Small Mass Ratio Limit of Boltzmann Equations in the Context of the Study of Evolution of Dust Particles in a Rarefied Atmosphere

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Abstract

We propose a model based on the coupling of two Boltzmann-like equations for the study of the evolution of dust particles in a rarefied atmosphere, such as it can be found in the context of safety studies for the ITER project of nuclear fusion.

When the typical size of a dust speck becomes too large, the numerical simulation of the system under study becomes too expensive and one needs to introduce an asymptotic model in which the mass ratio between molecules and dust speck tends to 0. This model is constituted of a coupling (by a drag force term) between a Boltzmann equation and a Vlasov equation.

A rigorous proof of the passage to the limit is given in the spatially homogeneous setting. It includes a new variant of Povzner's inequality in which the vanishing mass ratio is taken into account.

Key words: Boltzmann equation, Vlasov equation, Povzner's inequality

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1 Introduction

In the case of a loss of vacuum accident (LOVA) in the future nuclear fusion reactor ITER, the particles of dust produced by the abrasion of the wall by the plasma might be dispersed in the reactor and one needs to study their evolution.

This study can be performed by the use of macroscopic models (of Euler or Navier-Stokes type), Cf. [T]. However, those models are known to be inaccurate in a rarefied context, which occurs at the very beginning of the LOVA (later on, the pressure rapidly increases and the macroscopic models recover their validity).

Our proposition of modeling for the beginning of the LOVA consists in writing a kinetic-like system for the density of molecules and dust specks. The

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model that we present can be compared to related models used for example in the study of cometary flows (Cf. [F]).

The unknowns are the density $f_2 := f_2(t,x,v) \geq 0$ of molecules (of radius r_2 and mass m_2) which at time t and point x move with velocity v, and the density $f_1 := f_1(t,x,v,r) \geq 0$ of specks of dust (assumed to be spherical for the sake of simplicity) which at time t, point x, have velocity v and radius r. Here $t \in \mathbb{R}_+$, $x \in \Omega$ an open bounded and regular subset of $\mathbb{R}^3$, $v \in \mathbb{R}^3$ and $r \in [r_{min}, r_{max}]$ with $0 < r_{min} < r_{max}$. The equations write

$$\frac{\partial f_1}{\partial t} + v \cdot \nabla_x f_1 = R_1(f_1, f_2), \tag{1.1}$$

$$\frac{\partial f_2}{\partial t} + v \cdot \nabla_x f_2 = R_2(f_1, f_2) + Q(f_2, f_2), \tag{1.2}$$

where R_1, R_2, Q are collision kernels defined by

$$\begin{aligned} R_1(f_1,f_2)(v_1,r) \quad &= \quad \int_{\mathbb{R}^3}\int_{\mathbb{S}^2} [f_1(v_1',r)f_2(v_2') - f_1(v_1,r)f_2(v_2)] \\ &\qquad \times (r_2+r)^2\,|\omega\cdot(v_2-v_1)|\,d\omega dv_2, \end{aligned}$$

$$\begin{aligned} R_2(f_1,f_2)(v_2) \quad &= \quad \int_{\mathbb{R}^3}\int_{\mathbb{S}^2}\int_{r_{min}}^{r_{max}} [f_1(v_1',r)f_2(v_2') - f_1(v_1,r)f_2(v_2)] \\ &\qquad \times (r_2+r)^2\,|\omega\cdot(v_2-v_1)|\,drd\omega dv_1, \end{aligned}$$

with

$$\left\{ \begin{aligned} v_1' \quad &= \quad v_1 + \frac{2\varepsilon(r)}{1+\varepsilon(r)}[\omega\cdot(v_2-v_1)]\omega, \\ v_2' \quad &= \quad v_2 - \frac{2}{1+\varepsilon(r)}[\omega\cdot(v_2-v_1)]\omega, \end{aligned} \right. \tag{1.3}$$

and

$$\begin{aligned} Q(f_2,f_2)(v) \quad &= \quad \int_{\mathbb{R}^N}\int_{\mathbb{S}^{N-1}} [f_2(v')f_2(v_*') - f_2(v)f_2(v_*)] \\ &\qquad \times B\left(|v-v_*|, \frac{v-v_*}{|v-v_*|}\cdot\sigma\right) d\sigma dv_*, \end{aligned} \tag{1.4}$$

with

$$\left\{ \begin{aligned} v' \quad &= \quad \frac{v+v_*}{2} + \frac{|v-v_*|}{2}\sigma, \\ v_*' \quad &= \quad \frac{v+v_*}{2} - \frac{|v-v_*|}{2}\sigma. \end{aligned} \right. \tag{1.5}$$

In relations (1.3), $\varepsilon(r)$ represents the ratio of mass between a molecule and a dust speck of radius r (that is, $\varepsilon(r) = (r_{min}/r)^3\,\varepsilon(r_{min})$). We have assumed that the collision kernels R_1, R_2 corresponding to the interaction between molecules and specks of dust are of hard sphere type. This assumption is however not typical for collisions between molecules, and we consider instead that the cross section B is of variable hard sphere (VHS) type:

$$B(y,z) = C_{eff}\, y^{\alpha}, \tag{1.6}$$

with $C_{eff} > 0$ and $\alpha \in [0,1]$. This cross section is widely used in DSMC methods. Note that the rest of our paper would still hold if C_{eff} were a (smooth) function of z (that is, in the case of smoothly cutoff hard potentials).

The modeling assumptions underlying eq. (1.1), (1.2) include the absence of collisions between the dust specks. This is related to the value of the typical collision time (Cf. [B]) $t_{1,1}$ between two particles of dust, which is, in our context, much larger than the other time scales. Note also that the collision kernels R_1, R_2 could be modeled differently, since collisions between molecules and particles of dust are not necessarily conservative (that is, some kinetic energy can be lost). For more details about this possibility of modeling, we refer to [C2] and [C1].

The mathematical study of spatially homogeneous solutions to eq. (1.1) – (1.2) can be done in the same spirit as in [Ar]. It leads to the following Proposition. Its Proof is briefly sketched in Section 2.

Proposition 1.1 *: Let $f_{1,in} := f_{1,in}(v,r) \geq 0$ be an initial datum such that*

$$\int_{\mathbb{R}^3} \int_{r_{min}}^{r_{max}} f_{1,in}(v,r)\,(1 + |v|^2 + |\log f_{1,in}(v,r)|)\,drdv < +\infty,$$

and $f_{2,in} := f_{2,in}(v) \geq 0$ be an initial datum such that

$$\int_{\mathbb{R}^3} f_{2,in}(v)\,(1 + |v|^2 + |\log f_{2,in}(v)|)\,dv < +\infty.$$

Then for all $C_{eff} > 0$, $\alpha \in]0,1[$, $0 < r_{min} < r_{max}$ (constants appearing in the definition of R_1, R_2, Q), there exists a spatially homogeneous weak solution $(f_1 : (t,r,v) \in \mathbb{R}_+ \times \mathbb{R}^3 \times [r_{min}, r_{max}] \mapsto f_1(t,v,r) \geq 0, f_2 : (t,v) \in \mathbb{R}_+ \times \mathbb{R}^3 \mapsto f_2(t,v) \geq 0)$ to eq. (1.1) – (1.2) such that for all $T > 0$,

$$\sup_{t\in[0,T]} \int_{\mathbb{R}^3} \int_{r_{min}}^{r_{max}} f_1(t,v,r)\,(1 + |v|^2 + |\log f_1(t,v,r)|)\,drdv < +\infty,$$

$$\sup_{t\in[0,T]} \int_{\mathbb{R}^3} f_2(t,v)\,(1 + |v|^2 + |\log f_2(t,v)|)\,dv < +\infty.$$

It satisfies moreover (for all $t \in \mathbb{R}_+$), the conservation of mass

$$\text{for a.e } r \in [r_{min}, r_{max}], \quad \int_{\mathbb{R}^3} f_1(t,v,r)\,dv = \int_{\mathbb{R}^3} f_{1,in}(v,r)\,dv, \tag{1.7}$$

$$\int_{\mathbb{R}^3} f_2(t,v)\,dv = \int_{\mathbb{R}^3} f_{2,in}(v)\,dv, \tag{1.8}$$

and the following entropy inequality

$$\int_{\mathbb{R}^3}\int_{r_{min}}^{r_{max}} f_1(t,v_1,r)\ln\left(f_1(t,v_1,r)\right) drdv_1 + \int_{\mathbb{R}^3} f_2(t,v_2)\ln\left(f_2(t,v_2)\right) dv_2$$
$$\le \int_{\mathbb{R}^3}\int_{r_{min}}^{r_{max}} f_{1,in}(v_1,r)\ln\left(f_{1,in}(v_1,r)\right) drdv_1 + \int_{\mathbb{R}^3} f_{2,in}(v_2)\ln\left(f_{2,in}(v_2)\right) dv_2. \tag{1.9}$$

Finally, if for some $s \ge 1$,

$$\int_{\mathbb{R}^3}\int_{r_{min}}^{r_{max}} (1+|v|^2)^s \left[f_{1,in}(v,r)+f_{2,in}(v)\right] drdv < +\infty,$$

then one can find f_1, f_2 *in such a way that (for all* $T>0$*)*

$$\sup_{t\in[0,T]}\int_{\mathbb{R}^3}\int_{r_{min}}^{r_{max}} (1+|v|^2)^s \left[f_1(t,v,r)+f_2(t,v)\right] drdv < +\infty, \tag{1.10}$$

and the following relation of conservation of energy holds:

$$\begin{aligned}&\int_{\mathbb{R}^3}\int_{r_{min}}^{r_{max}} f_1(t,v,r)\,|v|^2\left(\frac{r}{r_{min}}\right)^3 drdv + \varepsilon_m\int_{\mathbb{R}^3} f_2(t,v)\,|v|^2\,dv\\ &= \int_{\mathbb{R}^3}\int_{r_{min}}^{r_{max}} f_{1,in}(v,r)\,|v|^2\left(\frac{r}{r_{min}}\right)^3 drdv + \varepsilon_m\int_{\mathbb{R}^3} f_{2,in}(v)\,|v|^2\,dv,\end{aligned} \tag{1.11}$$

where $\varepsilon_m = \varepsilon(r_{min})$*. By a weak solution, we mean here that for all* $T>0$,

$$(f_1,f_2)\in \mathrm{Lip}\left([0,T],\mathrm{L}^1\left(\mathbb{R}^3\times[r_{min},r_{max}]\right)\right)\times \mathrm{Lip}\left([0,T],\mathrm{L}^1\left(\mathbb{R}^3\right)\right), \tag{1.12}$$

and (f_1,f_2) *satifies for all* $t\in[0,T]$*, and a.e.* $v\in\mathbb{R}^3$, $r\in[r_{min},r_{max}]$,

$$f_1(t,v,r) = f_{1,in}(v,r) + \int_0^t R_1(f_1,f_2)(s,v,r)\,ds, \tag{1.13}$$

$$f_2(t,v) = f_{2,in}(v) + \int_0^t \Big(Q(f_2,f_2)(s,v) + R_2(f_1,f_2)(s,v)\Big)\,ds. \tag{1.14}$$

The set of (spatially inhomogeneous) eq. (1.1), (1.2) can be simulated at the numerical level by a DSMC method (Cf. [B], [N] for example). We refer to [C1] for numerical results in the context of an experiment related to a LOVA.

However, when the mass ratio between the molecules and the specks of dust becomes too small, the simulation becomes too expensive. Indeed, because of the intrisically explicit character of the DSMC method, the time step of the simulation must be at most of the same order of magnitude as the lowest of the time scales defined by the different types of collision. Here, it corresponds to the typical collision time $t_{1,2}$ between molecules and particles of dust (from the point of view of particles) which is related to the collision time between two molecules $t_{2,2}$ by the formula

$$t_{1,2} \approx \sqrt{\varepsilon(r)}\,t_{2,2}. \tag{1.15}$$

In order to perform computations on a time scale of the order of $t_{2,2}$, it is therefore necessary when the dust specks are "too big" (in practice, for the applications that we have in mind, when their typical radius is bigger than 10^{-8} m) to write down a model in which the mass ratio $\varepsilon(r)$ vanishes.

