# On the expression of the field scattered by a multimode plane

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summary :

The field scattered by a plane composed of several homogenous layers on a perfectly reflecting plane is generally given by its plane wave expansion (Fourier representation). We here develop another approach to express the field, more suitable for point source illumination. For this, we sum the contributions of modes corresponding to zeros $g_j$ (or poles $-g_j$) of the reflection coefficient of the imperfectly reflecting surface, developing an original integral representation and its expansions, valid for arbitrary mode which can be passive or active. Particular attention has been paid to active modes with $\mathrm{Re} g_j < 0$, and to the vicinity of the mode with $g_j = -1$. We then obtain novel exact expressions for the field in acoustics and for potentials in electromagnetism.

## *1) Introduction*

The field scattered by a structure composed of several homogeneous and planar layers on a perfectly reflecting plane [1]-[6] is usually given by its plane wave expansion (Fourier representation). In this expression, the reflection coefficient, which characterizes the structure, is a meromorphic function that can be modelled by a product of elementary reflections coefficients, with constants zeros $g_j$ (and poles $-g_j$), so that the field satisfies a multimode boundary condition, which is the product of elementary ones of impedance type, depending on $g_j$.

In practice, the Fourier expansion is suitable in far field or for simple plane wave illuminations, but is particularly complex to use for non-plane, in particular spherical, incident waves near the scatterer. Indeed, even if double Fourier integrals can be reduced to simple Fourier-Bessel integrals for point source illumination in 3D, numerical integration is quite lengthy because of the highly oscillatory nature of the integral and the calculus of Bessel functions. Moreover, in far field, the steepest descent method (or saddle point method) that is currently used for this integral [1]-[4], leads us to an expansion that is not strictly convergent but asymptotic, and poles of the reflection coefficient near steepest descent path can greatly complicate the calculus.

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So, we here consider another approach in acoustics and in electromagnetism to express the scattered field, developing an original integral representation and its expansions for the contribution of each mode depending on $g_j$. The roots $g_j$ can have positive or negative real parts even if the complete system is passive. Particular attention will be paid to active modes with $\mathrm{Re}g_j < 0$ and to the vicinity of the case $g_j = -1$ which is generally singular. The presence of active modes leads us to develop novel exact expressions for arbitrary $g_j$, for pressure field in acoustics and for potentials in electromagnetism, for arbitrary bounded sources. The analytic method so developed can be applied for the determination of coupling between antennas above an imperfectly reflective plane, or for the calculus of Green's functions for planar lines printed on a multilayer. Other methods that use analytic approximations in some specific cases or some discrete technics exist. So, for the radiation of a point source in 3D above a homogenous passive half-space or a passive monomode impedance plane, we can notice [1]-[4] and [7]-[12], while for determination of Green's functions for planar lines using asymptotics or discrete schemes, we recommend the reading of [5]-[6] and [13]-[16].
The paper is organized as follows. In section 2, we give a brief discussion on general properties of the field, and show how to derive a complex image representation for the one mode acoustic case. We transform the expression in section 3, then obtain a compact form for arbitrary $g_j$, from which we derive rapidly convergent series, and thus simple approximations. We detail in section 4 how to use this development in multimode acoustic case. The electromagnetic case is studied in section 5, where we develop novel compact expression of the potentials, by using the results we obtained in acoustic case.

## *2) Formulation in acoustic case : general properties and elementary problem*

### *2.1) General properties*

We consider the pressure field $u_s$ scattered by an imperfectly reflective plane illuminated by the incident field $u_{inc}$ radiated by a bounded source above the plane. The plane is defined by $z = 0$ in Cartesian coordinates $(x, y, z)$. A harmonic time dependence $e^{i\omega t}$, from now on assumed, is suppressed throughout. To simplify the notation, the field $u$ at the observation point $M(x, y, z)$ is thereafter denoted $u(z)$ or $u(\rho, z)$, $\rho = \sqrt{x^2 + y^2}$.
Some general properties are considered for the field:

(a) $u_s(\rho, z)$ satisfies the Helmholtz equation

$$(\Delta + k^2)u_s = 0 \text{ with } |\arg(ik)| \leq \pi/2, \tag{1}$$

and is regular like $u_{inc}(\rho, -z)$, in the domain $z > 0$;
(b) $u_s(\rho, z)$ is constituted of outgoing waves, and decreases at infinity for large $z$ or $\rho$, so that $u_s(\rho, z) = O(e^{-\delta|OM|})$, $\delta > 0$, as $z$ or $\rho \rightarrow \infty$, when $|\arg(ik)| < \pi/2$.
In other respects, we consider that, for any plane wave of incidence angle $\beta$ composing $u_{inc}$, the reflection coefficient $R(\beta)$ of the imperfectly reflective plane is a meromorphic function (see appendix A) that can be factorized as the product of elementary factors,

$$R(\beta) = \prod_j \frac{\cos\beta - g_j}{\cos\beta + g_j}, \tag{2}$$

attached to roots $\cos\beta = -g_j$ of the characteristic equation of the surface, so that

$$\prod_j \left(\frac{\partial}{\partial z} - ikg_j\right)u_s(z)|_{z=0} = -\prod_j \left(\frac{\partial}{\partial z} - ikg_j\right)u_{inc}(z)|_{z=0} \tag{3}$$

is satisfied [16]-[18]. In this multimode boundary condition, $g_j$ is denoted the impedance parameter of the mode $j$. Developing $R(\beta)$ in simple rational elements,

$$R(\beta) = 1 + \sum_j a_j \frac{1}{\cos\beta + g_j}, \tag{4}$$

we see that the reflection depends on a combination of elementary terms $(\cos\beta + g_j)^{-1}$ only differing in the constant $g_j$. The contribution of this term can be studied if we consider the problem with a one-mode boundary condition and $R(\beta) = 1 - 2g\frac{1}{\cos\beta + g}$ , for arbitrary $g$.
Therefore, we develop the expression of the field satisfying one-mode boundary condition in passive (Re$g > 0$) and active (Re$g < 0$) cases, then we consider its generalization for an arbitrary multimode condition in acoustics and in electromagnetism, respectively in sections 4 and 5.

***2.2) Elementary expression of the field in one-mode case***

The one-mode boundary condition is given by

$$\left(\frac{\partial}{\partial z} - ikg\right)u_s(z)|_{z=0} = -\left(\frac{\partial}{\partial z} - ikg\right)u_{inc}(z)|_{z=0} = \left(\frac{\partial}{\partial z} + ikg\right)u_{inc}(-z)|_{z=0}, \tag{5}$$

This type of condition, also called Robin or impedance boundary condition, is well-known in scattering theory for passive surfaces (Re$g>0$) [19]-[22]. For (5) when Re$g>0$, we can consider like Maliuzhinets in [7], the solution of

$$(\frac{\partial}{\partial z}-ikg)u_s(z)=(\frac{\partial}{\partial z}+ikg)u_{inc}(-z), \tag{6}$$

or

$$e^{ikgz}\frac{\partial}{k\partial z}(e^{-ikgz}u_s(z))=e^{-ikgz}\frac{\partial}{k\partial z}(e^{ikgz}u_{inc}(-z)), \tag{7}$$

which gives us,

$$\begin{aligned} u_s(z) &= \int_{-i\tau\infty}^{z} e^{ikg(z-z_1)}e^{-ikgz_1}\frac{\partial}{k\partial z_1}(e^{ikgz_1}u_{inc}(-z_1))kdz_1 \\ &= u_{inc}(-z)+2ig\int_{-i\tau\infty}^{0} e^{-ikgz_1}u_{inc}(-z_1-z)kdz_1 \end{aligned} \tag{8}$$

This representation, particularly simple, verifies (6) and thus (5), and satisfies the conditions (a) and (b). It has been described by Maliuzhinets in 1948 for Re$(g)>0$, for arg$(ik)=\pi/2$ with $\tau=1$, and is called the complex image expression of the field [2]-[4], [10], [12].

This expression can also be considered for Re$(g)<0$ with a new definition of $\tau\equiv\tau_g$. Radiated by bounded sources, the incident field at $(x,y,-z)$ satisfies $|r_0e^{ikr_0}u_{inc}(-z)|=O(1)$ as $z>0$, $r_0=\sqrt{x^2+y^2+z^2}$, and we can write [1]-[3],

$$\begin{aligned} u_{inc}(-z) &= \int_{\mathcal{D}'}(\int_0^{2\pi} W_0(\beta,\gamma)e^{-ik(x\cos\gamma+y\sin\gamma)\sin\beta}d\gamma)e^{-ikz\cos\beta}\sin\beta d\beta, \\ &= \int_{\mathcal{D}} V_0(\beta)e^{-ikz\cos\beta}d\beta, \end{aligned} \tag{9}$$

when $z>0$. In (9), $W_0$ is the spectrum of the plane wave expansion and $\mathcal{D}'$ is from $0$ to $+i\infty+\text{arg}(ik)$ with Re$(ik\sin\beta)=0$. We can then consider that $\mathcal{D}\equiv\mathcal{D}'$, or, in some conditions of parity, that $\mathcal{D}$ is from $-i\infty-\text{arg}(ik)$ to $+i\infty+\text{arg}(ik)$ with Re$(ik\sin\beta)=0$ (see remark 1). Applying the conditions (6) and (a)-(b), we then obtain

$$\begin{aligned} u_s(z) &= \int_{\mathcal{D}'}(\int_0^{2\pi} W_0(\beta,\gamma)e^{-ik(x\cos\gamma+y\sin\gamma)\sin\beta}d\gamma)\frac{\cos\beta-g}{\cos\beta+g}e^{-ikz\cos\beta}\sin\beta d\beta \\ &= u_{inc}(-z)+2ig\mathcal{I}_g \end{aligned} \tag{10}$$

where

$$\mathcal{I}_g = i\int_{\mathcal{D}} \frac{V_0(\beta)e^{-ikz\cos\beta}}{\cos\beta + g} d\beta, \tag{11}$$

The path $\mathcal{D}$ belongs to the line from $-i\infty - \arg(ik)$ to $+i\infty + \arg(ik)$ with $\mathrm{Re}(ik\sin\beta) = 0$. In consequence, we notice that,

$$\frac{1}{\cos\beta + g} = -i\int_{-i\tau_g\infty}^{0} e^{-ikz_1(\cos\beta + g)} k dz_1, \tag{12}$$

for arbitrary $g$ with $\tau_g = \mathrm{Im}g \neq 0$ when $\arg(ik) = 0$, for $\mathrm{Re}(g) > 0$ with $\tau_g = k^*$ when $|\arg(ik)| \leq \pi/2$, and for $\mathrm{Re}(g) < 0$ as $\pm\,\mathrm{Im}g < 0$ with $\tau_g = \pm\, ik^*$ when $\pm\arg(ik) \geq 0$. Using this expression in (11), we derive, after changing the order of integration,

$$\mathcal{I}_g = \int_{-i\tau_g\infty}^{0} e^{-ikgz_1} u_{inc}(-z_1 - z) k\, dz_1, \tag{13}$$

We notice that the integral term in (13) has poor convergence in the vicinity of $g \sim -1$, and that the parameter $\tau_g$ is not defined for arbitrary $g$ when $\arg(ik) = \pi/2$. Thus, we now seek more suitable expressions for arbitrary $g$ and $k$, for point source illumination.

Remark 1:
For $u_{inc}(z) = \frac{e^{-ikR(z)}}{kR(z)}$ with $R(z) = \sqrt{\rho^2 + (z-h)^2}$, $V_0$ is given by [24, eq. 6.616.2]

$$V_0(\beta) = -ie^{-ikh\cos\beta} J_0(k\rho\sin\beta)\sin\beta, \tag{14}$$

with $\mathrm{Re}(ik\sin\beta) = 0$ on $\mathcal{D}$ from $0$ to $i\infty + \arg(ik)$, or, from parity, $V_0(\beta) = -\frac{i}{2}e^{-ikh\cos\beta} H_0^{(2)}(k\rho\sin\beta)\sin\beta$ if $\mathcal{D}$ is from $-i\infty - \arg(ik)$ to $i\infty + \arg(ik)$.

