NL}}\tau(\overline{I_{\mathrm{m,r}}}\mathcal{E}_{\mathrm{r}} + I_{\mathrm{m,r}}\bar{\mathcal{E}}_{\mathrm{r}}), \\ 3e_0 = \frac{1}{2}\gamma_{\mathrm{NL}}\tau I_{\mathrm{d,r}} - |\mathcal{E}_{\mathrm{r}}|^2. \end{cases} \tag{39}$$

Therefore $I_{\mathrm{d,r}}$ obeys a first-order ODE[4] obtained by the elimination of $\mathcal{E}_{\mathrm{r}}$ and $I_{\mathrm{m,r}}$,

$$I_{\mathrm{d,r}}'^{\,2} + 2\gamma_{\mathrm{NL}}\tau I_{\mathrm{d,r}}^3 - 12e_0 I_{\mathrm{d,r}}^2 - 2(\gamma_{\mathrm{NL}}\tau)^{-1}K_0' I_{\mathrm{d,r}} + 4\big(K_1^2 + 3e_0K_0'\big)(\gamma_{\mathrm{NL}}\tau)^{-2} = 0. \tag{40}$$

The general solution $(I_{\mathrm{m,r}}, I_{\mathrm{d,r}}, \mathcal{E}_{\mathrm{r}})$ of (36.1) is singlevalued and expressible with the classical functions $\wp, \zeta, \sigma$ of Weierstrass,

$$\wp'^{\,2} = 4\wp^3 - g_2\wp - g_3 = 4(\wp - e_1)(\wp - e_2)(\wp - e_3), \quad \wp = -\zeta', \quad \zeta = (\log\sigma)'. \tag{41}$$

With the correspondence

$$K_0' = g_2 - 12e_0^2, \qquad K_1^2 = -\wp'(a)^2, \qquad -2e_0 = \wp(a), \tag{42}$$

the squared moduli and the gradient of their phases are doubly periodic functions,

$$\begin{cases} |I_{\mathrm{m,r}}|^2 = \dfrac{\wp'^2(\xi) - \wp'^2(a)}{4(\gamma_{\mathrm{NL}}\tau)^2(-\wp(\xi) + \wp(a))}, \quad I_{\mathrm{d,r}} = \dfrac{-2\wp(\xi) - \wp(a)}{\gamma_{\mathrm{NL}}\tau}, \quad |\mathcal{E}_{\mathrm{r}}|^2 = -\wp(\xi) + \wp(a), \\ (\arg I_{\mathrm{m,r}})' = -2K_1\dfrac{(-2\wp(\xi) - \wp(a))(-\wp(\xi) + \wp(a))}{\wp'^2(\xi) - \wp'^2(a)}, \quad (\arg\mathcal{E}_{\mathrm{r}})' = -\dfrac{K_1}{2(-\wp(\xi) + \wp(a))}, \\ \mathrm{e}^{\mathrm{i}(\arg I_{\mathrm{m,r}} - \arg\mathcal{E}_{\mathrm{r}})} = \dfrac{K_1 - \mathrm{i}\wp'(\xi)}{2\gamma_{\mathrm{NL}}\tau|\mathcal{E}_{\mathrm{r}}I_{\mathrm{m,r}}|}, \end{cases} \tag{43}$$

the five constants of integration being e_0, g_2, g_3 (actions), the origin of ξ and the common origin of the phase of $I_{\mathrm{m,r}}$ and $\mathcal{E}_{\mathrm{r}}$ (angles).

The complex amplitudes themselves $(I_{\mathrm{m,r}}, \mathcal{E}_{\mathrm{r}}, A_j)$ are also single-valued functions and their expression, analogous to the complex amplitude of the traveling wave of the nonlinear Schrödinger equation, involves the σ function of Weierstrass and is given in the appendix.