In order to do so, we perform a dimensional analysis leading to a non-dimensional form of the equations, in which appears a parameter p which is related to the mass ratio and which tends to infinity when this ratio vanishes. These equations write (in a spatially homogeneous context)

$$\frac{\partial f_{1,p}}{\partial t} = p\, c\, R_1^{a,p}(f_{1,p}, f_{2,p}), \tag{1.16}$$

$$\frac{\partial f_{2,p}}{\partial t} = c\, R_2^{a,p}(f_{1,p}, f_{2,p}) + Q^a(f_{2,p}, f_{2,p}), \tag{1.17}$$

where Q^a, $R_1^{a,p}$, $R_2^{a,p}$ are defined by

$$\begin{aligned} Q^a\left(f_2, f_2\right)(v) \quad &= \quad \int_{\mathbb{R}^3} \int_{\mathbb{S}^2} [f_2(v') f_2(v'_*) - f_2(v) f_2(v_*)] \\ &\qquad \times C^a_{eff} |v - v_*|^\alpha d\sigma dv^*, \end{aligned}$$

where C^a_{eff} is an adimensional constant [and v', v'_* satisfy (1.5)],

$$\begin{aligned} R_1^{a,p}(f_1, f_2)(v_1, r) \quad &= \quad \int_{\mathbb{R}^3} \int_{\mathbb{S}^2} \left[f_1(v'_{1,p}, r) f_2(v'_{2,p}) - f_1(v_1, r) f_2(v_2) \right] \\ &\qquad \times \left(\frac{1}{2\sqrt{\pi\, p\, c}} + r \right)^2 \left| \left(v_2 - \frac{v_1}{\xi\, p} \right) \cdot \omega \right| d\omega dv_2, \end{aligned}$$

and

$$\begin{aligned} R_2^{a,p}(f_1, f_2)(v_2) \quad &= \quad \int_{\mathbb{R}^3} \int_{\mathbb{S}^2} \int_1^{r_0} \left[f_1(v'_{1,p}, r) f_2(v'_{2,p}) - f_1(v_1, r) f_2(v_2) \right] \\ &\qquad \times \left(\frac{1}{2\sqrt{\pi\, p\, c}} + r \right)^2 \left| \left(v_2 - \frac{v_1}{\xi p} \right) \cdot \omega \right| dr d\omega dv_1, \end{aligned}$$

where c, ξ are adimensional constants, $r_0 = \frac{r_{max}}{r_{min}}$, and

$$\left\{ \begin{aligned} v'_{1,p} \quad &= \quad v_1 + \frac{2\,\xi\, p r^{-3}}{(\xi\, p)^2 + r^{-3}} \left[\omega \cdot \left(v_2 - \frac{v_1}{\xi\, p} \right) \right] \omega, \\ v'_{2,p} \quad &= \quad v_2 - \frac{2\,(\xi\, p)^2}{(\xi\, p)^2 + r^{-3}} \left[\omega \cdot \left(v_2 - \frac{v_1}{\xi\, p} \right) \right] \omega. \end{aligned} \right. \tag{1.18}$$

This dimensional analysis is briefly explained in Section 2, and fully detailed in [C2].

We rigorously show in this paper that in the limit $p \to \infty$, the solutions to eq. (1.16) – (1.17) given by Proposition 1.1 converge towards the solution of the following Vlasov-Boltzmann coupling:

$$\frac{\partial f_1}{\partial t} + K(f_2) \cdot \nabla_v f_1 = 0, \tag{1.19}$$

$$\frac{\partial f_2}{\partial t} = m(f_{1,in}) L(f_2) + Q^a(f_2, f_2), \tag{1.20}$$

where

$$m(f_{1,in}) = c \int_{\mathbb{R}^3} \int_1^{r_0} f_{1,in}(v_1, r)\, r^2 dr dv_1,$$

$$L(f_2)(t,v) = \int_{\mathbb{S}^2} \left[f_2\left(t, v - 2(\omega \cdot v)\omega\right) - f_2(t,v) \right] |v \cdot \omega| \, d\omega,$$

and

$$K(f_2)(t,r) = \frac{2\,\pi\, c}{r\,\xi} \int_{\mathbb{R}^3} |v_2|\, v_2 f_2(t, v_2) dv_2. \tag{1.21}$$

More precisely, we shall prove the

Theorem 1.1 *Let $c > 0$, $\xi > 0$, $C^a_{eff} > 0$, $\alpha \in [0,1]$, $r_0 > 1$ be the parameters appearing in Q^a, $R_1^{a,p}$, $R_2^{a,p}$. Let also $f_{1,in} := f_{1,in}(r,v) \geq 0$, $f_{2,in} := f_{2,in}(v) \geq 0$ be initial data such that*

$$\int_{\mathbb{R}^3} \int_1^{r_0} f_{1,in}(v,r)\, (1 + |v|^4 + |\log f_{1,in}(v,r)|)\, dr dv < +\infty,$$

$$\int_{\mathbb{R}^3} f_{2,in}(v)\, (1 + |v|^4 + |\log f_{2,in}(v)|)\, dv < +\infty. \tag{1.22}$$

*Then, if $(f_{1,p}, f_{2,p})$ denotes a family (indexed by p) of weak solutions to eq. (1.16) – (1.17) given by Proposition 1.1 (with $f_{1,p}(0,\cdot) = f_{1,in}$, $f_{2,p}(0,\cdot) = f_{2,in}$), one can extract a subsequence (still denoted by $(f_{1,p}, f_{2,p})$) which converges for all $T > 0$ in $L^\infty([0,T]; M^1(\mathbb{R}^3 \times [1, r_0]) \times L^1(\mathbb{R}^3))$ weak * towards a weak solution $(f_1, f_2) \in L^\infty([0,T]; M^1(\mathbb{R}^3 \times [1,r_0]) \times L^1(\mathbb{R}^3))$ to eq. (1.19) – (1.21).*

By a weak solution, we here mean that for all $\psi \in \mathcal{C}^2_c\left(\mathbb{R}_+ \times \mathbb{R}^3 \times [1, r_0]\right)$, we have

$$\begin{aligned} -\int_0^\infty \int_{\mathbb{R}^3} \int_1^{r_0} f_1(t,v,r) \frac{\partial \psi}{\partial t}(t,v,r)\, dr dv dt &= \int_{\mathbb{R}^3} \int_1^{r_0} f_{1,in}(v,r) \psi(0,v,r)\, dr dv \\ &+ \int_0^\infty \int_{\mathbb{R}^3} \int_1^{r_0} K(f_2)(t,r) \cdot \nabla_v \psi(t,v,r) f_1(t, v_1, r)\, dr dv dt, \end{aligned} \tag{1.23}$$

and for all $\varphi \in \mathcal{C}^2_c\left(\mathbb{R}_+ \times \mathbb{R}^3\right)$, we have

$$\begin{aligned} &-\int_0^\infty \int_{\mathbb{R}^3} f_2(t,v) \frac{\partial \varphi}{\partial t}(t,v)\, dv dt \\ &= \int_{\mathbb{R}^3} f_{2,in}(v) \varphi(0,v)\, dv + m\left(f_{1,in}\right) \int_0^\infty \int_{\mathbb{R}^3} L(f_2)(t,v) \varphi(t,v)\, dv dt \\ &\quad + \int_0^\infty \int_{\mathbb{R}^3} Q^a(f_2, f_2)(t,v) \varphi(t,v)\, dv dt. \end{aligned} \tag{1.24}$$

Note that in the formulas above, we have used the notation $f(\cdot, v, r)\, dr dv$ instead of $df(\cdot, v, r)$. This is justified in particular by the fact that this measure has a density, as stated in the Remark at the end of this paper.

Small ratio of mass limits in the context of kinetic equations are described in [D], in particular in the context of plasmas. Among the many references in this work, we wish to quote [DL1] and [DL2], in which some of the computations are close to the computations that we present here.

Our method of proof is based on uniform w.r.t. p a priori estimates including in particular moments estimates based on a new variant of Povzner's inequality, especially suited for collisions of particles with disparate masses. We refer for previous versions of this inequality (including inequalities devised for non cutoff or energy-dissipating kernels) to [P], [Bo], [L], [De], [MW], [DeM], [W], [GPV].

Unfortunately, the entropy estimate for $f_{1,p}$ is not uniform w.r.t. p (this uniformity holds only for $f_{2,p}$) so that the passage to the limit when $p \to \infty$ is done only in the sense of weak measures. Note that measure-valued solutions to the Boltzmann equation have been introduced in the context of steady solutions, (Cf. for example [Ce2]). Our own context is somehow more favorable, since when the initial datum is smooth enough, the equation obtained at the limit preserves in the evolution this smoothness.

The second section of this work is devoted to a brief Proof of Proposition 1.1 and to the exposition of the dimensional analysis leading to eq. (1.16) – (1.17). Then, in section 3, Theorem 1.1 is proven.

2 Preliminaries: Proof of Proposition 1.1 and Dimensional Analysis

We begin this section with a brief sketch of the Proof of Proposition 1.1. It mainly uses classical tools, and can be found in detail in [C2].

Sketch of the Proof of Proposition 1.1 : We first introduce the following approximation of eq. (1.1) – (1.2) (in the spatially homogeneous case),

$$\frac{\partial f_1^n}{\partial t} = \frac{R_1^n(f_1^n, f_2^n)}{1 + \frac{1}{n}\int\int f_1^n \, drdv + \frac{1}{n}\int f_2^n \, dv}, \tag{2.1}$$

$$\frac{\partial f_2^n}{\partial t} = \frac{R_2^n(f_1^n, f_2^n) + Q^n(f_2^n, f_2^n)}{1 + \frac{1}{n}\int f_2^n \, dv + \frac{1}{n}\int\int f_1^n \, drdv}, \tag{2.2}$$

$$f_1^n(0, v, r) = f_{1,in}(v, r)\, 1_{\{|v| \le n\}} + \frac{1}{n}\, e^{-|v|^2/2}, \tag{2.3}$$

$$f_2^n(0, v) = f_{2,in}(v)\, 1_{\{|v| \le n\}} + \frac{1}{n}\, e^{-|v|^2/2}, \tag{2.4}$$

with R_1^n, R_2^n, Q^n defined by [eq. (1.3), (1.5) and]

$$
\begin{aligned}
R_1^n(f_1,f_2)(v_1,r) &= \int_{\mathbb{R}^3}\int_{\mathbb{S}^2} [f_1(v_1',r)f_2(v_2') - f_1(v_1,r)f_2(v_2)] \\
&\qquad \times \min\left\{\frac{1}{n}, (1+r)^2\,|\omega\cdot(v_1-v_2)|\right\} d\omega dv_2,
\end{aligned}
$$

$$
\begin{aligned}
R_2^n(f_1,f_2)(v_2) &= \int_{\mathbb{R}^3}\int_{\mathbb{S}^2}\int_{r_{min}}^{r_{max}} [f_1(v_1',r)f_2(v_2') - f_1(v_1,r)f_2(v_2)] \\
&\qquad \times \min\left\{\frac{1}{n}, (1+r)^2\,|\omega\cdot(v_2-v_1)|\right\} drd\omega dv_1,
\end{aligned}
$$

and

$$
\begin{aligned}
Q^n(f_2,f_2)(v) &= \int_{\mathbb{R}^3}\int_{\mathbb{S}^2} [f_2(v')f_2(v_*') - f_2(v)f_2(v_*)] \\
&\qquad \times \min\left\{\frac{1}{n}, C_{eff}|v-v_*|^\alpha\right\} d\sigma dv_*.
\end{aligned}
\tag{2.5}
$$

We first observe that the operators in the r.h.s. of (2.1) – (2.2) are Lipschitz-continuous w.r.t. $L^1(\mathbb{R}^3\times[r_{min},r_{max}])\times L^1(\mathbb{R}^3)$ so that one can find a solution in $C^1(\mathbb{R}_+; L^1(\mathbb{R}^3\times[r_{min},r_{max}])\times L^1(\mathbb{R}^3))$ to system (2.1) – (2.4).

Moreover, it is easy to prove (thanks to some variant of the minimum principle) that $f_1^n, f_2^n \geq 0$, and one can check that the following uniform w.r.t. n a priori estimates hold:

$$
\sup_{t\geq 0, n\in\mathbb{N}^*} \int_{\mathbb{R}^3}\int_{r_{min}}^{r_{max}} f_1^n(t,v,r)\,drdv < +\infty, \tag{2.6}
$$

$$
\sup_{t\geq 0, n\in\mathbb{N}^*} \int_{\mathbb{R}^3} f_2^n(t,v)\,dv < +\infty, \tag{2.7}
$$

(deduced from the conservation of mass for molecules on one hand, and dust specks on the other hand)

$$
\begin{aligned}
\sup_{t\geq 0, n\in\mathbb{N}^*} \int_{\mathbb{R}^3} \bigg(\int_{r_{min}}^{r_{max}} & f_1^n(t,v,r)\,|\log f_1^n(t,v,r)|\,dr \\
& + f_2^n(t,v)\,|\log f_2^n(t,v)| \bigg)\, dv < +\infty,
\end{aligned}
\tag{2.8}
$$

(deduced from the entropy inequality)

$$
\sup_{t\geq 0, n\in\mathbb{N}^*} \int_{\mathbb{R}^3} \left(\int_{r_{min}}^{r_{max}} |v|^2\, f_1^n(t,v,r)\,dr + |v|^2\, f_2^n(t,v) \right) dv < +\infty, \tag{2.9}
$$

(deduced from the conservation of kinetic energy).

As a consequence, it is possible to extract from the sequence $(f_1^n, f_2^n)_{n\in\mathbb{N}^*}$ a subsequence which converges in $C(\mathbb{R}_+; L^1(\mathbb{R}^3 \times [r_{min}, r_{max}]; (1+|v|)\, dvdr) \times L^1(\mathbb{R}^3; (1+|v|)\, dv))$ weak towards a couple of functions (f_1, f_2) such that $f_1, f_2 \geq 0$, f_1, f_2 satisfies the bounds (2.6) – (2.9) [with f_1^n, f_2^n replaced by f_1, f_2, and without having to take the supremum w.r.t. $n \in \mathbb{N}^*$], and f_1, f_2 is a weak solution to eq. (1.1), (1.2), with initial datum $f_{1,in}$, $f_{2,in}$.