Remark 2:
The function $\mathcal{I}_g$ is multiform because of the cut due to poles of $(\cos\beta + g)^{-1}$ in (11) that can go through $\mathcal{D}$, and we have to pay attention to the condition (b) on the behaviour of $u_s$ at infinity when we modify the expression of the field.

### *3) Reduction of the integral expression for a point source in one-mode acoustic case*

The incident term $u_{inc}$ is generally a combination of elementary terms $e^{-ikR(z)}/kR(z)$, with $R(z) = \sqrt{(x-x')^2 + (y-y')^2 + (z-z')^2}$, radiated by a monopole at $(x', y', z')$, and we now consider the case of a monopole at $z' = h$ above the plane (figure 1).

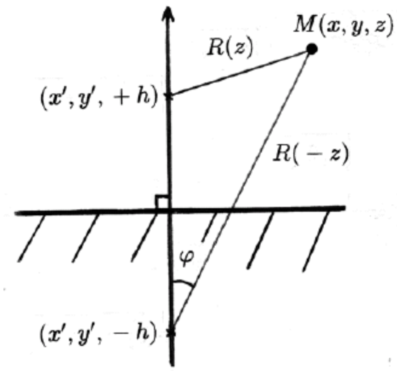


figure 1 : geometry and definition of $\varphi$ for the radiation at $M$

Letting $u_{inc}(z) = e^{-ikR(z)}/kR(z)$ and $\mathcal{I}_g = e^{ikg(z+h)}\mathcal{J}_g(\rho, -z-h)$ in previous expressions, we have

$$u_s(z) = e^{-ikR(-z)}/kR(-z) + 2ige^{ikg(z+h)}\mathcal{J}_g(\rho, -z-h)\,, \tag{15}$$

with

$$\mathcal{J}_g(\rho, -z-h) = \int_{-i\tau_g\infty}^{0} e^{-ikg(z_1+z+h)}\frac{e^{-ikR(-z_1-z)}}{kR(-z_1-z)}k\,dz_1, \tag{16}$$

where $R(-z) = \sqrt{\rho^2 + (z+h)^2}$, $z + h = R(-z)\cos\varphi$, $\rho = R(-z)\sin\varphi$. We now develop different exact expressions to cover any choice of active (Re$g < 0$) and passive (Re$g > 0$) modes, for $g = \sin\theta_1$ with $|\mathrm{Re}(\theta_1)| \leq \pi/2$, as $|\arg(ik)| \leq \pi/2$.
For this, we first explain how to obtain a correct integral form in 3.1. We begin with changing (16), as Maliuzhinets [7], into an expression more simply convergent but rather uneasy to use for Re$g < 0$, then we modify it and define a correct integral form in 3.1.2 for arbitrary $g$. We expand this latter form in different exact series in 3.2, 3.3, and 3.4.

### *3.1) An exact integral expression of $\mathcal{J}_g$ for arbitrary g as $|\arg(ik)| \leq \pi/2$*

#### *3.1.1) Simplication of (16) but difficulty concerning branch cut when Reg $< 0$*

A way to avoid the choice of $\tau$ parameter is to transform the integral (16) by taking $\nu = ik(R(-z_1-z) + g(h+z+z_1))$ as a new variable of integration. We then obtain,

$$\mathcal{J}_g(\rho, -z-h) = -\int_{\nu_z = ikR(-z)(1+g\cos\varphi)}^{\infty} \frac{e^{-\nu}}{\sqrt{\nu^2 - (ikR(-z)\sin\varphi\cos\theta_1)^2}}d\nu, \tag{17}$$

where the path is defined so $\sqrt{\nu_z^2 - (ik\rho\cos\theta_1)^2} = ikR(-z)(\cos\varphi + \sin\theta_1)$, $\nu_z$ being the value of $\nu$ for $z_1 = 0$, $g = \sin\theta_1$.

This expression, found in 1948 by Maliuzhinets for passive case [7] and rediscovered in 1951 by Ingard (see (10) in [8]), is very efficient when $\mathrm{Re}g > 0$. However, several difficulties exist for a correct use when $\mathrm{Re}g < 0$. The path of integration is straight when $\mathrm{Re}g > 0$, but it can turn around the branch points $\nu = \pm ik\rho\cos\theta_1$when $\mathrm{Re}g < 0$. Moreover, the cut $\mathrm{Re}(ik\cos\theta_1) = 0$ of $\mathcal{I}_g$ for $\mathrm{Re}g < 0$, which is due to poles of $(\cos\beta + g)^{-1}$ in (11) that can go through $\mathcal{D}$, does not appear clearly in (17).
Concerning this difficulty with integration path when $\mathrm{Re}g < 0$, we can consider the closed-form expression that we obtain when we let $\sin\varphi\cos\theta_1 = 0$ as $\mathrm{Re}g > 0$ in (17),

$$\mathcal{J}_g = -E_1(ikR(-z)(1 + g\cos\varphi)), \tag{18}$$

where $E_1$ is the exponential integral [25], and remark that this expression, already described in [7] and [11], cannot be used when $g = -1$, or when $\cos\varphi = 1$ if $\mathrm{Re}g < 0$ and $\mathrm{Re}(ik\cos\theta_1) < 0$, even though $\sin\varphi\cos\theta_1 = 0$.

To convince ourself of this, we give in figure 2 an example which shows that (18) for $\cos\varphi = 1$ is false in some region with $\mathrm{Re}g < 0$.

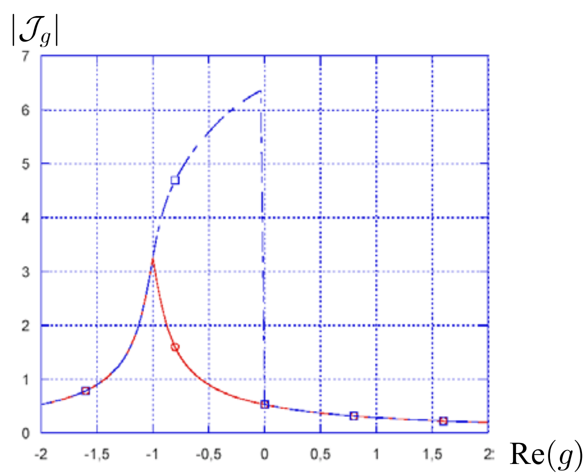


fig. 2) $|\mathcal{J}_g|$ when $\mathrm{Re}g$ varies, $\mathrm{Im}(g) = 0.1$, $\cos\varphi = 1.$, $z + h = 2.$, $k = 1.$, comparison between exact result (dashed line) and the expression (18) given in [7] and [11], incorrect for $\mathrm{Re}g < 0$ (continuous line).

### *3.1.2) A correct definition of the integral expression of $\mathcal{J}_g$ for arbitrary g*

**Proposition 3.1**. *A correct definition of $\mathcal{J}_g$ for arbitrary $g = \sin\theta_1$ is given by*

$$\mathcal{J}_g(\rho, -z-h) = -\int_{-ib}^{\infty} e^{-a\cosh t}dt = i\int_{b}^{i\infty} e^{-a\cos\alpha}d\alpha \tag{19}$$

*where* $a = \epsilon ikR(-z)\mathrm{sin}\varphi\mathrm{cos}\theta_1$, $\epsilon =$ *sign*$(Re(ik\mathrm{cos}\theta_1))$ *(*$Re(a) = 0$ *being considered as a limit case), and* $b$ *satisfies*

$$e^{\mp ib} = c_{\pm} = \frac{ikR(-z)}{a}(1 \pm \mathrm{sin}\theta_1)(1 \pm \mathrm{cos}\varphi), \tag{20}$$

*with* $|\mathrm{Re}b| < \pi$, *and* $e^{-2ib} = \frac{(1+\mathrm{sin}\theta_1)(1+\mathrm{cos}\varphi)}{(1-\mathrm{sin}\theta_1)(1-\mathrm{cos}\varphi)}$.

**Proof.** We now let $a = \epsilon ikR(-z)\mathrm{sin}\varphi\mathrm{cos}\theta_1$, $\epsilon = \mathrm{sign}(\mathrm{Re}(ik\mathrm{cos}\theta_1))$ ($\mathrm{Re}(a) = 0$ being considered as a limit case), $\nu_z = ikR(-z)(1 + \mathrm{sin}\theta_1\mathrm{cos}\varphi)$, and take a new variable of integration in (17), $t$ with $\nu = a\mathrm{cosh}t$, $a\mathrm{sinh}t = \sqrt{\nu^2 - a^2}$, or $\alpha = it$. Defining $b$ with $a\mathrm{cos}b = \nu_z$, $\quad -ia\mathrm{sin}b = a\sqrt{(\nu_z/a)^2 - 1} = ikR(-z)(\mathrm{cos}\varphi + \mathrm{sin}\theta_1)$, $\quad |\mathrm{Re}b| < \pi$, $|\mathrm{Re}(\theta_1)| \leq \pi/2$, $|\mathrm{arg}(ik)| \leq \frac{\pi}{2}$, we then obtain (19) where $b$ satisfies (20).
With this definition, the reader can verify by inspection that the expression of the field,

$$u_s(z) = u_{inc}(-z) - 2ge^{ikgR(-z)\mathrm{cos}\varphi}\int_b^{i\infty} e^{-a\mathrm{cos}\alpha}d\alpha \tag{21}$$

satisfies the conditions (a)-(b) and the boundary condition (5) for any choice of $g = \mathrm{sin}\theta_1$ as $|\mathrm{arg}(ik)| \leq \pi/2$, except for $g = -1$. In this latter case, the expression (21), like (11), is singular. Moreover, we notice that, as $g$ varies in complex plane, this expression has a correct cut as $\epsilon$ changes of sign for $\mathrm{Re}g < 0$, and is regular elsewhere (note: for $\mathrm{Re}g > 0$, the change of sign of $\epsilon$ does not induce a cut as $g$ varies).

This type of expression was previously described for a passive mode with $\mathrm{Re}(ik\mathrm{cos}\theta_1) > 0$ as an approximation for the scattering by the earth [9] ; it was also given in [11] for passive impedance case but it was with a definition of parameters which restricts its application, since, in particular, it gives (18) for $\mathrm{cos}\varphi = 1$, which is false in some region of $g$ with $\mathrm{Re}g < 0$ (see figure 2).
So, to our knowledge, it is the first time that this expression is given with a correct definition of $a$ and $b$ which permits the application for arbitrary $g$ as $|\mathrm{arg}(ik)| \leq \pi/2$.

A general property of the expression (19) is worth noticing. Using the integral expression of the modified Bessel function $K_0$ [25], we can write,

$$\begin{aligned}\mathcal{J}_g(\rho, -z-h) &= -i\int_{-i\infty}^{b} e^{-a\mathrm{cos}\alpha}d\alpha + i\int_{-i\infty}^{+i\infty} e^{-a\mathrm{cos}\alpha}d\alpha \\ &= -i\int_{-b}^{i\infty} e^{-a\mathrm{cos}\alpha}d\alpha - 2K_0(a)\end{aligned} \tag{22}$$

which is equivalent, by definition of $b$ and $a$, to

$$\mathcal{J}_g(\rho, -z-h) = -\mathcal{J}_{-g}(\rho, z+h) - 2K_0(a) \tag{23}$$

This relation between the values of $\mathcal{J}_g$, when we take $(-\theta_1, -\cos\varphi)$ in place of $(\theta_1, \cos\varphi)$, will be useful to derive other expressions of this function.

The figure 3 shows the agreement of $\mathcal{J}_g$ given by (19), and by Fourier-Bessel expansion when (11) is used with (14) [1]-[3].