An important particular case occurs for $\wp'(a) = 0$, all amplitudes then have constant phases. It proves convenient to first write this solution in complex form, in the symmetric notation of the Jacobi functions as introduced by Halphen [16],

$$h_\alpha(x) = \sqrt{\wp(x) - e_\alpha}, \qquad \alpha = 1, 2, 3, \lim_{x\to 0} x h_\alpha(x) = +1, \tag{44}$$

[4] When $\mathrm{Re}(\gamma) = 0$, $\omega \neq 0$, the ODE for $I_{\mathrm{d,r}}$ has second order and is studied in [6, equation (19.6)].

$$\begin{cases} \mathcal{E}_{\mathrm{r}} = -\mathrm{e}^{2\mathrm{i}\varphi_0}\mathrm{i}h_\alpha(\xi), \quad \bar{\mathcal{E}}_{\mathrm{r}} = -\mathrm{e}^{-2\mathrm{i}\varphi_0}\mathrm{i}h_\alpha(\xi), \\ I_{\mathrm{m,r}} = -\dfrac{1}{\gamma_{\mathrm{NL}}\tau}\,\mathrm{e}^{2\mathrm{i}\varphi_0}h_\beta(\xi)h_\gamma(\xi), \quad \overline{I_{\mathrm{m,r}}} = \dfrac{1}{\gamma_{\mathrm{NL}}\tau}\,\mathrm{e}^{-2\mathrm{i}\varphi_0}h_\beta(\xi)h_\gamma(\xi), \\ I_{\mathrm{d,r}} = \dfrac{1}{\gamma_{\mathrm{NL}}\tau}\left(-2h_\alpha^2(\xi) - 3e_\alpha\right), \quad e_0 = -\dfrac{e_\alpha}{2}, \\ A_{1,\mathrm{r}} = i_0\,\mathrm{e}^{+\mathrm{i}\varphi_0}(a_{12}h_\beta(\xi) + b_{12}h_\gamma(\xi)), \quad A_{2,\mathrm{r}} = i_0\,\mathrm{e}^{-\mathrm{i}\varphi_0}(a_{12}h_\gamma(\xi) + b_{12}h_\beta(\xi)), \\ \bar{A}_{3,\mathrm{r}} = i_0\,\mathrm{e}^{+\mathrm{i}\varphi_0}(a_{34}h_\beta(\xi) + b_{34}h_\gamma(\xi)), \quad \bar{A}_{4,\mathrm{r}} = i_0\,\mathrm{e}^{-\mathrm{i}\varphi_0}(a_{34}h_\gamma(\xi) + b_{34}h_\beta(\xi)), \\ \bar{A}_{1,\mathrm{r}} = i_0\,\mathrm{e}^{-\mathrm{i}\varphi_0}(A_{12}h_\beta(\xi) + B_{12}h_\gamma(\xi)), \quad \bar{A}_{2,\mathrm{r}} = i_0\,\mathrm{e}^{+\mathrm{i}\varphi_0}(-A_{12}h_\gamma(\xi) - B_{12}h_\beta(\xi)), \\ A_{3,\mathrm{r}} = i_0\,\mathrm{e}^{-\mathrm{i}\varphi_0}(A_{34}h_\beta(\xi) + B_{34}h_\gamma(\xi)), \quad A_{4,\mathrm{r}} = i_0\,\mathrm{e}^{+\mathrm{i}\varphi_0}(-A_{34}h_\gamma(\xi) - B_{34}h_\beta(\xi)), \\ i_0^2 = \dfrac{1}{\gamma_{\mathrm{NL}}\tau}, \end{cases} \tag{45}$$

with (α, β, γ) an arbitrary permutation of $(1, 2, 3)$ and the relations (A.8) for the eight constants in $A_{j,\mathrm{r}}$.