The Proof of Proposition 1.1 can be concluded by noticing that for all $s \geq 1$, estimates (1.10) and (1.11) is a consequence of an easy variant of Povzner's inequality (Cf. for example [MW]). Once again, we refer to [C2] for a completely detailed Proof of Proposition 1.1. ■

We now turn to the establishment of a non-dimensional version of eq. (1.1), (1.2). Our assumptions concern cases in which the number of dust particles is very small in front of the number of molecules, and in which the radiuses of different dust particles are of the same order of magnitude.

We introduce a time scale t° which is the typical collision time of two molecules (we refer to [C2] for non-dimensional versions of eq. (1.1), (1.2) with other time scales), a typical length scale L which corresponds to the mean free path of molecules, and, like in [DL1], two different scales V_1° and V_2° for the velocities of particles of dust and molecules respectively (they correspond to the thermal velocities of the species). We assume here that the order of magnitude of the kinetic temperature of the two species are identical and we denote them by T°. Under this assumption, V_1° and V_2° are defined by

$$V_1^\circ = \sqrt{\frac{8kT^\circ}{\pi m_1(r_{min})}} \quad \text{and} \quad V_2^\circ = \sqrt{\frac{8kT^\circ}{\pi m_2}},$$

where $m_1(r_{min})$ is the mass of a particle of dust of radius r_{min}, and m_2 is the mass of a molecule. These velocities are related by the formula

$$V_1^\circ = \sqrt{\varepsilon_m}\, V_2^\circ. \tag{2.10}$$

Contrary to the assumptions made in [DL1], we introduce here two different orders of magnitude n_1° and n_2° for the number density of the species, and we define by

$$\alpha^\circ = \frac{n_1^\circ}{n_2^\circ} \tag{2.11}$$

the ratio of these magnitudes. In the applications that we have in mind, this ratio is very small.

Then, we introduce the adimensional densities in the phase space:

$$\hat{f}_1(\bar{t}, \bar{x}, \hat{v}_1, \bar{r}) = \frac{(V_1^\circ)^3\, r_{min}}{n_1^\circ}\, f_1(t, x, v, r),$$

and

$$\check{f}_2(\bar{t}, \bar{x}, \check{v}_2) = \frac{(V_2^\circ)^3}{n_2^\circ}\, f_2(t, x, v_2),$$

where $\bar{x}$, $\bar{t}$, $\hat{v}_1$ and $\check{v}_2$ are the adimensional variables defined by

$$\bar{x} = \frac{x}{L}, \quad \bar{t} = \frac{t}{t^\circ}, \quad \bar{r} = \frac{r}{r_{min}}, \quad \hat{v}_1 = \frac{v_1}{V_1^\circ}, \quad \check{v}_2 = \frac{v_2}{V_2^\circ},$$

where

$$t^\circ = \frac{1}{4\pi\, n_2^\circ\, r_2^2\, V_2^\circ}, \quad L = t^\circ\, V_2^\circ,$$

and f_1, f_2 are solutions to eq. (1.1) - (1.5). The densities $(\hat{f}_1, \check{f}_2)$ are then solutions to the following system of equations

$$\frac{\partial \hat{f}_1}{\partial \bar{t}} + \sqrt{\varepsilon_m}\hat{v}_1 \cdot \nabla_{\bar{x}} \hat{f}_1 = \frac{1}{4\pi}\left(\frac{\eta}{\varepsilon_m}\right)^{2/3} \bar{R}_1(\hat{f}_1, \check{f}_2),$$

$$\frac{\partial \check{f}_2}{\partial \bar{t}} + \check{v}_2 \cdot \nabla_{\bar{x}} \check{f}_2 = \frac{\alpha^\circ}{4\pi}\left(\frac{\eta}{\varepsilon_m}\right)^{2/3} \bar{R}_2(\hat{f}_1, \check{f}_2) + \bar{Q}(\check{f}_2, \check{f}_2).$$

Here $\bar{R}_1$, $\bar{R}_2$ and $\bar{Q}$ are defined by

$$\begin{aligned} \bar{Q}\left(\check{f}_2, \check{f}_2\right)(\check{v}) &= \int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\int_{\mathbb{S}^2} \left[\check{f}_2(\check{v}')\check{f}_2(\check{v}'_*) - \check{f}_2(\check{v})\check{f}_2(\check{v}_*)\right] \\ &\qquad\qquad \times \frac{C_{eff}(V_2^\circ)^\alpha}{4\pi\, r_2^2\, V_2^\circ}\, |\check{v} - \check{v}_*|^\alpha d\sigma d\check{v}^*, \end{aligned}$$

with

$$\begin{cases} \check{v}' &= \dfrac{\check{v} + \check{v}_*}{2} + \dfrac{|\check{v} - \check{v}_*|}{2}\sigma, \\ \check{v}'_* &= \dfrac{\check{v} + \check{v}_*}{2} - \dfrac{|\check{v} - \check{v}_*|}{2}\sigma, \end{cases}$$

$$\begin{aligned} \bar{R}_1(\hat{f}_1, \check{f}_2)(\hat{v}_1, \bar{r}) &= \int_{\mathbb{R}^3}\int_{\mathbb{S}^2} \left[\hat{f}_1(\hat{v}'_1, \bar{r})\check{f}_2(\check{v}'_2) - \hat{f}_1(\hat{v}_1, \bar{r})\check{f}_2(\check{v}_2)\right] \\ &\qquad \times \left(\left(\tfrac{\varepsilon_m}{\eta}\right)^{1/3} + \bar{r}\right)^2 \left|\left(\check{v}_2 - \sqrt{\varepsilon_m}\hat{v}_1\right)\cdot\omega\right| d\omega d\check{v}_2, \end{aligned}$$

and

$$\begin{aligned} \bar{R}_2(\hat{f}_1, \check{f}_2)(\check{v}_2) &= \int_{\mathbb{R}^3}\int_{\mathbb{S}^2}\int_1^{r_0} \left[\hat{f}_1(\hat{v}'_1, \bar{r})\check{f}_2(\check{v}'_2) - \hat{f}_1(\hat{v}_1, \bar{r})\check{f}_2(\check{v}_2)\right] \\ &\qquad \times \left(\left(\tfrac{\varepsilon_m}{\eta}\right)^{1/3} + \bar{r}\right)^2 \left|\left(\check{v}_2 - \sqrt{\varepsilon_m}\hat{v}_1\right)\cdot\omega\right| dr d\omega d\hat{v}_1, \end{aligned}$$

with

$$\begin{cases} \hat{v}'_1 &= \hat{v}_1 + \dfrac{2\sqrt{\varepsilon_m}\bar{r}^{-3}}{1 + \varepsilon_m \bar{r}^{-3}}[\omega\cdot(\check{v}_2 - \sqrt{\varepsilon_m}\hat{v}_1)]\omega, \\ \check{v}'_2 &= \check{v}_2 - \dfrac{2}{1 + \varepsilon_m \bar{r}^{-3}}[\omega\cdot(\check{v}_2 - \sqrt{\varepsilon_m}\hat{v}_1)]\omega, \end{cases}$$

and $r_0 = \frac{r_{max}}{r_{min}}$. Finally, η is an adimensional constant defined by $\eta = \dfrac{3m_2}{4\pi\rho r_2^3}$, where ρ is the volumic mass of particles of dust and r_2 the radius of molecules (that is, $\left(\frac{\eta}{\varepsilon_m}\right)^{1/3} = \frac{r_{min}}{r_2}$). From now on, we also denote $C^a_{eff} := \frac{C_{eff}\,(V_2^\circ)^\alpha}{4\pi\, r_2^2\, V_2^\circ}$ (this parameter is of order 1 under our assumptions).

We now put ourselves in a spatially homogeneous context, and we establish the adimensional versions of various estimates (mass, energy, entropy).

We first notice that the adimensional versions of the relations of conservation of mass are similar to formulas (1.7) and (1.8): we get indeed, for a.e $\bar{t} \in \mathbb{R}_+$ and for all $\bar{r} \in [1, r_0]$:

$$\int_{\mathbb{R}^3} \hat{f}_1(\bar{t}, \hat{v}_1, \bar{r})\, d\hat{v}_1 = \int_{\mathbb{R}^3} \hat{f}_1(0, \hat{v}_1, \bar{r})\, d\hat{v}_1, \tag{2.12}$$

(where $r_0 = \frac{r_{max}}{r_{min}}$), and

$$\int_{\mathbb{R}^3} \check{f}_2(\bar{t}, \check{v}_2)\, d\check{v}_2 = \int_{\mathbb{R}^3} \check{f}_2(0, \check{v}_2)\, d\check{v}_2. \tag{2.13}$$

We also get

$$\begin{aligned}
&\int_{r_{min}}^{r_{max}} \int_{\mathbb{R}^3} f_1(t, v_1, r)\, |v_1|^2 \left(\frac{r}{r_{min}}\right)^3 dr dv_1 \\
&= n_1^\circ\, (V_1^\circ)^2 \int_1^{r_0} \int_{\mathbb{R}^3} \hat{f}_1(\bar{t}, \hat{v}_1, \bar{r})\, |\hat{v}_1|^2\, \bar{r}^3 d\bar{r} d\hat{v}_1,
\end{aligned}$$

$$\int_{\mathbb{R}^3} f_2(t, v_2)\, |v_2|^2\, dv_2 = n_2^\circ\, (V_2^\circ)^2 \int_{\mathbb{R}^3} \check{f}_2(\bar{t}, \check{v}_2)\, |\check{v}_2|^2\, d\check{v}_2.$$

Thanks to (2.10), (2.11), one deduces from the relation of conservation of energy (1.11) the following relation:

$$\begin{aligned}
&\alpha^\circ \int_1^{r_0} \int_{\mathbb{R}^3} \hat{f}_1(\bar{t}, \hat{v}_1, \bar{r})\, |\hat{v}_1|^2\, \bar{r}^3 d\bar{r} d\hat{v}_1 + \int_{\mathbb{R}^3} \check{f}_2(\bar{t}, \check{v}_2)\, |\check{v}_2|^2\, d\check{v}_2 \\
&= \alpha^\circ \int_1^{r_0} \int_{\mathbb{R}^3} \hat{f}_1(0, \hat{v}_1, \bar{r})\, |\hat{v}_1|^2\, \bar{r}^3 d\bar{r} d\hat{v}_1 + \int_{\mathbb{R}^3} \check{f}_2(0, \check{v}_2)\, |\check{v}_2|^2\, d\check{v}_2.
\end{aligned} \tag{2.14}$$

Moreover, since

$$\begin{aligned}
&\int_{\mathbb{R}^3} \int_{r_{min}}^{r_{max}} f_1(t, v_1, r) \ln\left(f_1(t, v_1, r)\right) dr dv_1 \\
&= n_1^\circ \int_{\mathbb{R}^3} \int_1^{r_0} \hat{f}_1(\bar{t}, \hat{v}_1, \bar{r}) \ln\left(\hat{f}_1(\bar{t}, \hat{v}_1, \bar{r})\right) d\bar{r} d\hat{v}_1 \\
&\quad + n_1^\circ \left(\ln\left(n_1^\circ\right) - \ln\left(\left(V_1^\circ\right)^3 r_{min}\right)\right) \int_{\mathbb{R}^3} \int_1^{r_0} \hat{f}_1(\bar{t}, \hat{v}_1, \bar{r}) d\bar{r} d\hat{v}_1,
\end{aligned}$$

and

$$
\begin{aligned}
\int_{\mathbb{R}^3} f_2(t,v_2)\ln\left(f_2(t,v_2)\right)dv_2 \quad &= \quad n_2^\circ \int_{\mathbb{R}^3} \check{f}_2(\bar{t},\check{v}_2)\ln\left(\check{f}_2(\bar{t},\check{v}_2)\right)d\check{v}_2 \\
&\qquad + n_2^\circ\left(\ln\left(n_2^\circ\right) - \ln\left(\left(V_2^\circ\right)^3\right)\right)\int_{\mathbb{R}^3} \check{f}_2(\bar{t},\check{v}_2)d\check{v}_2,
\end{aligned}
$$

[and thanks to relations (2.12) and (2.13)], the entropy inequality (1.9) leads to the following inequality:

$$
\begin{aligned}
&\alpha^\circ \int_{\mathbb{R}^3}\int_1^{r_0} \hat{f}_1(\bar{t},\hat{v}_1,\bar{r})\ln\left(\hat{f}_1(\bar{t},\hat{v}_1,\bar{r})\right)d\bar{r}d\hat{v}_1 + \int_{\mathbb{R}^3} \check{f}_2(\bar{t},\check{v}_2)\ln\left(\check{f}_2(\bar{t},\check{v}_2)\right)d\check{v}_2 \\
&\leq \alpha^\circ \int_{\mathbb{R}^3}\int_1^{r_0} \hat{f}_1(0,\hat{v}_1,\bar{r})\ln\left(\hat{f}_1(0,\hat{v}_1,\bar{r})\right)d\bar{r}d\hat{v}_1 + \int_{\mathbb{R}^3} \check{f}_2(0,\check{v}_2)\ln\left(\check{f}_2(0,\check{v}_2)\right)d\check{v}_2.
\end{aligned}
\tag{2.15}
$$

In the experiment that we consider, the typical value of α° is 10^{-6}, that of ε_m is 10^{-12}, and that of η is $6\cdot 10^{-2}$. Therefore, we consider that

$$
c := \frac{\alpha^\circ}{4\pi}\left(\frac{\eta}{\varepsilon_m}\right)^{2/3} \sim 1, \quad \text{and} \quad \frac{1}{\sqrt{\varepsilon_m}} \sim \frac{1}{\alpha^\circ} := p \to \infty. \tag{2.16}
$$

We now write $f_1(t,v,r)$ instead of $\hat{f}_1(\bar{t},\hat{v}_1,\bar{r})$, $f_2(t,v)$ instead of $\check{f}_2(\bar{t},\check{v}_2)$, Q^a instead of $\bar{Q}$, $R_1^{a,p}$ instead of $\bar{R}_1$, $R_2^{a,p}$ instead of $\bar{R}_2$. Then we have $\left(\frac{\varepsilon_m}{\eta}\right)^{1/3} = \frac{1}{2\sqrt{\pi\, p\, c}}$, and we write $\frac{1}{\sqrt{\varepsilon_m}} = \xi\, p$, with $\xi > 0$ fixed. We end up with system (1.16), (1.17).