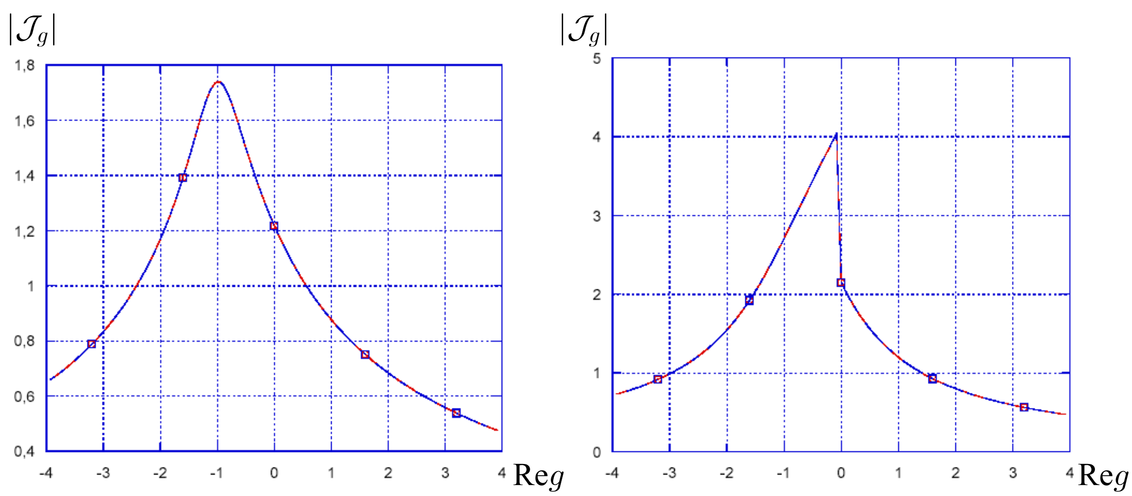


figure 3) Comparison of $\mathcal{J}_g$ given by (19) $(-\square-)$ and by Fourier-Bessel expansion when (11) is used with (14) $(-\circ-)$, when Re$g$ varies; left : $|\mathcal{J}_g|$ when $\mathrm{Im}(g) = -0.4$, $z+h = .2$, $\rho = .3$, $ik = .01 + i1.$; right : $|\mathcal{J}_g|$ when $\mathrm{Im}(g) = 1.2$, $z+h = 1.$, $\rho = 1.$, $ik = .01 + i1.$

Remark 3:

Some particular properties of $b$ are worth pointing out. So, we have

$$\left|\frac{\cos b - \epsilon}{\cos b + \epsilon}\right| = \left|\frac{1+\sin(\theta_1 - \varphi)}{1+\sin(\theta_1+\varphi)}\right| = \left|\frac{e^{\mathrm{Im}\theta_1} e^{i(\frac{\pi}{2} - \mathrm{Re}\theta_i)} + e^{-i\varphi}}{e^{\mathrm{Im}\theta_1} e^{i(\frac{\pi}{2} - \mathrm{Re}\theta_i)} + e^{i\varphi}}\right|^2 \leq 1 \tag{24}$$

where we have used that $|\mathrm{Re}\theta_i| \leq \pi/2$ ($\cos(\mathrm{Re}\theta_i) \geq 0$), $0 \leq \mathrm{Re}\varphi \leq \pi/2$ ($\sin\varphi \geq 0$), and $\epsilon a = ikR(-z)\sin\varphi\cos\theta_1$. We deduce from (24) that $\mathrm{Re}(\epsilon\cos b) \geq 0$, and

$$\frac{1-\epsilon}{4}\pi \leq |\mathrm{Re}b| \leq \frac{3-\epsilon}{4}\pi. \tag{25}$$

We remark that, if $\cos b - \epsilon = 0$, then $\cos\theta_1 = \sin\varphi$ and $\sin\theta_1 = -\cos\varphi$, so that, from $|\arg(ik)| \leq \pi/2$, $\epsilon = 1$ and $b = 0$, and thus $b$ is never equal to $\pm\pi$.

Besides, we notice that $\mathrm{sign}(\epsilon \mathrm{Re} b) = -\mathrm{sign}(\mathrm{Im}(\sin\theta_1))$, from $e^{-2ib} = \frac{(1+\sin\theta_1)(1+\cos\varphi)}{(1-\sin\theta_1)(1-\cos\varphi)}$ and (25). Thus, when $0 < \pm\arg(ik) < \frac{\pi}{2}$, we have $0 < \pm\epsilon\mathrm{Re}(b)\mathrm{sign}(\mathrm{Re}\theta_1) < \pi$ in the region $\pm\mathrm{Re}\theta_1\mathrm{Im}\theta_1 < 0$ where $\epsilon$ changes of sign. In this region, $\pm\epsilon\arg(a) > 0$ so that $\pm\epsilon\arg(a) = \frac{\pi}{2}$ when $\mathrm{Re}a = \epsilon 0^+$, and the zone $-\frac{\pi}{2} + \arg(a) < \mathrm{Re}\alpha < \frac{\pi}{2} + \arg(a)$ of convergence of $e^{-a\cos\alpha}$ when $\mathrm{Im}\alpha \to \infty$ becomes $0 \leq \pm\epsilon\mathrm{Re}\alpha < \pi$ when $\mathrm{Re}a = \epsilon 0^+$. Thus, we can deform the integration path from $i\infty$ to $i\infty \pm \epsilon\pi$ when $\mathrm{Re}\theta_1 > 0$ and obtain the continuity of $\mathcal{J}_g$ as $\epsilon$ changes of sign, while $\mathcal{J}_g$ is discontinuous when $\mathrm{Re}\theta_1 < 0$.

### *3.2) Two exact series for $\mathcal{J}_g$, rapidly convergent when $a$ or $R(-z)$ is small*

We present two exact series for $\mathcal{J}_g$ for arbitrary $g$ in this section. Their speed of convergence grows as $a$ or $R(-z)$ decreases.

#### *3.2.1) An exact series whose first two terms give exact expression for $g = 1$ and $\sin\varphi = 0$*

**Proposition 3.2.** *An exact expansion of $\mathcal{J}_g(\rho, -z-h)$ for arbitrary $g$ is given by,*

$$\mathcal{J}_g(\rho, -z-h) = -E_1(ikR(-z)\frac{(1+\sin\theta_1)(1+\cos\varphi)}{2}) - 1_{\Omega_g}2(K_0(a) - K_0(-a))$$
$$-\sum_{p\geq 1}\frac{(-ikR(-z)\frac{(1-\sin\theta_1)(1-\cos\varphi)}{2})^p}{p!}E_{p+1}(ikR(-z)\frac{(1+\sin\theta_1)(1+\cos\varphi)}{2}) \qquad (26)$$

*where $1_{\Omega_g}$ is the indicator function of the region $\Omega_g$ of the strip $|Re(\theta_1)| < \frac{\pi}{2}$, following*

$$1_{\Omega_g} = \frac{1-\epsilon}{2}U(-Re(\sin\theta_1))U(Re\theta_1 - (-\frac{\pi}{2} + \mathcal{G}(Im\theta_1))), \qquad (27)$$

*with $\mathcal{G}(x) = 2\arctan(\tan(\frac{\arg(ik)}{2})\tanh(\frac{x}{2}))$, $\epsilon = sign(Re(ik\cos\theta_1))$, and $U$ being the unit step function [25].*

**Proof.** We let $t = e^{-i\alpha}$ in (19), and write

$$\mathcal{J}_g(\rho, -z-h) = i\int_b^{i\infty} e^{-a\cos\alpha}d\alpha = -\int_{c_+}^{\infty}\frac{e^{-\frac{at}{2}}e^{-\frac{a}{2t}}}{t}dt \qquad (28)$$

where $a = \epsilon ikR(-z)\sin\varphi\cos\theta_1$, $\epsilon = \mathrm{sign}(\mathrm{Re}(ik\cos\theta_1))$ ($\mathrm{Re}(a) = 0$ being a limit case), $(t + 1/t)|_{t=c_+} = 2\nu_z/a$, $\nu_z = ikR(-z)(1 + \sin\theta_1\cos\varphi)$. We then develop the term $e^{-\frac{a}{2t}}$, and obtain for $\mathrm{Re}(ac_+) > 0$,

$$\mathcal{J}_g = -\sum_{p\geq 0}\frac{(-\frac{ac_-}{2})^p}{p!}\int_1^\infty \frac{e^{-\frac{ac_+}{2}t}}{t^{p+1}}dt = -\sum_{p\geq 0}\frac{(-\frac{ac_-}{2})^p}{p!}E_{p+1}(\frac{ac_+}{2}) \tag{29}$$

which gives us, for $\mathrm{Re}(ik(1+\sin\theta_1)) > 0$,

$$\mathcal{J}_g(\rho, -z-h) = -\sum_{p\geq 0}\frac{(-ikR(-z)\frac{(1-\sin\theta_1)(1-\cos\varphi)}{2})^p}{p!} \times E_{p+1}(ikR(-z)\frac{(1+\sin\theta_1)(1+\cos\varphi)}{2}) \tag{30}$$

where $E_n$ is the exponential integral of order $n$ [25].

To continue this expression for arbitrary $\sin\theta_1$ in complex plane, we need to take some precaution with the cut of (11) and of $E_{p+1}$. Indeed, the pole in the integrand of (11) can change of side with respect to the path of integration $\mathcal{D}$, which defines a cut in the expression of the field with respect to $g$. In (30), we have also a cut which is the one of $E_{p+1}(v)$ when $\mathrm{Im}(v) = 0$, $v < 0$. These cuts are different, but we can deform $\mathcal{D}$ so that the cuts coincide. We then capture the poles (the zeros $\beta = \theta_1 + \pi/2$ of $\cos\beta + \sin\theta_1$) that goes through the path during the deformation, and determine the residue terms that have to be added to (30).

We then obtain the exact expression (26) for arbitrary $g$, where $1_{\Omega_g}$ is the indicator function of the region $\Omega_g$ of the strip $|\mathrm{Re}(\theta_1)| < \frac{\pi}{2}$, on the right side of the path $\mathrm{Im}(ik(1+\sin\theta_1)) = 0$ with $\mathrm{Re}(ik(1+\sin\theta_1)) < 0$, where $\mathrm{Re}(ik\cos\theta_1) < 0$ and $\mathrm{Re}(\sin\theta_1) < 0$. The function $1_{\Omega_g}$ being equal to $1$ in $\Omega_g$ and zero elsewhere, can be expressed following (27). Since $|E_{n+1}(z)| < |E_n(z)|$ as $n$ is large, the series converges everywhere, except on the cut of exponential integral. This expression corresponds to the one given in works of IS Koh and JG Yook [11], except that the term $-1_{\Omega_g}2(K_0(a) - K_0(-a))$ has been added to render (30) valid everywhere.

#### *3.2.2) An exact series whose convergence is particularly rapid in vicinity of* $\mathbf{g = -1}$

**Proposition 3.3.** *A second expression of $\mathcal{J}_g(\rho, -z-h)$ with a better convergence in the vicinity of g $= -1$, is given by*

$$\mathcal{J}_g = E_1(ikR(-z)\frac{(1-\sin\theta_I)(1-\cos\varphi)}{2}) - 2K_0(a) + 1_{\Omega_{-g}}2(K_0(a) - K_0(-a)) + \sum_{p\geq 1}\frac{(-ikR(-z)\frac{(1+\sin\theta_I)(1+\cos\varphi)}{2})^p}{p!}E_{p+1}(ikR(-z)\frac{(1-\sin\theta_I)(1-\cos\varphi)}{2}) \tag{31}$$

**Proof.** For this, we consider (22), using (24) with $-b$, $c_-$ in place of $b$, $c_+$, or directly (26) in (23). This formula is more convergent than previous one (26) when $\mathrm{Im}b < 0$, i.e.

$$|\frac{(1+\sin\theta_1)(1+\cos\varphi)}{(1-\sin\theta_1)(1-\cos\varphi)}| = |\frac{1+\sin\theta_1\cos\varphi+(\sin\theta_1+\cos\varphi)}{1+\sin\theta_1\cos\varphi-(\sin\theta_1+\cos\varphi)}| < 1 \tag{32}$$

which is the case, for example, when $(1+\sin\theta_1\cos\varphi)$ and $(\sin\theta_1+\cos\varphi)$ are real and of opposite sign, and $\mathrm{Re}(\sin\theta_1) < 0$.