In terms of the real Jacobi functions, the complex solution (45) defines four bounded, physically admissible solutions (i.e., with positive square moduli for the amplitudes), in which the grating amplitude $\mathcal{E}_{\mathrm{r}}$ is, respectively, a cn, dn, sd, nd function (with the usual notation $k'^2 = 1 - k^2$),

$$\begin{cases} h_1(\xi) = \mathrm{i}rk\,\mathrm{cn}(r\xi, k), \quad h_2(\xi) = rk\,\mathrm{sn}(r\xi, k), \quad h_3(\xi) = \mathrm{i}r\,\mathrm{dn}(r\xi, k), \\ (\alpha, \beta, \gamma) = (1, 2, 3) : K_0' = r^4, \quad 6e_0 = r^2(1 - 2k^2), \\ (\alpha, \beta, \gamma) = (3, 2, 1) : K_0' = r^4k^4, \quad 6e_0 = r^2(k^2 - 2), \end{cases} \tag{46}$$

$$\begin{cases} h_1(\xi) = \mathrm{i}rkk'\,\mathrm{sd}(r\xi, k), \quad h_2(\xi) = rk\,\mathrm{cd}(r\xi, k), \quad h_3(\xi) = -\mathrm{i}rk'\,\mathrm{nd}(r\xi, k), \\ (\alpha, \beta, \gamma) = (1, 2, 3) : K_0' = r^4, \quad 6e_0 = r^2(1 - 2k^2), \\ (\alpha, \beta, \gamma) = (3, 2, 1) : K_0' = r^4k^4, \quad 6e_0 = r^2(k^2 - 2). \end{cases} \tag{47}$$

In these nine-parameter solutions, r, k are real, and $\lambda_{12}, \lambda_{34}$ must be taken real in (A.8) to ensure that A_j and $\bar{A}_j$ are complex conjugate.

A second important case is the degeneracy from doubly periodic to simply periodic. The subcase $\wp'(a) \neq 0$, which would correspond to the dark solitary wave

$$\begin{cases} \mathcal{E}_{\mathrm{r}} = \mathrm{i}\,\mathrm{e}^{2\mathrm{i}\varphi_0}(k\tanh(k\xi) - \mathrm{i}\kappa)\,\mathrm{e}^{\mathrm{i}\kappa\xi}, \quad \bar{\mathcal{E}}_{\mathrm{r}} = \mathrm{i}\,\mathrm{e}^{-2\mathrm{i}\varphi_0}(k\tanh(k\xi) + \mathrm{i}\kappa)\,\mathrm{e}^{-\mathrm{i}\kappa\xi}, \\ I_{\mathrm{m,r}} = -\dfrac{1}{\gamma_{\mathrm{NL}}\tau}\,\mathrm{e}^{2\mathrm{i}\varphi_0}(k^2 + \kappa^2 + \mathrm{i}k\kappa\tanh(k\xi) - k^2\tanh^2(k\xi))\,\mathrm{e}^{\mathrm{i}\kappa\xi}, \\ \overline{I_{\mathrm{m,r}}} = \dfrac{1}{\gamma_{\mathrm{NL}}\tau}\,\mathrm{e}^{-2\mathrm{i}\varphi_0}(k^2 + \kappa^2 - \mathrm{i}k\kappa\tanh(k\xi) - k^2\tanh^2(k\xi))\,\mathrm{e}^{-\mathrm{i}\kappa\xi}, \\ I_{\mathrm{d,r}} = \dfrac{1}{\gamma_{\mathrm{NL}}\tau}(-2k^2\tanh^2(k\xi) + 2k^2 + \kappa^2), \end{cases} \tag{48}$$

is unphysical since the square modulus $\mathcal{E}_{\mathrm{r}}\bar{\mathcal{E}}_{\mathrm{r}}$ is negative. As to the subcase $\wp'(a) = 0$, it defines the bright solitary wave obtained from the long wave limit $k^2 = 1$ in (46),

$$h_1(\xi) = h_3(\xi) = \mathrm{i}r\,\mathrm{sech}(r\xi), \qquad h_2(\xi) = r\tanh(r\xi), \quad K_0' = r^4, \quad K_1 = 0, \quad 6e_0 = -r^2, \tag{49}$$

with r, ξ_0, φ_0 being arbitrary. In this eight-parameter solution, r is real, and $\lambda_{12}, \lambda_{34}, \mu$ must be taken real to enforce the complex conjugation between A_j and $\bar{A}_j$.