Next section is devoted to the proof that when $p \to \infty$ in (1.16), (1.17), the solutions of this system converge towards the solutions of a Boltzmann-Vlasov coupling given by eq. (1.19), (1.20) [that is, Theorem 1.1].

3 Proof of Theorem 1.1

We now begin the

Proof of Theorem 1.1 : For the sake of readability, we only consider the case $\xi = 1$ (this changes nothing in the Proof). We first express what remains of the relations of conservation of mass, energy (and of the evolution of entropy) when $p \to \infty$ in eq. (1.16), (1.17), under the assumptions of Theorem 1.1. According to relations (2.12), (2.13), (2.14), (1.9) and to assumption (2.16), the following estimates hold, for all $p \in \mathbb{N}$, for all $t \in \mathbb{R}_+$, and for a.e $r \in [1, r_0]$:

$$
\int_{\mathbb{R}^3} f_{1,p}(t,v,r)\,dv = \int_{\mathbb{R}^3} f_{1,in}(v,r)\,dv, \tag{3.1}
$$

$$\int_{\mathbb{R}^3} f_{2,p}(t,v)\,dv = \int_{\mathbb{R}^3} f_{2,in}(v)\,dv, \tag{3.2}$$

$$\begin{aligned}
&\frac{1}{p}\int_{\mathbb{R}^3}\int_1^{r_0} f_{1,p}(t,v,r)\ln\left(f_{1,p}(t,v,r)\right) drdv + \int_{\mathbb{R}^3} f_{2,p}(t,v)\ln\left(f_{2,p}(t,v)\right) dv \\
&\le \frac{1}{p}\int_{\mathbb{R}^3}\int_1^{r_0} f_{1,in}(v,r)\ln\left(f_{1,in}(v,r)\right) drdv + \int_{\mathbb{R}^3} f_{2,in}(v)\ln\left(f_{2,in}(v)\right) dv,
\end{aligned} \tag{3.3}$$

and

$$\begin{aligned}
&\frac{1}{p}\int_1^{r_0}\int_{\mathbb{R}^3} f_{1,p}(t,v,r)\,|v|^2\,r^3 drdv + \int_{\mathbb{R}^3} f_{2,p}(t,v)\,|v|^2\,dv \\
&= \frac{1}{p}\int_1^{r_0}\int_{\mathbb{R}^3} f_{1,in}(v,r)\,|v|^2\,r^3 drdv + \int_{\mathbb{R}^3} f_{2,in}(v)\,|v|^2\,dv.
\end{aligned} \tag{3.4}$$

We consequently obtain the following bounds (for all $T > 0$)

$$\sup_{p\in\mathbb{N}^*}\sup_{t\in[0,T]}\int_{\mathbb{R}^3}\left(1+|v|+|v|^2\right) f_{2,p}(t,v)\,dv < +\infty, \tag{3.5}$$

and

$$\sup_{p\in\mathbb{N}^*}\sup_{t\in[0,T]}\int_{\mathbb{R}^3}\int_1^{r_0}\left(1+|v|+\frac{|v|^2}{p}\right) f_{1,p}(t,v,r)\,drdv < +\infty. \tag{3.6}$$

Estimate (3.5) is indeed a direct consequence of relations (3.1), (3.2) and (3.4). So is also the bound

$$\sup_{p\in\mathbb{N}^*}\sup_{t\in[0,T]}\int_{\mathbb{R}^3}\int_1^{r_0}\left(1+\frac{|v|^2}{p}\right) f_{1,p}(t,v,r)\,drdv < +\infty.$$

In order to obtain (3.6), we only have to prove the following bound:

$$\sup_{p\in\mathbb{N}^*}\sup_{t\in[0,T]}\int_{\mathbb{R}^3}\int_1^{r_0} |v| f_{1,p}(t,v,r)\,drdv < +\infty. \tag{3.7}$$

Let $p \in \mathbb{N}^*$ and $t \in [0,T]$. We have

$$\begin{aligned}
\int_1^{r_0}\int_{\mathbb{R}^3} f_{1,p}(t,v_1,r)\,|v_1|\,drdv_1 &= \int_1^{r_0}\int_{\mathbb{R}^3} f_{1,in}(v_1,r)\,|v_1|\,drdv_1 \\
&+ p\,c\int_1^{r_0}\int_{\mathbb{R}^3}\int_0^t R_1^{a,p}(f_{1,p},f_{2,p})(s,v_1,r)\,|v_1|\,dsdrdv_1,
\end{aligned}$$

with

$$\int_1^{r_0}\int_{\mathbb{R}^3} f_{1,in}(v_1,r)\,|v_1|\,drdv_1 \le \int_1^{r_0}\int_{\mathbb{R}^3} f_{1,in}(v_1,r)\left(1+|v_1|^2\right) drdv_1 < +\infty.$$

Thanks to the involutive character of the transformation $(v_1, v_2) \to \left(v'_{1,p}, v'_{2,p}\right)$, one can get:

$$
\begin{aligned}
& p \int_0^t \int_1^{r_0} \int_{\mathbb{R}^3} R_1^{a,p}(f_{1,p}, f_{2,p})(s, v_1, r)\, |v_1|\, dr dv_1 ds \\
&= p \int_0^t \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \int_{\mathbb{S}^2} \int_1^{r_0} f_{1,p}(s, v_1, r)\, f_{2,p}(s, v_2) \left| \omega \cdot \left(\frac{v_1}{p} - v_2 \right) \right| \\
&\quad \times \left(\frac{1}{2\sqrt{\pi\, pc}} + r \right)^2 \left(\left|v'_{1,p}\right| - |v_1| \right) dr d\omega dv_2 dv_1 ds.
\end{aligned}
$$

Noticing that

$$
\begin{aligned}
\left| \omega \cdot \left(\frac{v_1}{p} - v_2 \right) \right| \left(\left|v'_{1,p}\right| - |v_1| \right) \quad &\leq \quad \left| \frac{v_1}{p} - v_2 \right| \left| v'_{1,p} - v_1 \right| \\
&\leq \quad \frac{Cst}{p} \left(1 + \frac{|v_1|^2}{p} \right) \left(1 + |v_2|^2 \right),
\end{aligned}
$$

we get

$$
\begin{aligned}
& p \int_0^t \int_1^{r_0} \int_{\mathbb{R}^3} R_1^{a,p}(f_{1,p}, f_{2,p})(s, v_1, r)\, ds dr dv_1 \\
& \leq Cst \sup_{p \in \mathbb{N}^*} \sup_{t \in [0,T]} \int_{\mathbb{R}^3} \int_1^{r_0} \left(1 + \frac{|v_1|^2}{p} \right) f_{1,p}(t, v, r)\, dv dr \\
& \quad \times \sup_{p \in \mathbb{N}^*} \sup_{t \in [0,T]} \int_{\mathbb{R}^3} \left(1 + |v_2|^2 \right) f_{2,p}(t, v)\, dv,
\end{aligned}
$$

and estimate (3.7) (and therefore (3.5)) holds.

We now show that higher order moments can be bounded for $f_{2,p}$ (uniformly w.r.t. p), provided that they initially exist. More precisely, we define for $s \geq 1$, and $g_1 := g_1(t, v, r) \geq 0$, $g_2 := g_2(t, v) \geq 0$, the quantities

$$
M_{\gamma,p}(g_1, g_2)(t) = \int_{\mathbb{R}^3} (1 + |v|^\gamma) \left\{ g_2(t, v) + \frac{1}{p} \int_1^{r_0} r^{\frac{3\gamma}{2}} g_1(t, v, r)\, dr \right\} dv,
$$

and

$$
S_\gamma(g_1, g_2)(t) = \int_{\mathbb{R}^3} (1 + |v|^\gamma)\, g_2(t, v)\, dv + \int_{\mathbb{R}^3} \int_1^{r_0} (1 + |v|^\gamma)\, g_1(t, v, r)\, dr dv.
$$

Then the following Proposition holds:

Proposition 3.1 *Let $s \geq 1$. Then there exist constants $K_1, K_2, K_3 > 0$ which depend only on s, T, r_0, c, ξ and $C^a_{eff} > 0$, $\alpha \in [0, 1]$ in the cross section of Q^a, $R_1^{a,p}$, $R_2^{a,p}$, such that (for all $g_1 := g_1(t, v, r) \geq 0$, $g_2 := g_2(t, v) \geq 0$ such that the integrals make sense)*

$$
\int_{\mathbb{R}^3} \left(1 + |v|^{2s} \right) Q^a(g_2, g_2)(t, v)\, dv \leq K_1\, M_{2s,p}(g_2, g_2)(t) \tag{3.8}
$$

$$\times \left(M_{2,p}(g_2, g_2)(t) + M_{2s-2,p}(g_2, g_2)(t) \right).$$

$$\int_{\mathbb{R}^3} (1 + |v|^{2s}) \left\{ R_2^{a,p}(g_1, g_2)(t, v) + \int_1^{r_0} r^{3s} R_1^{a,p}(g_1, g_2)(t, v, r)\, dr \right\} dv$$
$$\leq K_2 \Big[M_{2s,p}(g_1, g_2)(t) S_1(g_1, g_2)(t) + p M_{2s-1,p}(g_1, g_2)(t) M_{2,p}(g_1, g_2)(t) \Big], \tag{3.9}$$

and

$$\int_{\mathbb{R}^3} \left(1 + |v|^{2s}\right) \left\{ R_2^{a,p}(g_1, g_2)(t, v) + \int_1^{r_0} r^{3s} R_1^{a,p}(g_1, g_2)(t, v, r)\, dr \right\} dv$$
$$\leq K_3 M_{2,p}(g_1, g_2)(t) M_{2s-1,p}(g_1, g_2)(t) + \frac{K_3}{p} S_1(g_1, g_2)(t) M_{2s,p}(g_1, g_2)(t)$$
$$+ \frac{K_3}{p^2} M_{2s+1,p}(g_1, g_2)(t) + \frac{K_3}{p} M_{3,p}(g_1, g_2)(t) M_{2s-2,p}(g_1, g_2)(t). \tag{3.10}$$

Proof of Proposition 3.1 : We use the classical Povzner's inequality to prove inequalities (3.8) and (3.9). More precisely, the inequality for (3.8) can be found in [De] for example. We have, for inequality (3.9):

$$\int_{\mathbb{R}^3} \left(1 + |v|^{2s}\right) \left(R_2^{a,p}(g_1, g_2)(t, v) + \int_1^{r_0} r^{3s} R_1^{a,p}(g_1, g_2)(t, v, r)\, dr \right) dv$$
$$\leq \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \int_1^{r_0} \int_{\mathbb{S}^2} \left(\left(r^3 |v'_{1,p}|^2 + |v'_{2,p}|^2\right)^s - r^{3s} |v_1|^{2s} - |v_2|^{2s} \right)$$
$$\times \left| \omega \cdot \left(\frac{v_1}{p} - v_2 \right) \right| \left(\frac{1}{2\sqrt{\pi\, pc}} + r \right)^2 g_1(t, v_1, r)\, g_2(t, v_2)\, dr d\omega dv_2 dv_1,$$

and since the couple of velocities $(v'_{1,p}, v'_{2,p})$ given by (1.18) satifies the relation

$$r^3 \left|v'_{1,p}\right|^2 + \left|v'_{2,p}\right|^2 = r^3 |v_1|^2 + |v_2|^2,$$

we get

$$\left(\left(r^3 |v'_{1,p}|^2 + |v'_{2,p}|^2\right)^s - r^{3s} |v_1|^{2s} - |v_2|^{2s} \right) \left| \omega \cdot \left(\frac{v_1}{p} - v_2 \right) \right|$$
$$\leq Cst(s, r_0) \left(|v_1|^{2s-1} |v_2| + |v_1| |v_2|^{2s-1} \right) \left| \frac{v_1}{p} - v_2 \right|$$
$$\leq Cst(s, r_0) \left(\frac{1}{p} |v_1|^{2s} |v_2| + |v_1| |v_2|^{2s} + \frac{1}{p} |v_1|^2 |v_2|^{2s-1} + |v_1|^{2s-1} |v_2|^2 \right)$$

and estimate (3.9) holds. This estimate only depends on moments M_k with $k \leq 2s$, but is not uniform w.r.t p. So it is not possible at this level to use it to establish a uniform estimate on the moments M_s.