Let us notice that, when $\cos\varphi = -\sin\theta_1$, the terms under summation are the same in (26) and (31), and $1_{\Omega_{\pm g}} = 0$. In this particular case, we have $b = 0$, and thus

$$\mathcal{J}_g = i\int_0^{i\infty} e^{-a\cos\alpha}d\alpha = -K_0(a). \tag{33}$$

We give in figure 4, two examples of application of (26) and (31). With only three terms, (26) is better for $\mathrm{Re}(g) > 0$, while, for (31), it is the case for $\mathrm{Re}(g) < 0$.

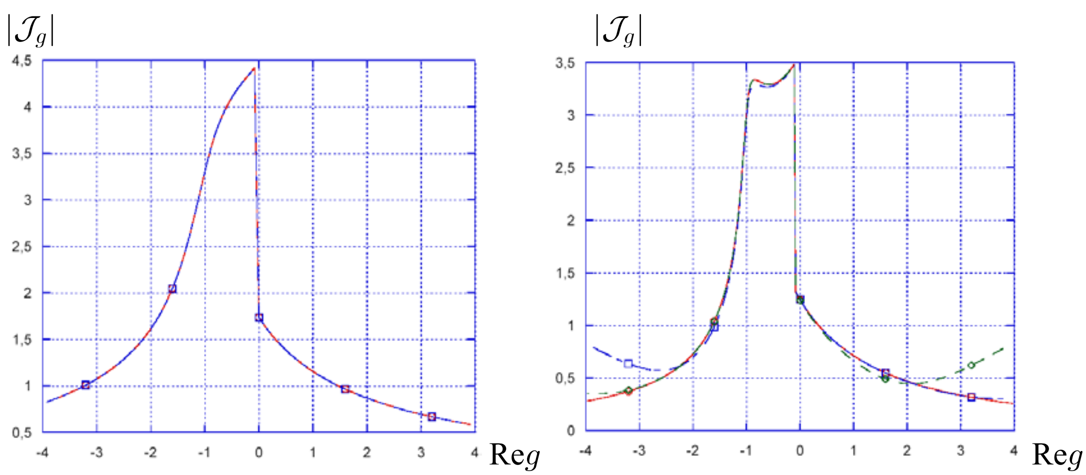


figure 4) Comparison of (26) and (31) and exact result $(-\circ-)$ when Re$g$ varies; left : $|\mathcal{J}_g|$ when $\mathrm{Im}(g) = 0.4$, $z+h = .2$, $\rho = .3$, $ik = .01 + i1$. (with all terms of (26) and (31)); right : $|\mathcal{J}_g|$ when $\mathrm{Im}(g) = 0.1$, $z+h = .02$, $\rho = 1.$, $ik = .01 + i1$. (with sums truncated to $p = 2$ in (26) and (31))

Remark 4 :
From $|\mathrm{Re}\theta_i| \leq \pi/2$, $\mathrm{sign}(\mathrm{Im}(a)) = -\mathrm{sign}(\arg(ik))$ when $\epsilon = -1$, and $\pm i\pi I_0(z) = K_0(z) - K_0(ze^{\pm i\pi})$, we have $\mathrm{sign}(\arg(ik))i\pi I_0(a) = K_0(a) - K_0(-a)$ when $1_{\Omega_g} = 1$. Notice that $K_0(z) = -i\pi H_0^{(2)}(-iz)/2$ when $-\pi/2 < \arg(z) \leq \pi$, $K_0(z) = i\pi H_0^{(1)}(iz)/2$ when $-\pi < \arg(z) \leq \pi/2$, and $I_0(z) = J_0(iz)$.

***3.3) a series rapidly convergent for arbitrary $R(-z)$ when $|\frac{2\sin\varphi\cos\theta_1}{(1+\sin(\theta_1+\varphi))}| < 1$, and asymptotic for large $R(-z)$***

We present an exact series, convergent for arbitrary $R(-z)$ when $|\frac{2a}{\nu_z+\epsilon a}| = |\frac{2\sin\varphi\cos\theta_1}{(1+\sin(\theta_1+\varphi))}| < 1$, which is also asymptotic for large $R(-z)$.

***3.3.1) a first order expression with $E_1(a\cos b - \epsilon a)$***

**Lemma 3.1.** *We can express $\mathcal{J}_g(\rho, -z-h)$ in the form*

$$\mathcal{J}_g = \mathcal{R}(\delta_\epsilon, \epsilon, 1_{\Omega_g^\varphi} w2\pi) - 1_{\Omega_g^\varphi} 2\pi i\delta w I_0(a) \tag{34}$$

*where*

$$\begin{aligned}\mathcal{R}(\delta_\epsilon, \epsilon, s) =& -e^{-\epsilon a} E_1(ikR(-z)(1+\sin(\theta_I-\varphi)))\frac{(1+\sin(\theta_I-\varphi))}{(\cos\varphi+\sin\theta_I)} \\ & -(1-\delta_\epsilon)K_0(a) - i\delta_\epsilon e^{-\epsilon a}\int_{\delta_\epsilon b}^{i\infty+s} \epsilon a(\frac{E_1(a\cos\alpha-\epsilon a)a\sin\alpha}{(a\cos\alpha-\epsilon a)^2})(\frac{\cos\alpha-\epsilon}{\sin\alpha})^3 d\alpha\end{aligned} \tag{35}$$

*with $s = 1_{\Omega_g^\varphi} w2\pi$. In this expression, we have*

$$1_{\Omega_g^\varphi} = \frac{1-\epsilon}{2} U(-Re(\sin\theta_1)) U(Re\theta_1 - \varphi - (-\frac{\pi}{2} + \mathcal{G}(Im\theta_1))), \tag{36}$$

*and $2\delta_\epsilon = (1+\epsilon)\delta_1 + (1-\epsilon)\delta$ with*

$$\delta = sign(\ln|\frac{(1+\sin\theta_1)(1+\cos\varphi)}{(1-\sin\theta_1)(1-\cos\varphi)}|),\ \delta_1 = sign(Re\theta_1 - \varphi - (-\frac{\pi}{2} + \mathcal{G}(Im\theta_1))), \tag{37}$$

*and $\delta w = sign(Re(b))$ which is equal to $sign(\arg(ik))$ in $\Omega_g^\varphi$.*

**Proof.** It is possible to express (19) and (22) in the compact form

$$\mathcal{J}_g(\rho, -z-h) = i\delta\int_{\delta b}^{i\infty} e^{-a\cos\alpha} d\alpha - (1-\delta)K_0(a), \tag{38}$$

with $\delta$ being equal to $+1$ or $-1$. We then choose the parameter $\delta$ in order to keep $\mathrm{Im}\alpha > 0$, and thus $\mathrm{Im}\delta b > 0$, which, from (20), implies

$$\delta = \mathrm{sign}(\ln|\frac{(1+\sin\theta_1)(1+\cos\varphi)}{(1-\sin\theta_1)(1-\cos\varphi)}|). \tag{39}$$

To develop the expression, we consider (24), which indicates that the singularity of $\sqrt{\nu_z^2 - a^2}$ the nearest of $\nu_z = a\cos b$ is $\epsilon a$, and we write,

$$\mathcal{J}_g = i\delta e^{-\epsilon a}\int_{\delta b}^{i\infty} (\frac{e^{-(a\cos\alpha - \epsilon a)}a\sin\alpha}{a\cos\alpha - \epsilon a})\frac{\cos\alpha - \epsilon}{\sin\alpha}d\alpha - (1-\delta)K_0(a) \tag{40}$$

where we have expressed the integrand as the product of two functions, respectively infinite and nul at $\cos\alpha = \epsilon$. We then choose to integrate by parts, restricting ourselves to not cross the cut of the exponential integral $E_1(v)$, $v < 0$, and we obtain

$$\mathcal{J}_g = \mathcal{R}(\delta, \epsilon, s = 0) \tag{41}$$

where

$$\begin{aligned}\mathcal{R}(\delta, \epsilon, s) = & -\delta e^{-\epsilon a}E_1(ikR(-z)(1+\sin(\theta_1 - \varphi)))\frac{(1+\sin(\theta_1-\varphi))}{\delta(\cos\varphi + \sin\theta_1)} \\ & -(1-\delta)K_0(a) - i\delta e^{-\epsilon a}\int_{\delta b}^{i\infty + s}\epsilon a(\frac{E_1(a\cos\alpha - \epsilon a)a\sin\alpha}{(a\cos\alpha - \epsilon a)^2})(\frac{\cos\alpha - \epsilon}{\sin\alpha})^3 d\alpha\end{aligned} \tag{42}$$

However, we can go further and transform the expression of $\mathcal{J}_g$ so that it becomes valid everywhere. For this, we need to distinguish the case $\epsilon = -1$ and the case $\epsilon = 1$.

In the case $\epsilon = -1$, the cuts of $E_1(a\cos\alpha - \epsilon a)$ from $\arg(a) \pm \pi + i\infty$ to $-\arg(a) \pm \pi - i\infty$ are centered on $\alpha = \pm\pi$. The path from $\delta b$ to $i\infty$ goes through the cuts when $\theta_1$ belongs to the domain denoted $\Omega_g^\varphi$ (figure 5). In this case, we then choose a new integral expression where the path of integration, going from $\delta b$ to $i\infty + w2\pi$, avoids to cross the cut. For this, we use that

$$i\delta\int_{\delta b}^{i\infty} e^{-a\cos\alpha}d\alpha = i\delta\int_{\delta b}^{i\infty + w2\pi} e^{-a\cos\alpha}d\alpha - 2\pi i\delta w I_0(a) \tag{43}$$

where $w = \text{sign}(\text{Re}(\delta b))$, and we obtain,

$$\mathcal{J}_g = \mathcal{R}(\delta, \epsilon, 1_{\Omega_g^\varphi}w2\pi) - 1_{\Omega_g^\varphi}2\pi i\delta w I_0(a) \tag{44}$$

as $\epsilon = -1$, where $1_{\Omega_g^\varphi}$ is the indicator function, equal to $1$ in $\Omega_g^\varphi$ (figure 5) and being nul elsewhere. The domain $\Omega_g^\varphi$ is a part of the region with $\text{Re}(\sin\theta_1) < 0$ and $\epsilon = -1$ which is limited by the curve $\text{Im}(\nu_z - \epsilon a) = 0$, $\text{Re}(\nu_z - \epsilon a) < 0$, so that (36) applies.

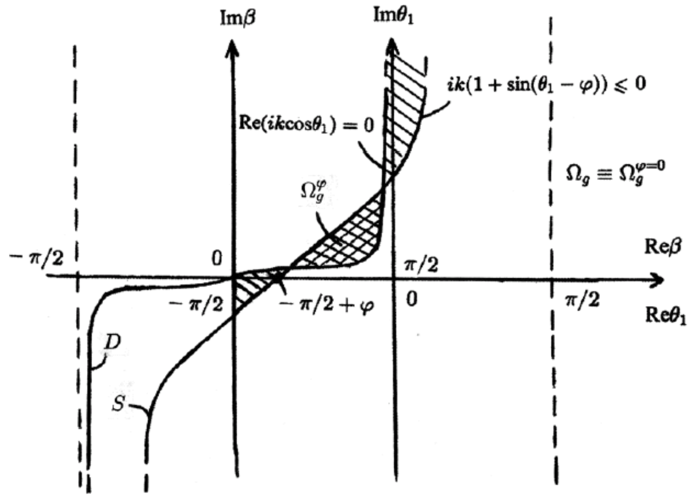


figure 5 : definition of $\Omega_g^{\varphi}$

In the case $\epsilon = 1$, the cut of $E_1(a\cos\alpha - \epsilon a)$ goes from $-\pi + \tau(\arg(a) + i\infty)$ to $\pi - \tau(i\infty + \arg(a))$ with $\tau = \text{sign}(\arg(a))$. To avoid the cut in this case, $\delta b$ has to be above it, so that we need to modify the definition of $\delta$. We then take, instead of $\delta$,

$$\delta_1 = \text{sign}(\text{Re}\theta_1 + \frac{\pi}{2} - \varphi - \mathcal{G}(\text{Im}\theta_1))) \tag{45}$$

Thus, we obtain in general (34), where $\delta w = \text{sign}(\arg(ik))$ in $\Omega_g^{\varphi}$ from remark 3. Let us notice that the integral term in the expression of $\mathcal{R}$ vanishes as $a = 0$, except in the case $b = 0$ where (33) applies.