Remark 2. Despite the similarity with the stationary value (23) for this bright profile of the grating amplitude, there is no limiting process yielding (23) from (49).

Remark 3. For those solutions displaying constant phases for the amplitudes, there must exist a value of the damped sine-Gordon variable u, equation (10), able to represent the solution. Up to the numerous additive and multiplicative constants in (10) and (45), this value is essentially given by

$$e^{iu} = h_A(\xi), \tag{50}$$

in which h_A and h_α are related by the Landen transformation [2, section 16.14.2]

$$\frac{d}{d\xi} \log h_A(\xi) = -\frac{h_B(\xi)h_C(\xi)}{h_A(\xi)} = h_\alpha(\xi), \tag{51}$$

the correspondence between the elliptic invariants $(e_\alpha, e_\beta, e_\gamma)$ and (e_A, e_B, e_C) being detailed in [2, section 16.14.1]. For the trigonometric degeneracy (49), the value is

$$e^{iu} = r \tanh r\xi, \tag{52}$$

and the Landen transformation reduces to the doubling of the argument with some shift,

$$\forall\, x : \tanh x - \frac{1}{\tanh x} = -2i\,\mathrm{sech}\left[2x + i\frac{\pi}{2}\right], \qquad \tanh x + \frac{1}{\tanh x} = 2\tanh\left[2x + i\frac{\pi}{2}\right]. \tag{53}$$

7. Conclusion

The four-wave mixing has been characterized by a lower-dimensional system of a deformed Maxwell–Bloch type. Then the three and only three possibly single-valued limits of the four-wave mixing model have been determined and integrated. These consist of (i) the stationary case for any τ and γ; (ii) the limiting case $1/\tau = 0$, $\mathrm{Re}(\gamma) = 0$ which is identified to the complex unpumped Maxwell–Bloch system; (iii) when $\mathrm{Re}(\gamma) = 0$, the reduction $\xi = \sqrt{2z}e^{-t/\tau}$ to an ODE system. Those solutions which are localized (typically Jacobi bounded functions sn, cn, dn, cd, nd, sd [2, section 16.2]) should improve both the design of the physical devices to be manufactured and the confidence in the numerical simulations. As is often the case with methods based on singularities, the present study cannot rule out possible closed form but multivalued solutions.

Moreover, the generic case $1/\tau \neq 0$, $\mathrm{Re}(\gamma) \neq 0$ has been shown to display a structure of singularities, i.e. of possible closed form solutions, quite similar to that of the cubic complex Ginzburg–Landau equation. These solutions will be investigated in a forthcoming paper.

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Appendix. Complex amplitudes of the integrable ξ reduction

By elimination from (36), both fields $\mathcal{E}_r$ and $\bar{\mathcal{E}}_r$ obey the same equation,

$$\left(\frac{d^2}{d\xi^2} - 2(\wp(\xi) - e_0)\right)\psi = 0, \qquad \psi = \mathcal{E}_r, \bar{\mathcal{E}}_r. \tag{A.1}$$

According to a classical result of Floquet, any linear differential equation with doubly periodic coefficients admits at least one solution which is doubly periodic of the second kind [16]. The elementary unit of such doubly periodic functions of the second kind has been introduced by Hermite under the name *élément simple* $\mathrm{H}(\xi, q, k)$ [16, vol. II, p. 506],

$$\mathrm{H}(\xi, q, k) = \frac{\sigma(\xi + q)}{\sigma(\xi)\sigma(q)} \mathrm{e}^{(k-\zeta(q))\xi}, \tag{A.2}$$