Therefore, we establish inequality (3.10) thanks to a new variant of Povzner's inequality. We use for that an other parametrisation of the post-collisional velocities in the operators $R_1^{a,p}$ and $R_2^{a,p}$ (Cf. [De] again):

$$\begin{aligned} R_1^{a,p}(g_1,g_2)(t,v_1,r) \quad = \quad & \int_{\mathbb{R}^3}\int_{\mathbb{S}^2} \frac{1}{2}\left(\frac{1}{2\sqrt{\pi\, pc}}+r\right)^2 \left|\frac{v_1}{p}-v_2\right| \\ & \times \left[g_1(t,v_{1,p}^{''},r)\, g_2(t,v_{2,p}^{''}) - g_1(t,v_1,r)\, g_2(t,v_2)\right] d\sigma dv_2, \end{aligned}$$

and

$$\begin{aligned} R_2^{a,p}(g_1,g_2)(t,v_2) \quad = \quad & \int_{\mathbb{R}^3}\int_{\mathbb{S}^2}\int_1^{r_0} \frac{1}{2}\left(\frac{1}{2\sqrt{\pi\, pc}}+r\right)^2 \left|\frac{v_1}{p}-v_2\right| \\ & \times \left[g_1(t,v_{1,p}^{''},r) g_2(t,v_{2,p}^{''}) - g_1(t,v_1,r) g_2(t,v_2)\right] drd\sigma dv_1, \end{aligned}$$

with

$$\left\{ \begin{aligned} v_{1,p}'' = \quad & \frac{p^2}{1+r^3p^2}\left[\left(v_1 r^3+\frac{v_2}{p}\right)-\frac{1}{p}\left|v_2-\frac{v_1}{p}\right|\sigma\right], \\ v_{2,p}'' = \quad & \frac{p^2}{1+r^3p^2}\left[\frac{1}{p}\left(v_1 r^3+\frac{v_2}{p}\right)+r^3\left|v_2-\frac{v_1}{p}\right|\sigma\right]. \end{aligned}\right. \tag{3.11}$$

We now establish the new variant of Povzner's inequality. We define, for $(v_1,v_2)\in\mathbb{R}^3\times\mathbb{R}^3$, $\sigma\in\mathbb{S}^2$, $r\in[1,r_0]$ and $s\geq 1$, the quantity

$$\psi^s_{v_1,v_2}(\sigma,r) = r^{3s}\left|v_{1,p}''\right|^{2s} + \left|v_{2,p}''\right|^{2s} - r^{3s}|v_1|^{2s} - |v_2|^{2s},$$

where $v_{1,p}''$ and $v_{2,p}''$ are given by (3.11), and we begin by introducing the vector $\sigma_0\in\mathbb{S}^2$ defined as:

$$\sigma_0 = -\frac{v_2-\frac{v_1}{p}}{\left|v_2-\frac{v_1}{p}\right|}.$$

Noticing that $r^3-\frac{1}{p^2}\geq 0$ for all $p\in\mathbb{N}^*$, $r\in[1,r_0]$, we get

$$\begin{aligned} & \psi^s_{v_1,v_2}(\sigma_0,r) \\ & = \left(\frac{p^2}{1+r^3p^2}\right)^{2s}\left[r^{3s}\left|\left(r^3-\frac{1}{p^2}\right)v_1+\frac{2}{p}v_2\right|^{2s} + \left|\frac{2r^3}{p}v_1-\left(r^3-\frac{1}{p^2}\right)v_2\right|^{2s}\right] \\ & \quad -r^{3s}|v_1|^{2s}-|v_2|^{2s} \\ & \leq \left(\frac{p^2}{1+r^3p^2}\right)^{2s}\left\{r^{3s}\left[\left(r^3-\frac{1}{p^2}\right)|v_1|+\frac{2}{p}|v_2|\right]^{2s}\right. \\ & \quad \left. +\left[\frac{2r^3}{p}|v_1|+\left(r^3-\frac{1}{p^2}\right)|v_2|\right]^{2s}\right\} - r^{3s}|v_1|^{2s}-|v_2|^{2s} \\ & \leq (a_s(p)-1)\left(r^{3s}|v_1|^{2s}+|v_2|^{2s}\right)+F(v_1,v_2), \end{aligned}$$

with

$$a_s(p) \quad = \quad \left(\frac{p^2}{1+r^3p^2}\right)^{2s} \left\{ \left(r^3 - \frac{1}{p^2}\right)^{2s} + r^{3s}\left(\frac{2}{p}\right)^{2s} \right\},$$

and

$$\begin{aligned}
&F(v_1, v_2) \\
&\leq Cst(s) \left(\frac{p^2}{1+r^3p^2}\right)^{2s} \\
&\quad \times \Bigg\{ |v_1|^{2s-1}|v_2| \left[r^{3s}\left(r^3 - \frac{1}{p^2}\right)^{2s-1} \frac{2}{p} + \left(\frac{2r^3}{p}\right)^{2s-1} \left(r^3 - \frac{1}{p^2}\right) \right] \\
&\quad + |v_1||v_2|^{2s-1} \left[r^{3s}\left(r^3 - \frac{1}{p^2}\right)\left(\frac{2}{p}\right)^{2s-1} + \left(\frac{2r^3}{p}\right)\left(r^3 - \frac{1}{p^2}\right)^{2s-1} \right] \Bigg\} \\
&\leq G_{s,p,r} \left(|v_1|^{2s-1}|v_2| + |v_1||v_2|^{2s-1} \right),
\end{aligned}$$

with

$$G_{s,p,r} = Cst(s) \left(\frac{p^2}{1+r^3p^2}\right)^{2s} r^{3(2s-1)} \left[\left(r^3 - \frac{1}{p^2}\right)^{2s-1} \frac{2}{p} + \left(\frac{2}{p}\right)^{2s-1} \left(r^3 - \frac{1}{p^2}\right) \right]$$

$$\leq \frac{Cst(s, r_0)}{p}.$$

Moreover,

$$a_s(p) \leq \left[\frac{\left(r^3p^2 - 1\right)^2 + 4r^3p^2}{\left(1 + r^3p^2\right)^2} \right]^s \leq 1,$$

consequently we have

$$\psi^s_{v_1,v_2}(\sigma_0, r) \leq \frac{Cst(s, r_0)}{p} \left[|v_1|^{2s-1} |v_2| + |v_1| |v_2|^{2s-1} \right]. \tag{3.12}$$

We then study the quantity $\psi^s_{v_1,v_2}(\sigma, r) - \psi^s_{v_1,v_2}(\sigma_0, r)$ for any $\sigma \in \mathbb{S}^2$. We denote here $v''_{1,p,\sigma}$ and $v''_{2,p,\sigma}$ the post-collisional velocities given by (3.11) corresponding to this vector σ, and v''_{1,p,σ_0} and v''_{2,p,σ_0} the post-collisional velocities given by (3.11) for $\sigma = \sigma_0$. We can write

$$\psi^s_{v_1,v_2}(\sigma, r) - \psi^s_{v_1,v_2}(\sigma_0, r) = r^{3s} \left(|v''_{1,p,\sigma}|^{2s} - |v''_{1,p,\sigma_0}|^{2s} \right) + |v''_{2,p,\sigma}|^{2s} - |v''_{2,p,\sigma_0}|^{2s},$$

with

$$\begin{aligned}
&|v''_{1,p,\sigma}|^{2s} - |v''_{1,p,\sigma_0}|^{2s} \\
&\leq \left(\frac{p^2}{1+r^3p^2}\right)^{2s} \left\{ \left[|v_1| \left(r^3 + \frac{1}{p^2}\right) + \frac{2}{p}|v_2| \right]^{2s} - \left| \left(r^3 - \frac{1}{p^2}\right)|v_1| - \frac{2}{p}|v_2| \right|^{2s} \right\},
\end{aligned}$$

and

$$|v''_{2,p,\sigma}|^{2s} - |v''_{2,p,\sigma_0}|^{2s} \leq \left(\frac{p^2}{1+r^3p^2}\right)^{2s} \left\{ \left[\frac{2r^3}{p}|v_1| + \left(\frac{1}{p^2}+r^3\right)|v_2|\right]^{2s} - \left|\frac{2r^3}{p}|v_1| - \left(r^3 - \frac{1}{p^2}\right)|v_2|\right|^{2s} \right\}.$$

Using the inequality

$$(a+b)^{2s} - (a-b)^{2s} = s\int_{(a-b)^2}^{(a+b)^2} x^{s-1}dx \leq 4s\,(a+b)^{2s-2}ab,$$

with $a, b > 0$ and $s > 1$, and the inequality

$$(a+b)^x \leq 2^x\,(a^x + b^x)\,,$$

with $x > 0$, we obtain:

$$\begin{aligned}
&|v''_{1,p,\sigma}|^{2s} - |v''_{1,p,\sigma_0}|^{2s} \\
&\leq \left(\frac{p^2}{1+r^3p^2}\right)^{2s} \frac{4sr^3}{p}\left[|v_1|\left(r^3+\frac{1}{p^2}\right)+\frac{2}{p}|v_2|\right]^{2s-2}|v_1|\left[\frac{1}{p}|v_1|+2|v_2|\right] \\
&\leq \left(\frac{p^2}{1+r^3p^2}\right)^{2s} \frac{Cst(s,r_0)}{p}\left[|v_1|^{2s-2}\left(r^3+\frac{1}{p^2}\right)^{2s-2}+\left(\frac{2}{p}\right)^{2s-2}|v_2|^{2s-2}\right] \\
&\quad\times |v_1|\left[\frac{1}{p}|v_1|+2|v_2|\right],
\end{aligned}$$

and

$$\begin{aligned}
&|v''_{2,p,\sigma}|^{2s} - |v''_{2,p,\sigma_0}|^{2s} \\
&\leq \left(\frac{p^2}{1+r^3p^2}\right)^{2s} \frac{4sr^3}{p}\left[\frac{2r^3}{p}|v_1|+\left(\frac{1}{p^2}+r^3\right)|v_2|\right]^{2s-2}\left[2r^3|v_1|+\frac{1}{p}|v_2|\right]|v_2| \\
&\leq \left(\frac{p^2}{1+r^3p^2}\right)^{2s} \frac{Cst(s,r_0)}{p}\left[\left(\frac{2r^3}{p}\right)^{2s-2}|v_1|^{2s-2}+\left(\frac{1}{p^2}+r^3\right)^{2s-2}|v_2|^{2s-2}\right] \\
&\quad\times \left[2r^3|v_1|+\frac{1}{p}|v_2|\right]|v_2|\,.
\end{aligned}$$

Then

$$r^{3s}\left(|v''_{1,p,\sigma}|^{2s} - |v''_{1,p,\sigma_0}|^{2s}\right) + |v''_{2,p,\sigma}|^{2s} - |v''_{2,p,\sigma_0}|^{2s} \tag{3.13}$$

$$\leq b_s(p)\left[r^{3s}|v_1|^{2s}+|v_2|^{2s}\right] + H(v_1,v_2), \tag{3.14}$$

where

$$b_s(p) = \frac{Cst(s,r_0)}{p^2}\left(\frac{p^2}{1+r^3p^2}\right)^2,$$

and

$$
\begin{aligned}
H(v_1,v_2) &= \left(\frac{p^2}{1+r^3p^2}\right)^2 \left\{ \frac{Cst(s,r_0)}{p}\left(|v_2|\,|v_1|^{2s-1}+|v_1|\,|v_2|^{2s-1}\right)\right.\\
&\quad \left. +\frac{Cst(s,r_0)}{p^2}\left(\frac{2}{p}\right)^{2s-2}\left[|v_1|^2\,|v_2|^{2s-2}+|v_2|^2\,|v_1|^{2s-2}\right]\right\}.
\end{aligned}
$$

Finally we estimate $\psi^s_{v_1,v_2}(\sigma,r)\left|\frac{v_1}{p}-v_2\right|$. Thank to (3.12) and (3.13), we obtain, for $s\geq 1$ and using the bound $1\leq r\leq r_0$,

$$
\begin{aligned}
\psi^s_{v_1,v_2}(\sigma,r) &\leq \frac{Cst(s,r_0)}{p^2}(|v_1|^{2s}+|v_2|^{2s})+\frac{Cst(s,r_0)}{p}\left[|v_1|^{2s-1}\,|v_2|+|v_2|^{2s-1}\,|v_1|\right]\\
&\quad +\frac{Cst(s,r_0)}{p^2}\left[|v_1|^{2s-2}\,|v_2|^2+|v_2|^{2s-2}\,|v_1|^2\right].
\end{aligned}
$$

Then

$$
\begin{aligned}
&\psi^s_{v_1,v_2}(\sigma,r)\left|\frac{v_1}{p}-v_2\right|\\
&\leq \frac{Cst(s,r_0)}{p^2}\left[\frac{1}{p}\,|v_1|^{2s+1}+|v_2|^{2s+1}\right]+\frac{Cst(s,r_0)}{p}\left[\frac{1}{p}\,|v_1|^{2s}\,|v_2|+|v_1|\,|v_2|^{2s}\right]\\
&\quad +\frac{Cst(s,r_0)}{p}\left[\frac{1}{p}\,|v_1|^2\,|v_2|^{2s-1}+|v_2|^2\,|v_1|^{2s-1}\right]\\
&\quad +\frac{Cst(s,r_0)}{p^2}\left[\frac{1}{p}\,|v_1|^3\,|v_2|^{2s-2}+|v_2|^3\,|v_1|^{2s-2}\right],
\end{aligned}
$$

and we finally obtain (3.10). This ends the Proof of Proposition 3.1. ■

Thanks to Proposition 3.1, we can prove the following (uniform w.r.t. p) bounds for the solutions of eq. (1.16), (1.17):