### *3.3.2) an expression of arbitrary order with $E_p(a\cos b - \epsilon a)$, $p \leq n$*

**Proposition 3.4.** *The function $\mathcal{J}_g(\rho, -z-h)$ can be developed following,*

$$\mathcal{J}_g = -e^{-ikR(-z)\sin\varphi\cos\theta_1} E_1(ikR(-z)(1+\sin(\theta_1-\varphi)))\frac{(1+\sin(\theta_1-\varphi))}{\cos\varphi+\sin\theta_1}$$
$$-(1-\delta_\epsilon)K_0(a) - 1_{\Omega_g^{\varphi}} 2\pi i\, sign(\arg(ik)) I_0(a) - \sum_{p=1}^{n-1} h_p - H_n \tag{46}$$

*where $h_p$ and $H_n$ are, respectively, the term of the series,*

$$h_p = (\frac{2\sin\varphi\cos\theta_1}{1+\sin(\theta_1+\varphi)})^p \frac{(1\times ...\times(2p-1))}{2^p p!} \times$$
$$\times e^{-ikR(-z)\sin\varphi\cos\theta_1} v_{p+1}(ikR(-z)(1+\sin(\theta_1-\varphi)))(\frac{1+\sin(\theta_1-\varphi)}{\cos\varphi+\sin\theta_1}) \tag{47}$$

*and the remaining integral term,*

$$H_n = -i\epsilon^n e^{-ikR(-z)\sin\varphi\cos\theta_1}\delta_\epsilon \frac{(1\times\ldots\times(2n-1))}{(n-1)!}\int_{\delta_\epsilon b}^{i\infty+s}\frac{v_n(a(\cos\alpha-\epsilon))}{(\cos\alpha+\epsilon)^n}d\alpha \quad (48)$$

*where* $\Omega_g^\varphi$*, s, and* $\delta_\epsilon$ *are defined in previous lemma and figure 5. The function* $v_n$*, detailed in appendix B, is given by* $v_n(t) = \sum_{m=0}^{n-1}\frac{(-1)^m(n-1)!}{m!(n-1-m)!}E_{m+1}(t)$.

*When* $|\frac{2\sin\varphi\cos\theta_1}{(1+\sin(\theta_1+\varphi))}| < 1$ *and as* $n\to\infty$*, Im(*$\delta_\epsilon b$*)* $>0$ *and the term* $H_n$ *vanishes, and the expansion becomes an absolutely convergent series. Moreover, for large* $a = \epsilon ikR(-z)\sin\varphi\cos\theta_1$*,* $a^n H_n$ *is small, and the expansion is asymptotic, except when* $b=0$ *and* $\cos\varphi+\sin\theta_1 = 0$*.*

**Proof.** We continue to iterate the integration by parts in (34). For this, we notice that

$$\frac{\partial}{\partial\tau}\left(\frac{(\tau-\epsilon)^c}{(\tau+\epsilon)^c}\right) = \frac{2\epsilon c}{(\tau-\epsilon)^2}\frac{(\tau-\epsilon)^{c+1}}{(\tau+\epsilon)^{c+1}} \quad (49)$$

and define the function $v_n$ satisfying

$$v_n(z) = \int_z^\infty \frac{e^{-t}}{t^n}(t-z)^{n-1}dt = \sum_{m=0}^{n-1}\frac{(-1)^m(n-1)!}{m!(n-1-m)!}E_{m+1}(z) \quad (50)$$

where $|\arg z| < \pi$. After $n$ integration by parts, we obtain (46), where $a = \epsilon ikR(-z)\sin\varphi\cos\theta_1$, $\epsilon = \mathrm{sign}(\mathrm{Re}(ik\cos\theta_1))$, $e^{\mp ib} = \frac{ikR(-z)}{a}(1\pm\sin\theta_1)(1\pm\cos\varphi)$, $\delta w = \mathrm{sign}(\mathrm{Re}(b)) = \mathrm{sign}(\arg(ik))$ in $\Omega_g^\varphi$, $s = 1_{\Omega_g^\varphi}w2\pi$, $2\delta_\epsilon = (1+\epsilon)\delta_1 + (1-\epsilon)\delta$.

The function $v_n$ has remarkable properties (see appendix B), in particular,

$$\begin{aligned} v_n(z) &= \frac{e^{-z}(n-1)!}{z^n}(1+O(\frac{1}{z})) \text{ as } z \text{ is large, } n \text{ fixed} \\ v_n(z) &\sim 2e^{-z}(e^{z/2}K_0(2\sqrt{nz})) \text{ as } n \text{ is large, } z \text{ fixed} \\ v_n(z) &= -\ln z - 2\gamma - \Psi(n) + O(z\ln z) \text{ as } z \text{ is small} \end{aligned} \quad (51)$$

Consequently, $v_{p+1}(z)$ is bounded as $p$ increases and $z$ is fixed, so that the series is absolutely convergent as

$$|\frac{2\sin\varphi\cos\theta_1}{(1+\sin(\theta_1+\varphi))}| = |\frac{2a}{\nu_z+\epsilon a}| = \frac{2}{|\cos b+\epsilon|} < 1 \quad (52)$$

This is confirmed by the fact that the remainder integral term $H_n$ is $O(\frac{(2n)!}{2^{2n}n!(n-1)!}(\frac{2}{\cos b+\epsilon})^n)$ as $n$ increases, since $\mathrm{Im}(\delta_\epsilon b) > 0$ when (52) is satisfied.

In other respects, when $a$ is large and $b \neq 0$, we can consider the asymptotic behaviour of $v_n(z)$ as $z$ becomes large and $n$ is fixed. In this case, $h_{n-1}$ is $O(\frac{1}{a^n})$ and the remainder $H_n$ is $O(\frac{1}{a^{n+1}})$ if $\cos b \neq \epsilon$. Thus, $a^n H_n$ is small which implies that the expansion is also asymptotic for large $a$, except when $b = 0$ and $\cos\varphi + \sin\theta_1 = 0$ (see remark 3).

We give in figure 6, two examples of application of (46). Notice the convergence when (52) is satisfied, and a correct asymptotic behaviour, except when $\cos\varphi + \sin\theta_1 \sim 0$.

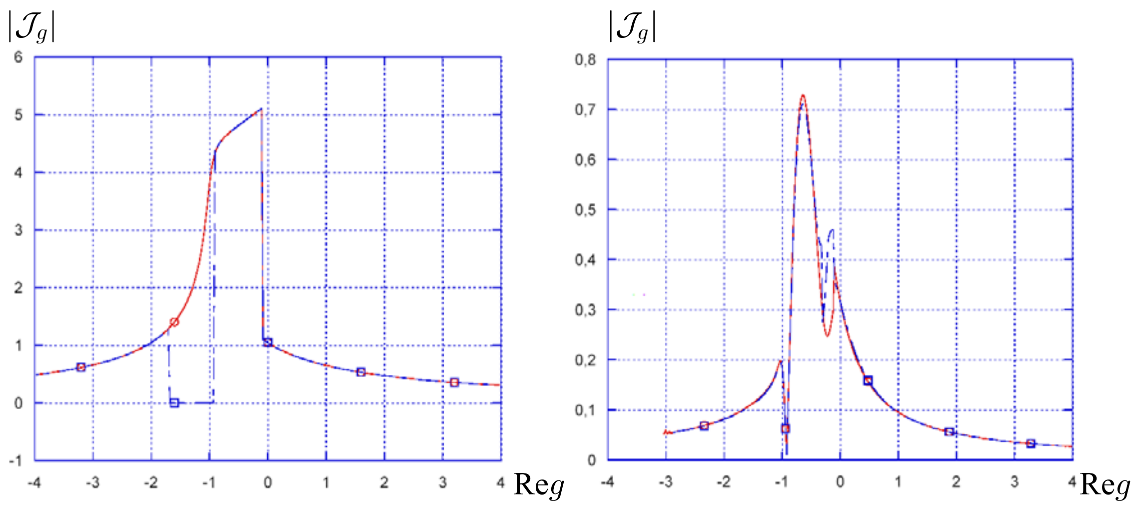


figure 6) comparison of (46) with exact result ( $-\circ-$ ) when Re$g$ varies. left : $|\mathcal{J}_g|$ when $\mathrm{Im}(g) = 0.1$, $z + h = .6$, $\rho = .2$, $ik = .01 + i1$. (with $n \to \infty$ and $H_n = 0$ in (46) / convergence except when $|\cos b + \epsilon| < 2$); right : $|\mathcal{J}_g|$ when $\mathrm{Im}(g) = 0.1$, $z + h = 3.$, $\rho = 9.$, $ik = .01 + i1$. (with only $n = 2$ and $H_n = 0$ / asymptotic except in the vicinity of $\sin\theta_1 = -\cos\varphi$)

### *3.4) a series rapidly convergent for arbitrary $R(-z)$ when $|\frac{(1+\sin(\theta_1+\varphi))}{2\sin\varphi\cos\theta_1}| < 1$*

To complete the expression (46) given in previous section, we need some expansion of (19) rapidly convergent for arbitrary $R(-z)$ when $|\nu_z + \epsilon a| < 2|a|$, or better still, in the larger domain $|\nu_z - \epsilon a| < 2|a|$, since, from (24), $|\nu_z - \epsilon a| \leq |\nu_z + \epsilon a|$. We then choose to develop the expression when $|\cos b - \epsilon| < 2$ and $b$ is in a vicinity of 0 or $\pm\pi$.

**Proposition 3.5.** *When $|\frac{\nu_z - \epsilon a}{2a}| = |\frac{\cos b - \epsilon}{2}| < 1$, and thus as $|\frac{\nu_z + \epsilon a}{2a}| = |\frac{(1+\sin(\theta_1+\varphi))}{2\sin\varphi\cos\theta_1}| < 1$, two convergent expansions of (19) apply. If $\epsilon = 1$, we are in vicinity of $b = 0$ and we can write*

$$\mathcal{J}_g(\rho,\,-z-h) = i(\int_0^{i\infty} e^{-a\cos\alpha}d\alpha - \int_0^b e^{-a\cos\alpha}d\alpha)$$
$$= -(K_0(a) + i\int_0^b e^{-a\cos\alpha}d\alpha), \tag{53}$$

*where,*

$$\int_0^b e^{-a\cos\alpha}d\alpha = \sum_{m\geq 0}\frac{ic_m}{(2a)^{m+1/2}}e^{-a}\gamma(m+1/2,(-i\sqrt{2a}\sin(b/2))^2) \tag{54}$$

*while, if $\epsilon = -1$, in vicinity of $b = v\pi$, $v = sign(Reb)$, we have*

$$\mathcal{J}_g(\rho,\,-z-h) = i(\int_{v\pi}^0 e^{-a\cos\alpha}d\alpha + \int_0^{i\infty} e^{-a\cos\alpha}d\alpha - \int_{v\pi}^b e^{-a\cos\alpha}d\alpha)$$
$$= -(i\pi vI_0(a) + K_0(a) + i\int_{v\pi}^b e^{-a\cos\alpha}d\alpha), \tag{55}$$

*where,*

$$\int_{v\pi}^b e^{-a\cos\alpha}d\alpha = -v\sum_{m\geq 0}\frac{(-1)^m c_m}{(2a)^{m+1/2}}e^{a}\gamma(m+1/2,(\sqrt{2a}\cos(b/2))^2) \tag{56}$$

*In these expressions, $c_m = (-1)^m(2m)!/((m!)^2 2^{2m})$ is the binomial coefficient of the function $(1+t)^{-1/2}$ for $|t|<1$, and $\gamma(m+1/2,z)$ is the incomplete gamma function [25, p.262], related to error function $erf(x)$ by*

$$\gamma(1/2,x^2) = \sqrt{\pi}erf(x),\ \gamma(\alpha+1,x^2) = \alpha\gamma(\alpha,x^2) - x^{2\alpha}e^{-x^2}. \tag{57}$$

**Proof.** In the case $\epsilon = 1$, we use $\cos\alpha = 1 - 2(\sin(\alpha/2))^2$, and obtain, when $|\cos b - 1| < 2$,

$$\int_0^b e^{-a\cos\alpha}d\alpha = \int_0^{\alpha=b}\frac{ie^{-a\cos\alpha}}{-i\sin(\alpha/2)(1+(-i\sin(\alpha/2))^2)^{1/2}}d(-i\sin(\alpha/2))^2$$
$$= \int_0^{(-i\sin(b/2))^2}\frac{ie^{-a}e^{-2at}}{t^{1/2}(1+t)^{1/2}}dt = \sum_{m\geq 0} ic_m e^{-a}\int_0^{(-i\sin(b/2))^2} e^{-2at}t^{m-1/2}dt$$

while, when $\epsilon = -1$, we use $\cos\alpha = 2(\cos(\alpha/2))^2 - 1$, and obtain, when $|\cos b + 1| < 2$,

$$\int_{v\pi}^{b} e^{-a\cos\alpha} d\alpha = -\int_{\alpha=v\pi}^{\alpha=b} \frac{v e^{-a\cos\alpha}}{\cos(\alpha/2)(1-(\cos(\alpha/2))^2)^{1/2}} d(\cos(\alpha/2))^2$$
$$= -\int_0^{(\cos(b/2))^2} \frac{v e^{a} e^{-2at}}{t^{1/2}(1-t)^{1/2}} dt = -v\sum_{m\geq 0} c_m e^{a} \int_0^{(\cos(b/2))^2} e^{-2at} \frac{(-t)^m}{t^{1/2}} dt$$

with $v = \text{sign}(\text{Re}b)$, which gives us (54) and (56).