chosen to have as only singularity a simple pole with residue 1 at the origin. Lamé indeed proved that equation (A.1) admits the two solutions $\mathrm{H}(\xi, -a, 0)$ and $\mathrm{H}(\xi, +a, 0)$, which are generically linearly independent. Hence the complex amplitudes

$$\begin{cases} \mathcal{E}_{\mathrm{r}} = -\mathrm{i}\,\mathrm{e}^{2\mathrm{i}\varphi_0}\mathrm{H}(\xi, -a, 0), & I_{\mathrm{m,r}} = -\dfrac{\mathrm{i}}{2\gamma_{\mathrm{NL}}\tau}\, \mathrm{e}^{2\mathrm{i}\varphi_0} \dfrac{\wp'(\xi) - \wp'(a)}{\wp(\xi) - \wp(a)} \mathrm{H}(\xi, -a, 0), \\ \bar{\mathcal{E}}_{\mathrm{r}} = -\mathrm{i}\mathrm{e}^{-2\mathrm{i}\varphi_0}\mathrm{H}(\xi, a, 0), & \overline{I_{\mathrm{m,r}}} = -\dfrac{\mathrm{i}}{2\gamma_{\mathrm{NL}}\tau}\, \mathrm{e}^{-2\mathrm{i}\varphi_0} \dfrac{\wp'(\xi) + \wp'(a)}{\wp(\xi) - \wp(a)} \mathrm{H}(\xi, a, 0), \end{cases} \tag{A.3}$$

in which the five constants of integration are e_0, g_2, g_3, φ_0 and the origin of ξ.

Given the values (A.3) of $\mathcal{E}_{\mathrm{r}}(\xi), \bar{\mathcal{E}}_{\mathrm{r}}(\xi)$, each variable X, Y of the linear system (16) also obeys a second-order linear differential equation with doubly periodic coefficients, e.g.,

$$\left(\frac{\mathrm{d}^2}{\mathrm{d}\xi^2} - \frac{1}{2}\frac{\wp'(\xi) - \wp'(a)}{\wp(\xi) - \wp(a)}\frac{\mathrm{d}}{\mathrm{d}\xi} - (\wp(\xi) - \wp(a))\right) X = 0, \qquad X = A_{1,\mathrm{r}}. \tag{A.4}$$

This equation has the same features as (A.1): unique singularity $\xi = 0$ of the Fuchsian type, Fuchs indices equal to -1, 1, absence of logarithms in the general solution. A direct search for solutions of the elementary type (A.2) provides the two solutions, generically linearly independent,

$$X = \mathrm{H}(\xi, +a/2 \pm h, 0), \qquad Y = \mathrm{H}(\xi, -a/2 \pm h, 0), \qquad \wp(h) = \wp(a/2) - 2\frac{\wp'^2(a/2)}{\wp''(a/2)}. \tag{A.5}$$

Taking account of the first integrals

$$A_{1,\mathrm{r}}\bar{A}_{1,\mathrm{r}} + A_{2,\mathrm{r}}\bar{A}_{2,\mathrm{r}} = \text{constant}, \quad A_{3,\mathrm{r}}\bar{A}_{3,\mathrm{r}} + A_{4,\mathrm{r}}\bar{A}_{4,\mathrm{r}} = \text{constant}, \tag{A.6}$$