Proposition 3.2 *Under the assumptions of Theorem 1.1, the moment of order 3 of $f_{2,p}$ is uniformly bounded (w.r.t. p) for all $T>0$, more precisely:*

$$
\sup_{t\in[0,T],p\in\mathbb{N}^*}\int_{\mathbb{R}^3}\int_1^{r_0}\left(\frac{1}{p}\,f_{1,p}(t,v,r)+f_{2,p}(t,v)\right)(1+|v|^3)\,drdv<+\infty. \quad (3.15)
$$

Proof of Proposition 3.2 : Thanks to (3.5), (3.6), we know that (for all $T>0$)

$$
S:=\sup_{t\in[0,T]}\sup_{p\in\mathbb{N}^*}S_1\left(f_{1,p},f_{2,p}\right)<+\infty, \quad (3.16)
$$

and

$$
M_2:=\sup_{t\in[0,T]}\sup_{p\in\mathbb{N}^*}M_{2,p}\left(f_{1,p},f_{2,p}\right)(t)<+\infty. \quad (3.17)
$$

The Proof will be divided in several steps. We first notice thanks to (3.8) and (3.9) used with $s = 3/2$ that (for all $T > 0$)

$$\sup_{p\in\mathbb{N}^*}\sup_{t\in[0,T]}\frac{1}{p}M_{3,p}(f_{1,p},f_{2,p})(t) < +\infty. \tag{3.18}$$

Using the same inequalities, but with $s = 2$, we then obtain:

$$\sup_{p\in\mathbb{N}^*}\sup_{t\in[0,T]}\frac{1}{p^2}M_{4,p}(f_{1,p},f_{2,p})(t) < +\infty. \tag{3.19}$$

This allows us to prove, thanks to inequality (3.10) used with $s = 3/2$ that:

$$\sup_{p\in\mathbb{N}^*}\sup_{t\in[0,T]}M_{3,p}(f_{1,p},f_{2,p})(t) < +\infty. \tag{3.20}$$

Let us now give a few more details about the successive bounds:

Bound on $\frac{1}{p}M_{3,p}(f_{1,p},f_{2,p})(t)$*:* Since $f_{1,p}$ and $f_{2,p}$ are solutions of the equations (1.16) and (1.17), we have, for all $s \geq 1$:

$$\begin{aligned}
&M_{2s,p}(f_{1,p},f_{2,p})(t)\\
&= M_{2s,p}(f_{1,in},f_{2,in}) + \int_0^t\int_{\mathbb{R}^3}\left(1+|v_2|^{2s}\right)Q^a(f_{1,p},f_{2,p})(\tau,v_2)\,dv_2d\tau\\
&\quad +c\int_0^t\int_{\mathbb{R}^3}\left(1+|v_2|^{2s}\right)R_2^{a,p}(f_{1,p},f_{2,p})(\tau,v_2)\,dv_2d\tau\\
&\quad +c\int_0^t\int_{\mathbb{R}^3}\int_1^{r_0} r^{3s}\left(1+|v_1|^{2s}\right)R_1^{a,p}(f_{1,p},f_{2,p})(\tau,v_1,r)\,dr dv_1 d\tau.
\end{aligned}$$

Thank to (3.8) and (3.9), we obtain the following bound when $s \in [1,2]$ (with the notations (3.16) and (3.17))

$$\begin{aligned}
M_{2s,p}(f_{1,p},f_{2,p})(t) \quad &\leq \quad M_{2s,p}(f_{1,in},f_{2,in}) + p\,c\,K_2M_2\int_0^t M_{2s-1,p}(f_{1,p},f_{2,p})(\tau)\,d\tau\\
&\quad + (K_1M_2 + c\,K_2S)\int_0^t M_{2s,p}(f_{1,p},f_{2,p})(\tau)\,d\tau. \qquad (3.21)
\end{aligned}$$

Taking $s = \frac{3}{2}$ in (3.21), we obtain

$$\begin{aligned}
M_{3,p}(f_{1,p},f_{2,p})(t) \quad &\leq \quad M_{3,p}(f_{1,in},f_{2,in}) + p\,c\,K_2M_2^2T\\
&\quad + (K_1M_2 + c\,K_2S)\int_0^t M_{3,p}(f_{1,p},f_{2,p})(\tau)\,d\tau.
\end{aligned}$$

Then, thanks to Gronwall's lemma, we can deduce that for all $t \in [0,T]$:

$$\frac{1}{p}M_{3,p}(f_{1,p},f_{2,p})(t) \leq \left[c\,K_2M_2^2T + M_{3,1}(f_{1,in},f_{2,in})\right]\exp\left[(K_1M_2 + cK_2S)\,t\right], \tag{3.22}$$

so that relation (3.18) holds.

Bound on $\frac{1}{p^2} M_{4,p}(f_{1,p}, f_{2,p})(t)$*:* Using now inequality (3.21) with $s = 2$, we see that for some constant $K_4 > 0$, for all $t \in [0, T]$,

$$\begin{aligned}
&M_{4,p}(f_{1,p}, f_{2,p})(t) - M_{4,p}(f_{1,p}, f_{2,p})(0) \\
&\leq K_4 \left\{ \int_0^t M_{4,p}\left(f_{1,p}, f_{2,p}\right)(\tau)\, d\tau + p \int_0^t M_{3,p}\left(f_{1,p}, f_{2,p}\right)(\tau)\, d\tau \right\}.
\end{aligned}$$

Using (3.22) and Gronwall's lemma, we see that estimate (3.19) holds.

Bound on $M_{3,p}(f_{1,p}, f_{2,p})(t)$. We here use the bound (3.10), and obtain, for all $s \in [1, 2]$, the following estimate:

$$\begin{aligned}
&M_{2s,p}(f_{1,p}, f_{2,p})(t) \\
&\leq M_{2s,p}(f_{1,in}, f_{2,in}) + c\, \frac{K_3}{p^2} \int_0^t M_{2s+1,p}(f_{1,p}, f_{2,p})(\tau)\, d\tau \\
&\quad + \left(c\, \frac{K_3}{p} S + M_2 K_1 \right) \int_0^t M_{2s,p}(f_{1,p}, f_{2,p})(\tau)\, d\tau \\
&\quad + c\, K_3 M_2 \int_0^t M_{2s-1,p}(f_{1,p}, f_{2,p})(\tau)\, d\tau \\
&\quad + c\, \frac{K_3}{p} \int_0^t M_{3,p}(f_{1,p}, f_{2,p})(\tau) M_{2s-2,p}(f_{1,p}, f_{2,p})(\tau)\, d\tau. \qquad (3.23)
\end{aligned}$$

Taking $s = \frac{3}{2}$ in the previous estimate, we get:

$$\begin{aligned}
M_{3,p}(f_{1,p}, f_{2,p})(t) \quad \leq \quad & M_{3,p}(f_{1,in}, f_{2,in}) + c\, \frac{K_3}{p^2} \int_0^t M_{4,p}(f_{1,p}, f_{2,p})(\tau)\, d\tau \\
& + \left(c\, \frac{K_3}{p} S + M_2 K_1 \right) \int_0^t M_{3,p}(f_{1,p}, f_{2,p})(\tau)\, d\tau + c\, K_3 M_2^2 T \\
& + c\, \frac{K_3}{p} S \int_0^t M_{3,p}(f_{1,p}, f_{2,p})(\tau)\, d\tau,
\end{aligned}$$

so that thanks to (3.19) and Gronwall's lemma, we get estimate (3.20) (and (3.15)). ■

We now are in a position to pass to the limit in (the weak form of) eq. (1.16), (1.17). We first notice that thanks to estimates (3.3), (3.5) and (3.6), the sequences $(f_{1,p}, f_{2,p})_{p \in \mathbb{N}^*}$ converge up to extraction to measure-valued functions (f_1, f_2) in $L^\infty(\mathbb{R}_+; M^1(\mathbb{R}^3 \times [1, r_0]) \times L^1(\mathbb{R}^3))$ weak * and the following estimate holds:

$$\sup_{t \in [0,T]} \int_{\mathbb{R}^3} \int_1^{r_0} (1 + |v|) f_1(t, v, r)\, dv dr < \infty, \qquad (3.24)$$

$$\sup_{t\in[0,T]} \int_{\mathbb{R}^3} (1+|v|^2) f_2(t,v)\, dv < \infty. \tag{3.25}$$

In the sequel, we keep the notation $f_1(t,v,r)$ for the measure-valued function $f_1 : \mathbb{R}_+ \to M^1(\mathbb{R}^3 \times [1, r_0])$ as in (3.24), though this measure might a priori not have a density w.r.t. Lebesgue's measure.

Moreover, thanks to assumption (1.22), and bound (3.15), moments of order lower or equal to 3 of $f_{2,p}$ are bounded w.r.t p.

In order to conclude the Proof of Theorem 1.1, it remains to show that (f_1, f_2) is a weak solution to eq. (1.19), (1.20). We study for that the convergence of the weak form of kernels $R_1^{a,p}(f_{1,p}, f_{2,p})$, $R_2^{a,p}(f_{1,p}, f_{2,p})$ and $Q^a(f_{2,p}, f_{2,p})$, when $p \to \infty$, in the following Proposition.

Proposition 3.3 *Under the assumptions of Theorem 1.1, we can extract from $(f_{1,p}, f_{2,p})_{p\in\mathbb{N}^*}$ a subsequence such that (for all $T > 0$):*

1. *for all $\psi \in C_c^2\left([0,T] \times \mathbb{R}^3 \times [1, r_0]\right)$,*

$$\begin{aligned} &\lim_{p\to\infty} \int_0^T \int_{\mathbb{R}^3} \int_1^{r_0} p\, c\, R_1^{a,p}(f_{1,p}, f_{2,p}) \psi \, drdvdt \\ &= \int_0^T \int_{\mathbb{R}^3} \int_1^{r_0} K(f_2) \cdot \nabla_v \psi \, f_1 \, drdvdt; \end{aligned} \tag{3.26}$$

2. *for all $\varphi \in C_c^1\left([0,T] \times \mathbb{R}^3\right)$,*

$$\lim_{p\to\infty} \int_0^T \int_{\mathbb{R}^3} c\, R_2^{a,p}(f_{1,p}, f_{2,p}) \varphi \, dvdt = \int_0^T \int_{\mathbb{R}^3} m(f_{1,in}) L(f_2) \varphi \, dvdt$$

3. *for all $\varphi \in C_c\left([0,T] \times \mathbb{R}^3\right)$,*

$$\lim_{p\to\infty} \int_0^T \int_{\mathbb{R}^3} Q^a(f_{2,p}, f_{2,p}) \varphi \, dvdt = \int_0^T \int_{\mathbb{R}^3} Q^a(f_2, f_2) \varphi \, dvdt.$$

Proof of Proposition 3.3 :

1. Let $\psi \in C_c^2\left([0,T] \times \mathbb{R}^3 \times [1, r_0]\right)$. Denoting

$$I_{1,p} := p\, c \int_0^T \int_{\mathbb{R}^3} \int_1^{r_0} \psi(s,v,r) R_1^{a,p}(f_{1,p}, f_{2,p})(s,v,r) \, drdvds,$$

and

$$I_1 = \int_0^T \int_{\mathbb{R}^3} \int_1^{r_0} K(f_2)(r,s) \cdot \nabla_v \psi(s,v,r) f_1(s,v,r) \, drdvds,$$

where $K(f_2)$ is given by (1.21), we prove that

$$\lim_{p\to\infty} I_{1,p} = I_1.$$

Thank to the involutive character of the transformation $(v_1, v_2) \to (v'_{1,p}, v'_{2,p})$, $I_{1,p}$ can be written under the form :

$$\begin{aligned} I_{1,p} &= c \int_0^T \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \int_{\mathbb{S}^2} \int_1^{r_0} \left(\frac{1}{2\sqrt{\pi\, p\, c}} + r \right)^2 f_{1,p}(s, v_1, r) f_{2,p}(s, v_2) \\ &\quad \times \left| \omega \cdot \left(\frac{v_1}{p} - v_2 \right) \right| p \left(\psi(s, v'_{1,p}, r) - \psi(s, v_1, r) \right) dr d\omega dv_2 dv_1 ds \end{aligned}$$

with

$$v'_{1,p} = v_1 + \frac{2pr^{-3}}{p^2 + r^{-3}} \left[\omega \cdot \left(v_2 - \frac{v_1}{p} \right) \right] \omega,$$

and thanks to the relation

$$\int_{\mathbb{S}^2} (\mathbf{a} \cdot \omega) (\mathbf{b} \cdot \omega) |\mathbf{a} \cdot \omega| \, d\omega = \pi \, |\mathbf{a}| \, (\mathbf{a} \cdot \mathbf{b}),$$

for $a \in \mathbb{R}^3$ and $b \in \mathbb{R}^3$, I_1 can be written under the form:

$$\begin{aligned} I_1 &= 2c \int_0^T \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \int_{\mathbb{S}^2} \int_1^{r_0} r^2 \, |v_2 \cdot \omega| \, (v_2 \cdot \omega) \, f_2(s, v_2) \, (\omega \cdot \nabla_{v_1} \psi(s, v_1, r)) \\ &\quad \times \frac{1}{r^3} f_1(v_1, r, s) \, dr d\omega dv_1 dv_2 ds. \end{aligned}$$