We give in figure 7, two examples of application of (53)-(56). We then notice the excellent convergence when $|\nu_z - \epsilon a| < 2|a|$.

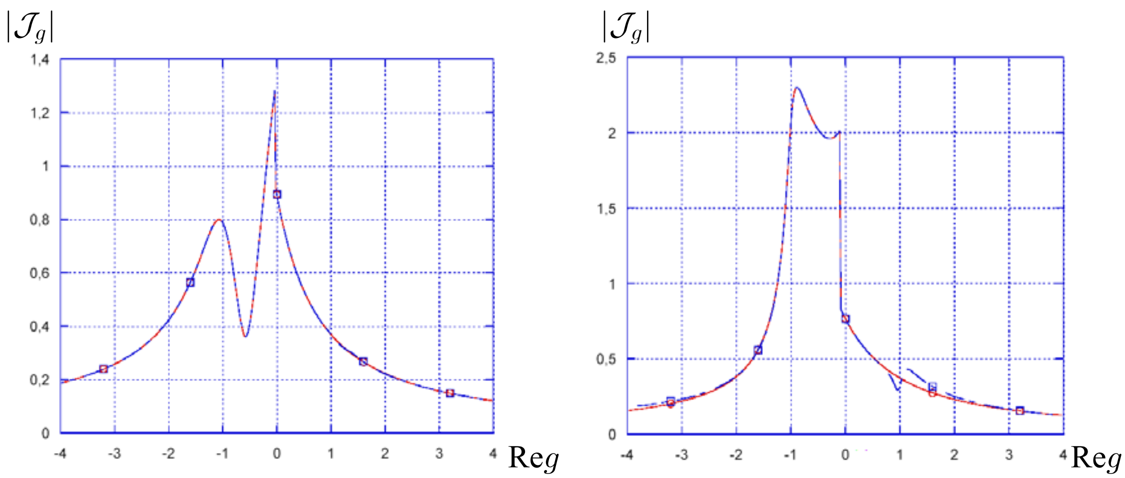


figure 7) comparison of (53)-(56) with exact result $(-\circ-)$. left : $|\mathcal{J}_g|$ when $\text{Im}(g) = 0.4$, $z+h = 2.$, $\rho = 3.$, $ik = .01 + i1.$ (with all terms in (53)-(56)), right : $|\mathcal{J}_g|$ when $\text{Im}(g) = 0.1$, $z+h = .6$, $\rho = 1.8$, $ik = .01 + i1.$ (with sums truncated to $m = 2$ in (53)-(56))

Remark 5 :
We can truncate the series for $(1+t)^{-1/2}$, noticing that, for $|t| < 1$,

$$(1+t)^{-1/2} = \sum_{m=0}^{N-1} c_m t^m + (-1)^N \frac{2}{\pi} \int_0^{\pi/2} \frac{(t\sin^2 x)^N}{1+t\sin^2 x} dx \tag{58}$$

where the integral term is $O(\frac{t^N}{(1+t)^{1/2}})$.

***4) The generalization to a multimode boundary condition in acoustics***

Among the zeros $g_j$ of the reflection coefficient used in (2) and (3), a finite number can be with nul real part when there is no loss, while the ones with large real parts, which have small influence on the field, can be neglected [16]-[18].

Therefore, we truncate the boundary condition (3) at the order $N$ with $|\mathrm{Re}(g_{j+1})| > |\mathrm{Re}(g_j)|$, and consider that,

$$\prod_{j=1}^{N}(\frac{\partial}{\partial z} - ikg_j)u_s(z)|_{z=0} = -\prod_{j=1}^{N}(\frac{\partial}{\partial z} - ikg_j)u_{inc}(z)|_{z=0} \tag{59}$$

The reflection coefficient is then given by

$$R(\varphi) = \frac{\cos\varphi - Z_r(\varphi)}{\cos\varphi + Z_r(\varphi)} = \prod_{j=1}^{N}\frac{\cos\varphi - g_j}{\cos\varphi + g_j} \tag{60}$$

where $\varphi$ is the angle with the normal $\widehat{z}$, and $|R(\varphi)| \leq 1$ for real $\varphi$ if the plane is passive. The sign of $\mathrm{Re}g_j$ can be positive or negative when $\mathrm{Re}g_j \neq 0$. Indeed, let us consider a common physical example without loss with $\arg(ik) = \pi/2$, where $|R(\varphi)| = 1$ and $Z_r(\varphi)$ is purely imaginary for real $\varphi$, and $Z_r(\varphi)^* = -Z_r(\varphi^*)$ (see appendix A). In this case, we have $R(\varphi) = (R(\varphi^*)^{-1})^*$, and thus $g_j$ and $(-g_j)^*$ are zeros of $R(\varphi)$. These zeros have real parts of opposite sign when $\mathrm{Re}g_j \neq 0$, which confirms our assertion.

***4.1) An expression of the solution***

**Proposition 4.1.** *The field scattered by a multimode plane illuminated by a monopole, satisfying the condition (a)-(b) and the boundary condition (59), when* $u_{inc}(z) = e^{-ikR(z)}/kR(z)$ *with* $R(z) = \sqrt{\rho^2 + (z-h)^2}$, $|\arg(ik)| \leq \pi/2$*, is given by*

$$u_s(z) = u_{inc}(-z) - i\sum_{j=1}^{N} a_j e^{ikg_j(z+h)}\mathcal{J}_{g_j}(\rho, -z-h) \tag{61}$$

*where the coefficients* $a_j$ *satisfy*

$$\prod_{j=1}^{N}\frac{\cos\beta - g_j}{\cos\beta + g_j} = 1 + \sum_{j=1}^{N} a_j \frac{1}{\cos\beta + g_j} \quad , \quad \frac{a_j}{2g_j} = -\prod_{i \neq j}^{N} \frac{g_j + g_i}{g_j - g_i}. \tag{62}$$

**Proof.** As in (9) of section 2, we consider the representation of $u_{inc}(-z)$ for bounded sources, for $z > 0$,

$$u_{inc}(-z) = \int_{\mathcal{D}} V_0(\beta)e^{-ikz\cos\beta}d\beta. \tag{63}$$

We first let $\arg(ik) = 0$, $\mathrm{Im}g_j \neq 0$. Using (59) and (62), we obtain

$$\begin{aligned} u_s(z) &= \int_{\mathcal{D}} V_0(\beta) e^{-ikz\cos\beta} \prod_{j=1}^{N} \frac{\cos\beta - g_j}{\cos\beta + g_j} d\beta \\ &= \int_{\mathcal{D}} V_0(\beta) e^{-ikz\cos\beta} d\beta + \sum_{j=1}^{N} a_j \int_{\mathcal{D}} \frac{V_0(\beta) e^{-ikz\cos\beta}}{\cos\beta + g_j} d\beta \end{aligned} \tag{64}$$

then, writing,

$$\frac{1}{\cos\beta + g_j} = -i \int_{-i\tau_j\infty}^{0} e^{-ikz_j(\cos\beta + g_j)} k dz_j \tag{65}$$

with $\tau_j = \mathrm{Im} g_j$, we derive

$$u_s(z) = u_{inc}(-z) - \sum_{j=1}^{N} \frac{a_j}{2g_j} (2ig_j \int_{-i\tau_j\infty}^{0} e^{-ikg_j z_j} u_{inc}(-z_j - z) k\, dz_j). \tag{66}$$

As expected, the integral terms in (66) are similar to the one in (8) for single mode case. For an illumination by a monopole at $(x', y', z' = h)$, when $u_{inc}(-z) = e^{-ikR(-z)}/kR(-z)$ with $R(-z) = \sqrt{\rho^2 + (z+h)^2}$, these terms are given by $e^{ikg(z+h)} \mathcal{J}_g(\rho, -z-h)$ for $g = g_j$, which has been studied in section 3 for arbitrary $g$ and $k$, so that we can write (61) in general for $|\arg(ik)| \leq \pi/2$.
This expression can be used, by superposition principle, for arbitrary sources which are combinations of monopoles.

***5) The generalization to electromagnetic case with multimode boundary conditions***

The problem is more complex in electromagnetism because of the polarization of the field. However, it can be reduced for a large class of multimode boundary conditions. Following R.F. Harrington [23, p.131] in 1961 (see also [1, p.19] in 1964), we can write the electric field $E$ and the magnetic field $H$ satisfying the Maxwell equations, with two scalar potentials $\mathcal{E}$ and $\mathcal{H}$, following

$$\begin{aligned} E &= -ik\mathrm{rot}(\mathcal{H}\widehat{z}) + (\mathrm{grad}(\mathrm{div}(.)) + k^2)(\mathcal{E}\widehat{z}) \\ \sqrt{\frac{\mu_0}{\epsilon_0}} H &= ik\mathrm{rot}(\mathcal{E}\widehat{z}) + (\mathrm{grad}(\mathrm{div}(.)) + k^2)(\mathcal{H}\widehat{z}) \end{aligned} \tag{67}$$

where $(\Delta + k^2)\mathcal{E} = 0$ and $(\Delta + k^2)\mathcal{H} = 0$ outside the sources, with $k = \omega\sqrt{\mu_0\epsilon_0}$, the constants $\epsilon_0$ and $\mu_0$ being respectively the permittivity and the permeability of the medium above the plane. These equations, developed, give us

$$E_x = \frac{\partial^2 \mathcal{E}}{\partial x \partial z} - ik\frac{\partial \mathcal{H}}{\partial y}\,, \ \ \sqrt{\frac{\mu_0}{\epsilon_0}}H_x = \frac{\partial^2 \mathcal{H}}{\partial x \partial z} + ik\frac{\partial \mathcal{E}}{\partial y},$$
$$E_y = \frac{\partial^2 \mathcal{E}}{\partial y \partial z} + ik\frac{\partial \mathcal{H}}{\partial x}\,, \ \ \sqrt{\frac{\mu_0}{\epsilon_0}}H_y = \frac{\partial^2 \mathcal{H}}{\partial y \partial z} - ik\frac{\partial \mathcal{E}}{\partial x},$$
$$E_z = \frac{\partial^2 \mathcal{E}}{\partial z^2} + k^2\mathcal{E}\,, \ \ \sqrt{\frac{\mu_0}{\epsilon_0}}H_z = \frac{\partial^2 \mathcal{H}}{\partial z^2} + k^2\mathcal{H}. \tag{68}$$

Thereafter, we denote $(\mathcal{E}_{inc}, \mathcal{H}_{inc})$ and $(\mathcal{E}_s,\mathcal{H}_s)$ the potentials corresponding to the incident field and the scattered field, and write $(E,\sqrt{\frac{\mu_0}{\epsilon_0}}H)(z) = \mathcal{L}(\widehat{z}\mathcal{E},\widehat{z}\mathcal{H})(z)$.