the general solution for the complex amplitudes can be parametrized as

$$\begin{cases} A_{1,\mathrm{r}} = i_0\,(a_{12}\mathrm{H}(\xi, +a/2 + h, 0) + b_{12}\mathrm{H}(\xi, +a/2 - h, 0))\,\mathrm{e}^{+\mathrm{i}\varphi_0}, \\ A_{2,\mathrm{r}} = i_0\,(a_{12}\mathrm{H}(\xi, -a/2 + h, 0) + b_{12}\mathrm{H}(\xi, -a/2 - h, 0))\,\mathrm{e}^{-\mathrm{i}\varphi_0}, \\ \bar{A}_{3,\mathrm{r}} = i_0\,(a_{43}\mathrm{H}(\xi, +a/2 + h, 0) + b_{43}\mathrm{H}(\xi, +a/2 - h, 0))\,\mathrm{e}^{+\mathrm{i}\varphi_0}, \\ \bar{A}_{4,\mathrm{r}} = i_0\,(a_{43}\mathrm{H}(\xi, -a/2 + h, 0) + b_{43}\mathrm{H}(\xi, -a/2 - h, 0))\,\mathrm{e}^{-\mathrm{i}\varphi_0}, \\ \bar{A}_{1,\mathrm{r}} = i_0\,(A_{12}\mathrm{H}(\xi, -a/2 - h, 0) + B_{12}\mathrm{H}(\xi, -a/2 + h, 0))\,\mathrm{e}^{-\mathrm{i}\varphi_0}, \\ \bar{A}_{2,\mathrm{r}} = i_0\,(-A_{12}\mathrm{H}(\xi, +a/2 - h, 0) - B_{12}\mathrm{H}(\xi, +a/2 + h, 0))\,\mathrm{e}^{+\mathrm{i}\varphi_0}, \\ A_{3,\mathrm{r}} = i_0\,(A_{34}\mathrm{H}(\xi, -a/2 - h, 0) + B_{34}\mathrm{H}(\xi, -a/2 + h, 0))\,\mathrm{e}^{-\mathrm{i}\varphi_0}, \\ A_{4,\mathrm{r}} = i_0\,(-A_{43}\mathrm{H}(\xi, +a/2 - h, 0) - B_{43}\mathrm{H}(\xi, +a/2 + h, 0))\,\mathrm{e}^{+\mathrm{i}\varphi_0}, \\ i_0^2 = \dfrac{1}{\gamma_{\mathrm{NL}}\tau}, \end{cases} \tag{A.7}$$

with

$$\begin{cases} a_{12} = \cos\mu\cosh\lambda_{12}\,\mathrm{e}^{+\mathrm{i}\alpha_{12}}, & A_{12} = \cos\mu\cosh\lambda_{12}\,\mathrm{e}^{-\mathrm{i}\alpha_{12}}, \\ b_{12} = \mathrm{i}\cos\mu\sinh\lambda_{12}\,\mathrm{e}^{+\mathrm{i}\beta_{12}}, & B_{12} = \mathrm{i}\cos\mu\sinh\lambda_{12}\,\mathrm{e}^{-\mathrm{i}\beta_{12}}, \\ a_{34} = \sin\mu\cosh\lambda_{34}\,\mathrm{e}^{+\mathrm{i}\alpha_{34}}, & A_{34} = \sin\mu\cosh\lambda_{34}\,\mathrm{e}^{-\mathrm{i}\alpha_{34}}, \\ b_{34} = \mathrm{i}\sin\mu\sinh\lambda_{34}\,\mathrm{e}^{+\mathrm{i}\beta_{34}}, & B_{34} = \mathrm{i}\sin\mu\sinh\lambda_{34}\,\mathrm{e}^{-\mathrm{i}\beta_{34}}, \\ \mathrm{e}^{\mathrm{i}(\alpha_{12}-\beta_{12}-\alpha_{34}+\beta_{34})} = \pm 1, & \tan^2\mu = \pm\dfrac{\sinh(2\lambda_{12})}{\sinh(2\lambda_{34})}. \end{cases} \tag{A.8}$$

The five additional integration constants are three of the four constant real phases $\alpha_{12}, \beta_{12}, \alpha_{34}, \beta_{34}$, plus the two complex constants $\lambda_{12}, \lambda_{34}$. Finally, the conditions that $\bar{A}_{j,\mathrm{r}}$ be the complex conjugate of $A_{j,\mathrm{r}}$ puts on $\lambda_{12}, \lambda_{34}$ some constraints which depend on the choice for $\mathrm{H}(\xi, \pm a/2 \pm h, 0)$, see the text.

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