We now write the difference $I_{1,p} - I_1$ as the following sum:

$$I_{1,p} - I_1 = J^1_{1,p} + J^2_{1,p} + J^3_{1,p} + J^4_{1,p} + J^5_{1,p},$$

where

$$\begin{aligned} J^1_{1,p} &= p\, c \int_0^T \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \int_{\mathbb{S}^2} \int_1^{r_0} \left[\left(\frac{1}{2\sqrt{\pi\, p\, c}} + r \right)^2 - r^2 \right] f_{1,p}(s, v_1, r) f_{2,p}(s, v_2) \\ &\quad \times \left| \omega \cdot \left(\frac{v_1}{p} - v_2 \right) \right| \left[\psi(s, v'_{1,p}, r) - \psi(s, v_1, r) \right] dr d\omega dv_2 dv_1 ds, \end{aligned}$$

$$\begin{aligned} J^2_{1,p} &= p\, c \int_0^T \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \int_{\mathbb{S}^2} \int_1^{r_0} \left[\left| \omega \cdot \left(\frac{v_1}{p} - v_2 \right) \right| - |\omega \cdot v_2| \right] f_{2,p}(s, v_2) \\ &\quad \times r^2 f_{1,p}(s, v_1, r) \left[\psi(s, v'_{1,p}, r) - \psi(s, v_1, r) \right] dr d\omega dv_2 dv_1 ds, \end{aligned}$$

$$\begin{aligned} J^3_{1,p} &= c \int_0^T \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \int_{\mathbb{S}^2} \int_1^{r_0} r^2 f_{1,p}(s, v_1, r) f_{2,p}(s, v_2) \, |\omega \cdot v_2| \\ &\quad \times \left[p \left(\psi(s, v'_{1,p}, r) - \psi(s, v_1, r) \right) - \frac{2}{r^3} \omega \cdot \nabla_{v_1} \psi(s, v_1, r) \, (v_2 \cdot \omega) \right] \\ &\quad \times dr d\omega dv_1 dv_2 ds, \end{aligned}$$

$$J_{1,p}^4 = 2\,c\int_0^T \int_{\mathbb{S}^2}\int_{\mathbb{R}^3}\int_1^{r_0} \frac{1}{r}\omega\cdot\nabla_{v_1}\psi(s,v_1,r)f_1(s,v_1,r)drdv_1$$
$$\times\int_{\mathbb{R}^3}\left(f_{2,p}(s,v_2)-f_2(s,v_2)\right)(v_2\cdot\omega)\left|\omega\cdot v_2\right| dv_2 d\omega ds,$$

$$J_{1,p}^5 = 2\,c\int_0^T \int_{\mathbb{S}^2}\int_{\mathbb{R}^3}\int_1^{r_0} \frac{1}{r}\omega\cdot\nabla_{v_1}\psi(s,v_1,r)\left(f_{1,p}(s,v_1,r)-f_1(s,v_1,r)\right)drdv_1$$
$$\times\int_{\mathbb{R}^3} f_{2,p}(s,v_2)\left|\omega\cdot v_2\right|(v_2\cdot\omega)\, dv_2 d\omega ds.$$

Noticing that

$$\begin{aligned} p\left(\psi(s,v'_{1,p},r)-\psi(s,v_1,r)\right) &\leq p\left\|\nabla\psi\right\|_\infty \left|v'_{1,p}-v_1\right| \\ &\leq 2\left\|\nabla\psi\right\|_\infty \left|v_2-\frac{v_1}{p}\right|, \end{aligned}$$

and thank to bounds (3.5) and (3.6), it is easy to prove that

$$\lim_{p\to\infty} J_{1,p}^1 = \lim_{p\to\infty} J_{1,p}^2 = 0.$$

Moreover,

$$\begin{aligned} &\psi(s,v'_{1,p},r)-\psi(s,v_1,r) \\ &= \left(v'_{1,p}-v_1\right)\cdot\nabla_{v_1}\psi(s,v_1,r) + T(v_1,v_2,r,s,p) \\ &= \frac{2pr^{-3}}{p^2+r^{-3}}\left[\omega\cdot\left(v_2-\frac{v_1}{p}\right)\right](\omega\cdot\nabla_{v_1}\psi(s,v_1,r)) + T(v_1,v_2,r,s,p), \end{aligned}$$

with, since $\psi\in C_c^2\left([0,T]\times\mathbb{R}^3\times[1,r_0]\right)$,

$$\begin{aligned} \left|T(v_1,v_2,r,s,p)\right| &\leq \frac{\|D_v^2\psi\|_\infty}{2}\left|v'_{1,p}-v_1\right|^2 \\ &\leq \frac{Cst\,\|D_v^2\psi\|_\infty}{p^2}\left|v_2-\frac{v_1}{p}\right|^2. \end{aligned}$$

Then

$$\begin{aligned} &\left[p\left(\psi(s,v'_{1,p},r)-\psi(s,v_1,r)\right)-\frac{2}{r^3}\left(\omega\cdot\nabla_{v_1}\psi(s,v_1,r)\right)(v_2\cdot\omega)\right]\left|\omega\cdot v_2\right| \\ &\leq \frac{Cst(r_0)\,\|D_v\psi\|_\infty}{r^3}\left(\frac{1}{p^2}|v_2|^2+\frac{|v_1|}{p}\,|v_2|\right) \\ &\quad +\frac{Cst(r_0)\,\|D_v^2\psi\|_\infty}{p}\left(|v_2|^3+\frac{|v_2||v_1|^2}{p^2}+2\frac{|v_1||v_2|^2}{p}\right), \end{aligned}$$

and thank to bounds (3.5), (3.6), and (3.15), we see that $\lim_{p\to\infty} J_{3,p}=0$. Morever, one can write

$$J_{1,p}^4 = \int_0^T\int_{\mathbb{R}^3} h(s,v_2)\left(f_{2,p}(s,v_2)-f_2(s,v_2)\right)\left(1+|v_2|^2\right)dv_2 ds$$

with $h \in L^\infty\left([0,T]\times\mathbb{R}^3\right)$; so that thanks to the weak convergence of $(f_{2,p})_{p\in\mathbb{N}^*}$ and to the bound (3.15), we get $\lim\limits_{p\to\infty} J^4_{1,p} = 0$. It remains to prove that $\lim\limits_{p\to\infty} J^5_{1,p} = 0$. We write for that

$$J^5_{1,p} = \int_0^T \int_{\mathbb{R}^3}\int_1^{r_0} \frac{2\,c}{r}\nabla_{v_1}\psi(s,v_1,r)\cdot k_p(s)\left(f_{1,p}(s,v_1,r)-f_1(s,v_1,r)\right)drdv_1ds,$$

where

$$k_p(s) = \int_{\mathbb{R}^3}\int_{\mathbb{S}^2} \omega f_{2,p}(s,v_2)\,|\omega\cdot v_2|\,(v_2\cdot\omega)\,d\omega dv_2.$$

Thanks to estimate (3.5), the sequence $(k_p)_{p\in\mathbb{N}^*}$ is bounded in $L^\infty([0,T])$. Moreover, for all $h>0$ and $t\in[0,T]$ such that $t+h\le T$, the following estimate holds for all $p\in\mathbb{N}^*$:

$$\begin{aligned}
&\int_0^t |k_p(s+h)-k_p(s)|\,ds\\
&\le Cst\int_0^t\left|\int_s^{s+h}\int_{\mathbb{R}^3}\left[cR^a_{2,p}(f_{1,p},f_{2,p})(\tau,v_2)+Q^a(f_{2,p},f_{2,p})(\tau,v_2)\right]\right.\\
&\qquad\qquad \left.\times\int_{\mathbb{S}^2}\omega\ |\omega\cdot v_2|\,(v_2\cdot\omega)\ d\omega\,dv_2d\tau\right|ds\\
&\le ht\,Cst\sup_{\tau\in[0,T],p\in\mathbb{N}^*}\int_{\mathbb{R}^3} f_{2,p}(\tau,v_2)\left(|v_2|^3+1\right)dv_2\\
&\quad\times\left[\sup_{\tau\in[0,T],p\in\mathbb{N}^*}\int_{\mathbb{R}^3} f_{1,p}(\tau,v_1,r)\left(1+\frac{|v_1|}{p}+\frac{|v_1|^2}{p^2}+\frac{|v_1|^3}{p^3}\right)drdv_1\right.\\
&\qquad\left.+\sup_{\tau\in[0,T],p\in\mathbb{N}^*}\int_{\mathbb{R}^3} f_{2,p}(\tau,v_2)\left(|v_2|^3+1\right)dv_2\right].
\end{aligned}$$

Using estimate (3.15), we deduce then from Riesz-Fréchet-Kolmogorov's Theorem that $\{k_p,\,p\in\mathbb{N}^*\}$ strongly converges (up to a subsequence) in $\mathrm{L}^1([0,T])$. But

$$\int_{\mathbb{R}^3}\int_1^{r_0}\frac{2\,c}{r}\nabla_{v_1}\psi(s,v_1,r)\left(f_{1,p}(s,v_1,r)-f_1(s,v_1,r)\right)\,drdv_1$$

tends to 0 in $L^\infty([0,T])$ weak *. This allows us to conclude that $\lim\limits_{p\to\infty} J^5_{1,p}=0$.

2. Let $\varphi\in C^1_c\left([0,T]\times\mathbb{R}^3\right)$. We can write

$$\int_{\mathbb{R}^3}\int_0^T \varphi(s,v)R^{a,p}_2(f_{1,p},f_{2,p})(s,v)\,dvds = I^+_{2,p}-I^-_{2,p},$$

where we denote

$$\begin{aligned}
I^+_{2,p} \;=\; & c\int_0^T\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\int_{\mathbb{S}^2}\int_1^{r_0}\varphi(s,v'_{2,p})f_{1,p}(s,v_1,r)f_{2,p}(s,v_2)\\
&\times\left(\frac{1}{2\sqrt{\pi\,p\,c}}+r\right)^2\left|\omega\cdot\left(\frac{v_1}{p}-v_2\right)\right|drd\omega dv_1dv_2ds,
\end{aligned}$$

and

$$
\begin{aligned}
I_{2,p}^{-} &= c\int_0^T\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\int_{\mathbb{S}^2}\int_1^{r_0}\varphi(s,v_2)f_{1,p}(s,v_1,r)f_{2,p}(s,v_2)\\
&\quad\times\left(\frac{1}{2\sqrt{\pi\,p\,c}}+r\right)^2\left|\omega\cdot\left(\frac{v_1}{p}-v_2\right)\right|drd\omega dv_1dv_2ds.
\end{aligned}
$$

Denoting

$$
\begin{aligned}
I_2^{+} &= c\int_0^T\left(\int_{\mathbb{R}^3}\int_1^{r_0}r^2f_{1,in}(v,r)dvdr\right)\\
&\quad\times\int_{\mathbb{R}^3}\int_{\mathbb{S}^2}\varphi(s,v_2-2\,(\omega\cdot v_2)\,\omega)\;f_2(s,v_2)\;|\omega\cdot v_2|\;d\omega dv_2ds,
\end{aligned}
$$

and

$$
\begin{aligned}
I_2^{-} &= c\int_0^T\left(\int_{\mathbb{R}^3}\int_1^{r_0}r^2f_{1,in}(v,r)dvdr\right)\\
&\quad\times\int_{\mathbb{R}^3}\int_{\mathbb{S}^2}\varphi(s,v_2)f_2\,(s,v_2)\,|\omega\cdot v_2|\,d\omega dv_2ds,
\end{aligned}
$$

we prove that $\lim_{p\to\infty}I_{2,p}^{+}=I_2^{+}$ (the proof can then be easily adapted to show that $\lim_{p\to\infty}I_{2,p}^{-}=I_2^{-}$).