We then consider the class of multimode boundary conditions on an isotropic plane,

$$\prod_{j=1}^{N}(\frac{\partial}{\partial z} - ikg_j^e)E_{z,tot}|_{z=0} = 0,$$
$$\prod_{j=1}^{P}(\frac{\partial}{\partial z} - ikg_j^h)H_{z,tot}|_{z=0} = 0, \tag{69}$$

which corresponds to the reflection coefficients for the principal polarizations $TM$ (components of electric field $E$ in the plane of incidence) and $TE$ (components of magnetic field $H$ in the plane of incidence), given by,

$$R_{TM}(\varphi) = \prod_{j=1}^{N} \frac{\cos\varphi - g_j^e}{\cos\varphi + g_j^e},$$
$$R_{TE}(\varphi) = \prod_{j=1}^{P} \frac{\cos\varphi - g_j^h}{\cos\varphi + g_j^h}, \tag{70}$$

where $\varphi$ is the angle of observation with the normal $\widehat{z}$ [1]-[3]. This class of problem corresponds to the reflection by a substrate with different layers composed of isotropic media, or more generally, of uniaxial anisotropic media with principal axis along $z$ [1]-[3]. From the symmetry at normal incidence, we notice that the condition

$$R_{TE}(0) = \,-\,R_{TM}(0) \ \text{ i.e. } \prod_{j=1}^{N}\frac{1-g_j^e}{1+g_j^e} = \,-\prod_{j=1}^{P}\frac{1-g_j^h}{1+g_j^h}, \tag{71}$$

is satisfied, which implies, for monomode conditions ($N = P = 1$), that $g_1^e = 1/g_1^h$. Following the boundary conditions on the field, we have the conditions

$$(\frac{\partial^2}{\partial z^2}+k^2)\prod_{j=1}^{N}(\frac{\partial}{\partial z}-ikg_j^e)\mathcal{E}_s(z)=(\frac{\partial^2}{\partial z^2}+k^2)\prod_{j=1}^{N}(\frac{\partial}{\partial z}+ikg_j^e)\mathcal{E}_{inc}(-z)$$
$$(\frac{\partial^2}{\partial z^2}+k^2)\prod_{j=1}^{P}(\frac{\partial}{\partial z}-ikg_j^h)\mathcal{H}_s(z)=(\frac{\partial^2}{\partial z^2}+k^2)\prod_{j=1}^{P}(\frac{\partial}{\partial z}+ikg_j^h)\mathcal{H}^{inc}(-z) \quad (72)$$

for $z \geq 0$. We can then choose to search the two scalar potentials $\mathcal{E}_s$ and $\mathcal{H}_s$, satisfying the Helmholtz equation as $z > 0$, regular and vanishing as $z \to \infty$ when $|\arg(ik)| < \pi/2$, which satisfy

$$\prod_{j=1}^{N}(\frac{\partial}{\partial z}-ikg_j^e)\mathcal{E}_s(z)=\prod_{j=1}^{N}(\frac{\partial}{\partial z}+ikg_j^e)\mathcal{E}_{inc}(-z)$$
$$\prod_{j=1}^{P}(\frac{\partial}{\partial z}-ikg_j^h)\mathcal{H}_s(z)=\prod_{j=1}^{P}(\frac{\partial}{\partial z}+ikg_j^h)\mathcal{H}_{inc}(-z) \quad (73)$$

Consequently, we can use an approach similar to the one previously used in acoustics, with

$$u \leftrightarrow \mathcal{E} \text{ with } g_j \leftrightarrow g_j^e,\ u \leftrightarrow \mathcal{H} \text{ with } g_j \leftrightarrow g_j^h \quad (74)$$

in (66), with scattered field components verifying the conditions (a)-(b), if we have a correct definition of the scalar potentials $(\mathcal{E}_{inc},\ \mathcal{H}_{inc})$. We then study the expression of the potentials $(\mathcal{E}_{inc},\ \mathcal{H}_{inc})$ attached to the incident field radiated by an electric or a magnetic dipole of arbitrary orientation.

### *5.1) A new expression of potentials $(\mathcal{E}_{inc},\mathcal{H}_{inc})$ for arbitrary source*

We begin with searching $(\mathcal{E}_{inc},\mathcal{H}_{inc})$ for arbitrary dipole, reducing the expression to a finite combination of known special functions, then we express potentials for any source.

#### *5.1.1) An artificial parameter g to reduce the determination of $(\mathcal{E}_{inc},\mathcal{H}_{inc})$*

Let us consider the incident field $(E, H)$ at $r$ of coordinates $(x, y, z)$, radiated by an electric or magnetic dipole source, $J = J_0\delta(r - r')$ or $M = M_0\delta(r - r')$, of arbitrary direction at $r'$ of coordinates $(x', y',\ z')$, where we first take $z' = 0$ to simplify. We have, from appendix D,

$$E = \mathrm{rot}(G{*}M) + \frac{i}{\omega\epsilon_0}(\mathrm{grad}(\mathrm{div}(.)) + k^2)(G{*}J)$$
$$\sqrt{\frac{\mu_0}{\epsilon_0}}H = -\sqrt{\frac{\mu_0}{\epsilon_0}}\mathrm{rot}(G{*}J) + \frac{i}{k}(\mathrm{grad}(\mathrm{div}(.)) + k^2)(G{*}M) \qquad (75)$$

where $G = -\frac{e^{-ik\sqrt{\rho^2+z^2}}}{4\pi\sqrt{\rho^2+z^2}}$ with $\rho = \sqrt{x^2+y^2}$, and $*$ is the convolution product.

We search the scalar potentials $\mathcal{E}_{inc}$ and $\mathcal{H}_{inc}$, satisfying the Helmholtz equation as $\pm z > 0$ and vanishing at infinity when $|\arg(ik)| < \pi/2$ when $\pm z \to \infty$, which gives us

$$E = (\mathrm{grad}(\mathrm{div}(.)) + k^2)(\mathcal{E}_{inc}\widehat{z}) - ik\mathrm{rot}(\mathcal{H}_{inc}\widehat{z})$$
$$\sqrt{\frac{\mu_0}{\epsilon_0}}H = ik\mathrm{rot}(\mathcal{E}_{inc}\widehat{z}) + (\mathrm{grad}(\mathrm{div}(.)) + k^2)(\mathcal{H}_{inc}\widehat{z}). \qquad (76)$$

To simplify, we first take $M = 0$. We choose to use an original approach in the sense of the limit. So, we introduce an artificial parameter $g$, and consider the equations

$$E_z(z) = \frac{i}{\omega\epsilon_0}\widehat{z}\,(\mathrm{grad}(\mathrm{div}(.)) + k^2)(J){*}G = (\frac{\partial^2}{\partial z^2} + (kg)^2)\mathcal{E}_g(z),$$
$$\sqrt{\frac{\mu_0}{\epsilon_0}}\,H_z(z) = -\sqrt{\frac{\mu_0}{\epsilon_0}}\,\widehat{z}\,\mathrm{rot}(J){*}G = (\frac{\partial^2}{\partial z^2} + (kg)^2)\mathcal{H}_g(z), \qquad (77)$$

with $(\Delta + k^2)(\mathcal{E}_g, \mathcal{H}_g) = 0$, that $\mathcal{E}_{inc}$ and $\mathcal{H}_{inc}$ satisfy when $g \to 1$ from (74)-(75).

**Lemma 5.1.** *One of solutions of (77) satisfies,*

$$(\mathcal{E}_g, \mathcal{H}_g) = \frac{i}{\omega\epsilon_0}\frac{(\widehat{z}\,(grad(div(.)) + k^2)(J_0.), ik\,\widehat{z}\,rot(J_0.))}{8\pi igk}\mathcal{W}_g$$
$$= \frac{i}{\omega g\epsilon_0}(\frac{J_{0x}}{8\pi ik}(\partial_{zx}, -ik\partial_y) + \frac{J_{0y}}{8\pi ik}(\partial_{zy}, ik\partial_x) + \frac{J_{0z}}{8\pi ik}(\partial_{z^2} + k^2, 0))\mathcal{W}_g \qquad (78)$$

*where,*

$$\mathcal{W}_g = (e^{\pm ikgz}E_I(ik(|r-r'| \pm z)) + e^{\mp ikgz}(E_I(ik(|r-r'| \mp z)) - \qquad (79)$$
$$- 2K_0(ik\sqrt{1-g^2}\rho)) + O(g-1)),$$

*is a function whose factor $K_0(ik\sqrt{1-g^2}\rho))$ diverges as $g \to 1$.*

**Proof.** To solve (77), we begin by noticing that, for $\pm z > 0$, we can write $G(z)$ in the form

$$G(z) = \int_0^{+i\infty+\arg(ik)} V_0(\beta) e^{\mp ikz\cos\beta} d\beta =$$
$$= (\frac{\partial^2}{\partial z^2} + (kg)^2) \int_0^{+i\infty+\arg(ik)} V_0(\beta) \frac{e^{\mp ikz\cos\beta}}{2gk^2} (\frac{1}{\cos\beta + g} - \frac{1}{\cos\beta - g}) d\beta \qquad (80)$$

Comparing (77) and the second line of (80), we use (11)-(13), and obtain the following solution, as $\pm z > 0$,

$$(\mathcal{E}_g, \mathcal{H}_g) = \frac{\frac{i}{\omega\epsilon_0}(\widehat{z}\,(\text{grad}(\text{div}(.)) + k^2)(J_0.), ik\,\widehat{z}\,\text{rot}(J_0.))}{2igk^2}$$
$$(\int_{-i\tau_g\infty}^0 e^{-ikgz_j} \times G(z_j \pm z) k d(z_j) - \int_{-i\tau_{-g}\infty}^0 e^{ikgz_j} \times G(z_j \pm z) k d(z_j)) \qquad (81)$$

for $\arg(ik) = 0$. We can then consider $\mathcal{J}_g$ in (16) and its expansions (26) and (31) to express (81) for arbitrary $k$. So, as $g - 1$ is small, we can write (78) with (79).

We remark the divergence of the expression as $g \to 1$. However, we can modify (79) and suppress diverging factors which have no influence on the field.