Thanks to relation (3.1), we notice that for all $t\in[0,T]$,

$$
\int_{\mathbb{R}^3}\int_1^{r_0}f_{1,p}(t,v,r)r^2\,drdv=\int_{\mathbb{R}^3}\int_1^{r_0}f_{1,in}(v,r)r^2\,drdv
$$

and we write the difference $I_{2,p}^{+}-I_2^{+}$ as the following sum:

$$
I_{2,p}^{+}-I_2^{+}=J_{2,p}^1+J_{2,p}^2+J_{2,p}^3+J_{2,p}^4,
$$

where

$$
\begin{aligned}
J_{2,p}^1 &= c\int_0^T\int_{\mathbb{R}^3}\int_1^{r_0}f_{1,in}(v_1,r)r^2drdv_1\int_{\mathbb{R}^3}\int_{\mathbb{S}^2}|\omega\cdot v_2|\\
&\quad\times\varphi\left(s,v_2-2\,(\omega\cdot v_2)\,\omega\right)[f_{2,p}(s,v_2)-f_2(s,v_2)]\,d\omega dv_2ds,
\end{aligned}
$$

$$
\begin{aligned}
J_{2,p}^2 &= c\int_0^T\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\int_{\mathbb{S}^2}\int_1^{r_0}\varphi(s,v_{2,p}')f_{1,p}(s,v_1,r)f_{2,p}(s,v_2)\\
&\quad\times\left(\left(\frac{1}{2\sqrt{\pi\,p\,c}}+r\right)^2-r^2\right)|\omega\cdot v_2|\,drd\omega dv_1dv_2ds,
\end{aligned}
$$

$$\begin{aligned} J_{2,p}^{3} &= c\int_0^T\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\int_{\mathbb{S}^2}\int_1^{r_0}\varphi(s,v_{2,p}')f_{1,p}(s,v_1,r)f_{2,p}(s,v_2)\left(\frac{1}{2\sqrt{\pi\, pc}}+r\right)^2 \\ &\quad\times\left(\left|\omega\cdot\left(\frac{v_1}{p}-v_2\right)\right|-|\omega\cdot v_2|\right)drd\omega dv_1dv_2ds, \end{aligned}$$

$$\begin{aligned} J_{2,p}^{4} &= c\int_0^T\int_{\mathbb{R}^3}\int_1^{r_0}\int_{\mathbb{R}^3}\int_{\mathbb{S}^2}\left(\varphi(s,v_{2,p}')-\varphi\left(s,v_2-2\left(\omega\cdot v_2\right)\omega\right)\right) \\ &\quad\times r^2\left|\omega\cdot v_2\right| f_{2,p}(s,v_2)f_{1,p}(s,v_1,r)\,drdv_1d\omega dv_2ds. \end{aligned}$$

Firstly we have:

$$\begin{aligned} J_{2,p}^{1} &= \int_1^{r_0}\int_{\mathbb{R}^3}f_{1,in}(v_1,r)dv_1r^2dr \\ &\quad\times\int_0^T\int_{\mathbb{R}^3}b(s,v_2)\left[f_{2,p}(s,v_2)-f_2(s,v_2)\right](1+|v_2|)\,dv_2ds \end{aligned}$$

with

$$b(s,v_2)=\frac{1}{(1+|v_2|)}\int_{\mathbb{S}^2}\varphi\left(s,v_2-2\left(\omega\cdot v_2\right)\omega\right)|\omega\cdot v_2|\,d\omega,$$

and since $b\in L^\infty([0,T]\times\mathbb{R}^3)$, the convergence of $(f_{2,p})_{p\in\mathbb{N}^*}$ in $L^\infty\left([0,T];L^1(\mathbb{R}^3,(1+|v|)\,dv)\right)$ weak * implies that $\lim\limits_{p\to\infty}J_{2,p}^1=0$.

Then, we can observe that

$$\begin{aligned} \left|J_{2,p}^{2}\right| &\le \frac{Cst}{\sqrt{p}}\left\|\varphi\right\|_\infty T\int_{\mathbb{R}^3}\int_1^{r_0}f_{1,in}(v_1,r)\,drdv_1 \\ &\quad\times\sup_{\tau\in[0,T],p\in\mathbb{N}^*}\int_{\mathbb{R}^3}f_{2,p}(\tau,v_2)\left|v_2\right|\,dv_2, \end{aligned}$$

and

$$\begin{aligned} \left|J_{2,p}^{3}\right| &\le \frac{Cst}{p}\left\|\varphi\right\|_\infty T\sup_{\tau\in[0,T],p\in\mathbb{N}^*}\int_{\mathbb{R}^3}\int_1^{r_0}f_{1,p}(\tau,v_1,r)\left|v_1\right|drdv_1 \\ &\quad\times\sup_{\tau\in[0,T],p\in\mathbb{N}^*}\int_{\mathbb{R}^3}f_{2,p}(\tau,v_2)\,dv_2, \end{aligned}$$

so that $\lim\limits_{p\to\infty}J_{2,p}^2=0$ and $\lim\limits_{p\to\infty}J_{2,p}^3=0$.

Finally, we have the following estimate:

$$\begin{aligned} \left|J_{2,p}^{4}\right| &\le c\,r_0^2\|\nabla\varphi\|_\infty\int_0^T\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\int_1^{r_0}f_{1,p}(s,v_1,r)f_{2,p}(s,v_2)\left|v_2\right| \\ &\quad\times\int_{\mathbb{S}^2}\left|v_{2,p}'-\left(v_2-2\left(\omega\cdot v_2\right)\omega\right)\right|d\omega drdv_2dv_1ds, \end{aligned}$$

with

$$\left|v'_{2,p} - (v_2 - 2\,(\omega \cdot v_2)\,\omega)\right| \;=\; \left|-\frac{2p^2}{p^2+r^{-3}}\left[\omega \cdot \left(v_2 - \frac{v_1}{p}\right)\right]\omega + 2\,(\omega \cdot v_2)\,\omega\right|$$
$$\leq \frac{Cst}{p}\,(1+|v_1|)\,(1+|v_2|)\,.$$

We conclude that $\lim\limits_{p\to\infty} J^4_{2,p} = 0$.

3. Let $\varphi \in C_c\left([0,T]\times\mathbb{R}^3\right)$. Let us write:

$$\int_0^T \int_{\mathbb{R}^3} \varphi(s,v_2)\left[Q^a(f_{2,p},f_{2,p})(s,v) - Q^a(f_2,f_2)(s,v)\right] dvds = A_{1,p} - A_{2,p},$$

where we denote

$$A_{1,p} = \int_0^T \int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\int_{\mathbb{S}^2}\left[f_{2,p}(s,v)f_{2,p}(s,v_*) - f_2(s,v)f_2(s,v_*)\right]$$
$$\times\varphi(s,v')\,C^a_{eff}\,|v-v_*|^\alpha\, d\sigma dv_* dvds,$$

and

$$A_{2,p} = \int_0^T \int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\int_{\mathbb{S}^2}\left[f_{2,p}(s,v)f_{2,p}(s,v_*) - f_2(s,v)f_2(s,v_*)\right].$$
$$\times\varphi(s,v)\,C^a_{eff}\,|v-v_*|^\alpha\, d\sigma dv_* dvds.$$

We prove here that $\lim\limits_{p\to\infty} A_{1,p} = 0$ (the proof can easily be adapted to show that $\lim\limits_{p\to\infty} A_{2,p} = 0$). We write $A_{1,p}$ as the following sum:

$$A_{1,p} = J^1_{3,p} + J^2_{3,p},$$

where

$$J^1_{3,p} = \int_0^T \int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\int_{\mathbb{S}^2} f_2(s,v)\left[f_{2,p}(s,v_*) - f_2(s,v_*)\right],$$
$$\times\varphi(s,v')\,C^a_{eff}\,|v-v_*|^\alpha\, d\sigma dv_* dvds$$

and

$$J^2_{3,p} = \int_0^T \int_{\mathbb{R}^3} \kappa_p(s,v)\left[f_{2,p}(s,v) - f_2(s,v)\right] dvds,$$

with

$$\kappa_p(s,v) = \int_{\mathbb{R}^3}\int_{\mathbb{S}^2} \varphi(s,v')\,C^a_{eff}\,|v-v_*|^\alpha\, f_{2,p}(s,v_*) d\sigma dv_*.$$

Since $(f_{2,p})_{p\in\mathbb{N}^*}$ converges to f_2 in $L^\infty(\left([0,T];L^1\left(\mathbb{R}^3,(1+|v|)\,dv\right)\right)$ weak *, it follows that $\lim\limits_{p\to\infty} J^1_{3,p} = 0$.

Then, using the weak formulations of Q^a and $R_2^{a,p}$, we observe that

$$
\begin{aligned}
|\partial_s \kappa_p(s,v)| \quad \leq \quad & Cst\,||\varphi||_{L^\infty} \int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\int_1^{r_0} \left(|v|^\alpha + |v_2|^\alpha + |\frac{v_1}{p}|^\alpha\right)\left(|v_2| + |\frac{v_1}{p}|\right) \\
& \qquad\qquad \times f_{1,p}(s,v_1,r)\, f_{2,p}(s,v_2)\, dr dv_2 dv_1 \\
& + Cst\,||\varphi||_{L^\infty} \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \left(|v|^\alpha + |v_2|^\alpha + |w|^\alpha\right)\left(|v_2|^\alpha + |w|^\alpha\right) \\
& \qquad\qquad \times f_{2,p}(s,w)\, f_{2,p}(s,v_2)\, dv_2 dw \\
& + Cst\,||\partial_s\varphi||_{L^\infty}(1+|v|^\alpha) \int_{\mathbb{R}^3} f_{2,p}(s,v_2)\,(1+|v_2|^\alpha)\; dv_2.
\end{aligned}
$$

As a consequence, we can extract from $(\kappa_p)_{p\in\mathbb{N}^*}$ a subsequence which converges a.e. in $[0,T]\times\mathbb{R}^3$. Since moreover

$$
|\kappa_p(s,v)| \leq Cst\,||\varphi||_{L^\infty}(1+|v|^\alpha) \int_{\mathbb{R}^3} f_{2,p}(s,v_2)\,(1+|v_2|^\alpha)\; dv_2,
$$

the weak * convergence of $(f_{2,p})_{p\in\mathbb{N}^*}$ in $L^\infty([0,T];L^1(\mathbb{R}^3,(1+|v|)\,dv))$ implies that $\lim_{p\to\infty} J_{3,p}^2 = 0$. This ends the proof of Proposition 3.3. ■

We can then deduce from Proposition 3.3 that f_1 and f_2 are weak solution of (1.19) – (1.21), in the sense given by (1.23) and (1.24). This ends also the Proof of Theorem 1.1. ■

Remark: Note that $f_{1,in}$ being a function (that is, not only a measure), the solution of the equation

$$
\frac{\partial f_1}{\partial t} + \mathrm{div}_v\,(K(f_2)f_1) = 0
$$

is itself a function, given by

$$
f_1(t,v,r) = f_{1,in}\left(v - \int_0^t K(f_2)(s,r)ds,\, r\right).
$$

References

[Ar] L. Arkeryd, *On the Boltzmann equation, I and II*, Arch. Rat. Mech. Anal., **45**, (1972), 1–34.

[B] G.A. Bird *Monte Carlo Simulation of Gas Flows*, Annual Review of Fluid Mechanics, **10**, (1978), 11-31.

[Bo] A. V. Bobylev, *Moment inequalities for the Boltzmann equation and applications to spatially homogeneous problems*, J. Statist. Phys., **88**, (1997), 1183–1214.

[Ce1] C. Cercignani, *The Boltzmann equation and its applications*, (1988), Springer, Berlin.

[Ce2] C. Cercignani, *Measure solutions of the steady Boltzmann equation in a slab*, Commun. Math. Phys., **142**, n.2, (1991), 285–296.

[C1] F. Charles, *Kinetic modelling and numerical simulations using particle methods for the transport of dust in a rarefied gas*, In Rarefied Gas Dynamics: Proceedings of the 26th International Symposium on Rarefied Gas Dynamics, **1084**, (2008), 409–414.

[C2] F. Charles, PhD Thesis, in Preparation.

[D] P. Degond, *Asymptotic continuum models for plasmas and disparate mass gaseous binary mixtures*, in Material substructures in complex bodies: from atomic level to continuum, G. Capriz, P-M. Mariano (eds), Elsevier, Amsterdam, (2007).

[DL1] P. Degond and B. Lucquin-Desreux, *The asymptotics of collision operators for two species of particles of disparate masses*, Math. Models Methods Appl. Sci., **6**, n.3, (1996), 405–410.

[DL2] P. Degond and B. Lucquin-Desreux, *Transport coefficients of plasmas and disparate mass binary gases*, Transport Th. Statist. Phys., **25**, n.6, (1996), 595–633.

[De] L. Desvillettes, *Some applications of the method of moments for the homogeneous Boltzmann and Kac equations*, Arch. Rat. Mech. Anal., **123**, (1993), 387–404.

[DeM] L. Desvillettes, C. Mouhot, *About Lp Estimates for the Spatially Homogeneous Boltzmann Equation*, Ann. Inst. H. Poincaré, Anal. Non linéaire, **22**, (2005), 127–142.

[F] A. Frezzotti, S. Ostmo, T. Ytrhus, *Kinetic theory of steady evaporation from a spherical condensed phase containing inert solid particles*, Physics of fluids, **9**, (1997), 211-225.

[GPV] I. Gamba, V. Panferov, C. Villani, *On the Boltzmann equation for diffusively excited granular media*, Commun. Math. Phys., **246**, n.3, (2004), 503-541.

[L] X. Lu, *Conservation of energy, entropy identity, and local stability for the spatially homogeneous Boltzmann equation*, J. Statist. Phys., **96**, (1999), 765–796.

[MW] S. Mischler, B. Wennberg, *On the spatially homogeneous Boltzmann equation*, Ann. Inst. H. Poincaré, Anal. Non linéaire, **16**, n.4, (1999), 467–501.

[N] K. Nanbu, *Theoretical basis of the direct simulation Monte Carlo method* In Rarefied Gas Dynamics : Proceedings of the 15th International Symposium on Rarefied Gas Dynamics, (1986), 369-383.

[P] A.J. Povzner, *On the Boltzmann equation in the kinetic theory of gases*, Mat. Sb., **58**, (100), (1962), 65–86.

[T] K. Takase, *Three-dimensional numerical simulation of dust mobilization and air ingress characteristics in a fusion reactor during a LOVA event*, Fusion Engineering and Design, **54**, n. 3-4, (2001), 605–615, Elsevier.

[W] B. Wennberg, *Entropy dissipation and moment production for the Boltzmann equation*, J. Statist. Phys., **86**, (1997), 1053–1066.