### *5.1.2) Expression of $(\mathcal{E}_{inc}, \mathcal{H}_{inc})$, and suppression of diverging factors*

**Proposition 5.1.** *A definite expression of the potentials $\mathcal{E}_{inc}(r)$ and $\mathcal{H}_{inc}(r)$ in the region $\pm z > 0$, for the field radiated by arbitrary bounded sources $J$ and $M$ in the domain $\mp z > 0$, $|\arg(ik)| \leq \pi/2$, when (76) and (77) apply, is given by*

$$(\mathcal{E}_{inc}, \mathcal{H}_{inc}) = \frac{\widehat{z}}{8\pi k^2}(\sqrt{\frac{\mu_0}{\epsilon_0}}(grad(div(J)) + k^2 J, ik\; rot(J)) +$$
$$+ (-ik\,rot(M), grad(div(M)) + k^2 M)) * \mathcal{W} = \frac{\widehat{z}}{8\pi k^2}\mathcal{L}(\sqrt{\frac{\mu_0}{\epsilon_0}}J, M) * \mathcal{W}, \qquad (82)$$

*where*

$$\mathcal{W}(r) = (e^{ik|z|} E_I(ik(|r| + |z|)) + e^{-ik|z|}(E_I(ik(|r| - |z|)) + 2\ln\rho)) \qquad (83)$$

*with $|r| = \sqrt{\rho^2 + z^2}$ and $\rho = \sqrt{x^2 + y^2}$ at $r(x, y, z)$.*

**Proof.** The electric field $E_g$, attached to the potentials $(\mathcal{E}_g, \mathcal{H}_g)$, as $\pm z > 0$, satisfies

$$E_g = \frac{i}{\omega\epsilon_0 8\pi ik}(\,\widehat{x}(J_{0x}(\partial_{z^2}\partial_{x^2} - k^2\partial_{y^2}) + J_{0y}(\partial_{xy}(\partial_{z^2} + k^2)) + J_{0z}(\partial_{xz}(\partial_{z^2} + k^2))$$
$$+\,\widehat{y}(J_{0x}(\partial_{xy}(\partial_{z^2} + k^2)) + J_{0y}(\partial_{z^2}\partial_{y^2} - k^2\partial_{x^2}) + J_{0z}(\partial_{yz}(\partial_{z^2} + k^2))$$
$$+\,\widehat{z}(J_{0x}(\partial_{zx}(\partial_{z^2} + k^2)) + J_{0y}(\partial_{zy}(\partial_{z^2} + k^2)) + J_{0z}((\partial_{z^2} + k^2)(\partial_{z^2} + k^2))\mathcal{W}_g \quad (84)$$

when $M_0 = 0$, and the magnetic field is given by $H_g = \frac{i}{\omega\mu}\text{rot}(E_g)$. We can complete (84), from $(\Delta + k^2)\mathcal{W}_g = 0$, with

$$(\partial_{z^2}\partial_{x^2} - k^2\partial_{y^2})\mathcal{W}_g = (\partial_{x^2} + k^2)(\partial_{z^2} + k^2)\mathcal{W}_g$$
$$(\partial_{z^2}\partial_{y^2} - k^2\partial_{x^2})\mathcal{W}_g = (\partial_{y^2} + k^2)(\partial_{z^2} + k^2)\mathcal{W}_g \quad (85)$$

so that, using $(\partial_{z^2} + (kg)^2)\mathcal{W}_g = G$, we recover the known expression of field in function of $J_0G$ as $g \to 1$.

We then use (84)-(85) and verify that some part of $\mathcal{W}_g$ given in previous lemma, which diverges when $g \to 1^-$, can be regularised. So, considering $\mathcal{W}_g$ and the behaviour of $K_0$ as $g \to 1^-$, we begin by writing

$$e^{\mp ikgz}K_0(ik\sqrt{1-g^2}\rho)) = (e^{\mp ikgz}K_0(ik\sqrt{1-g^2}\rho)) + e^{\mp ikz}\text{ln}\rho) -$$
$$-\,e^{\mp ikz}\text{ln}\rho \quad (86)$$

The contribution of $(e^{\mp ikgz}K_0(ik\sqrt{1-g^2}\rho)) + e^{\mp ikz}\text{ln}\rho)$ to the field given by (84)-(85) vanishes when $g \to 1^-$. Indeed, we notice that, as $g \to 1^-$,

$$(\Delta + k^2)(e^{\mp ikgz}K_0(ik\sqrt{1-g^2}\rho)) + e^{\mp ikz}\text{ln}\rho)) =$$
$$= -2\pi(e^{\mp ikgz} - e^{\mp ikz})\delta(x)\delta(y) \to 0, \quad (87)$$

where $\delta(w)$ is the Dirac delta function [1], and

$$(\frac{\partial^2}{\partial z^2} + k^2)(e^{\mp ikgz}K_0(ik\sqrt{1-g^2}\rho)) + e^{\mp ikz}\text{ln}\rho) =$$
$$= k^2(1-g^2)K_0(ik\sqrt{1-g^2}\rho) \to 0. \quad (88)$$

We can then formally suppress this term in (86), as we let tend $g \to 1^-$, so that we obtain

$$(\mathcal{E}_{inc}, \mathcal{H}_{inc})(r) = \frac{i\widehat{z}}{\omega\epsilon_0}\frac{(\text{grad}(\text{div}(J_0.)) + k^2(J_0.), ik\,\text{rot}(J_0.))}{8\pi ik}\mathcal{W}(r - r') \quad (89)$$

where

$$\mathcal{W}(r) = (e^{\pm ikz}E_1(ik(|r| \pm z)) + e^{\mp ikz}(E_1(ik(|r| \mp z)) + 2\text{ln}\rho)) \text{ as } \pm z > 0 \quad (90)$$

From $\partial_w E_1(w) = -e^{-w}/w$, the reader can verify by inspection that the conditions

$$(\frac{\partial^2}{\partial z^2}+k^2)\frac{\mathcal{W}(r)}{8\pi ik} = -\frac{e^{-ik|r|}}{4\pi|r|},\ (\Delta+k^2)\mathcal{W}(r)=0 \qquad (91)$$

are satisfied when $\pm z > 0$, and that all derivatives of $\mathcal{W}$ are regular in these domains. In the same manner, this method can also be applied with $M$ instead of $J$, and generalised for arbitrary combination of dipoles in the domain $\mp z > 0$, so that we obtain (82) with (83) as $\pm z > 0$.

It is worth noticing that we have $\Delta_{xy}\ln(\rho)=0$ for $\rho\neq 0$, which implies that $\ln\rho$ in (90) has no influence on the expression of the field except by its singularity at $\rho=0$. It is then possible to consider $\mathcal{W}(r)$ as the generalized function

$$\begin{aligned}\mathcal{W}(r) \equiv\ & e^{ikz}(E_1(ik(|r|+z))+2f_{\epsilon_1}(|r|+z)\ln\rho)+e^{-ikz}(E_1(ik(|r|-z))+\\ & +2f_{\epsilon_1}(|r|-z)\ln\rho)\end{aligned} \qquad (92)$$

where $f_{\epsilon_1}^{(n)}(w)\to 0$ when $\epsilon_1\to 0^+$ for $w\geq 0$, except that $f_{\epsilon_1}(0)=1$, $\epsilon_1>0$ (for example, $f_{\epsilon_1}(w)=e^{-(e^{-1/w^2})/\epsilon_1}$). This latter expression is even in $z$, and it can be used in place of previous one in (83) to obtain the scalar potentials in the whole space outside the sources.

### *5.2) Expression of the potentials $(\mathcal{E}_s,\mathcal{H}_s)$ for a multimode plane with $\mathcal{J}_{g_j^{e,h}}$*

We have obtained an expression for the potentials $(\mathcal{E}_{inc},\mathcal{H}_{inc})$ attached to the radiation of arbitrary sources, and we can now express the scalar potentials $\mathcal{E}_s$ and $\mathcal{H}_s$ which satisfy the boundary conditions (73), for scattered field components verifying (69) and (a)-(b), with $\mathcal{J}_g$ defined in section 3.

**Proposition 5.2.** *The potentials $\mathcal{E}_s(r)$ and $\mathcal{H}_s(r)$ at $r(x,y,z)$, for arbitrary bounded sources $J$ and $M$ above the multimode plane, $|\arg(ik)|\leq\pi/2$, verify as $z\geq 0$,*

$$\mathcal{E}_s(x,y,z) = \mathcal{E}_{inc}(x,y,\,-z)\;+((\frac{\widehat{z}}{\omega\epsilon_0}\frac{grad(div(J))+k^2J}{8\pi k}+\frac{\widehat{z}}{k}\frac{(-ik\,rot(M))}{8\pi k})*$$

$$*\sum_{\epsilon=-1,1}\sum_{j=1}^{N}\frac{a_j^e}{(g_j^e-\epsilon)}(\mathcal{V}_\epsilon+\epsilon\mathcal{K}_{g_j^e}))(x,y,\,-z)$$

$$=\mathcal{E}_{inc}(x,y,\,-z)\;+((\frac{\widehat{z}}{\omega\epsilon_0}\frac{grad(div(J))+k^2J}{8\pi k}+\frac{\widehat{z}}{k}\frac{(-ik\,rot(M))}{8\pi k})*$$

$$*\sum_{\epsilon=-1,1}((\prod_{j=1}^{N}\frac{\epsilon+g_j^e}{\epsilon-g_j^e}-1)\mathcal{V}_\epsilon+\sum_{j=1}^{N}\frac{\epsilon a_j^e\mathcal{K}_{g_j^e}}{(g_j^e-\epsilon)}))(x,y,\,-z) \tag{93}$$

*and*

$$\mathcal{H}_s(x,y,z) = \mathcal{H}_{inc}(x,y,\,-z)+((\frac{\widehat{z}}{\omega\epsilon_0}\frac{(ik\;rot(J))}{8\pi k}+\frac{\widehat{z}}{k}\frac{(grad(div(M))+k^2M)}{8\pi k})*$$

$$*\sum_{\epsilon=-1,1}\sum_{j=1}^{P}\frac{a_j^h}{(g_j^h-\epsilon)}(\mathcal{V}_\epsilon+\epsilon\mathcal{K}_{g_j^h}))(x,y,\,-z)$$

$$=\;-\;\mathcal{H}_{inc}(x,y,\,-z)+((\frac{\widehat{z}}{\omega\epsilon_0}\frac{(ik\;rot(J))}{8\pi k}+\frac{\widehat{z}}{k}\frac{(grad(div(M))+k^2M)}{8\pi k})*$$

$$*\sum_{\epsilon=-1,1}((\prod_{j=1}^{P}\frac{\epsilon+g_j^h}{\epsilon-g_j^h}+1)\mathcal{V}_\epsilon+\sum_{j=1}^{P}\frac{\epsilon a_j^h\mathcal{K}_{g_j^h}}{(g_j^h-\epsilon)}))(x,y,\,-z) \tag{94}$$

where $\mathcal{V}_\epsilon$, $\mathcal{K}_g$ and the $a_j^{e,h}$ satisfy

$$\mathcal{V}_\epsilon(\rho,z)=e^{-\epsilon ikz}(E_I(ik(|r|-\epsilon z))+(1-\epsilon)\ln\rho),\;\mathcal{K}_g(\rho,z)=e^{-ikgz}\mathcal{J}_g(\rho,z)$$
$$\prod_{j=1}^{N,P}\frac{\cos\beta-g_j^{e,h}}{\cos\beta+g_j^{e,h}}=1+\sum_{j=1}^{N,P}a_j^{e,h}\frac{1}{\cos\beta+g_j^{e,h}}\;,\;\frac{a_j^{e,h}}{2g_j^{e,h}}=\;-\;\prod_{i\neq j}^{N,P}\frac{g_j^{e,h}+g_i^{e,h}}{g_j^{e,h}-g_i^{e,h}}. \tag{95}$$

**Proof.** We first let arg$(ik)=0$. As indicated in (74), we can then use (66) of the section 4 on acoustic case, and write,

$$(\mathcal{E}_s,\mathcal{H}_s)(z)=(\mathcal{E}_{inc},\mathcal{H}_{inc})(-z)$$
$$-\,i\int_{-i\tau_j\infty}^{0}(\sum_{j=1}^{N}a_j^e e^{-ikg_j^ez_j}\mathcal{E}_{inc},\sum_{j=1}^{P}a_j^h e^{-ikg_j^hz_j}\mathcal{H}_{inc})(-z_j-z)kd(z_j) \tag{96}$$

where the $a_j^{e,h}$ satisfy (95).

If we consider $J=J_0\delta(r-r')$, $M=M_0\delta(r-r')$, with $r'(x',y',\,z'=h)$ in (91), the principal terms we need to reduce are of the type,

$$
\begin{aligned}
I_j^{e,h} &= -ia_j^{e,h}\int_{-i\tau_j\infty}^{0} e^{-ikg_j^{e,h}z_j}\mathcal{W}(-z_j - z - h)kd(z_j) \\
&= -ia_j^{e,h}(\sum_{\epsilon=-1,1} e^{i\epsilon k(z+h)}\int_{-i\tau_j\infty}^{0} e^{-ik(g_j^{e,h}-\epsilon)z_j}E_1(ik(R(-z-z_j-h)+ \\
&+\epsilon(z+z_j+h)))kd(z_j) + 2e^{-ik(z+h)}\ln\rho\int_{-i\tau_j\infty}^{0} e^{-ik(g_j^{e,h}+1)z_j}\,kd(z_j))
\end{aligned}
\tag{97}
$$

that we can express, after integration by parts, in the form,

$$
\begin{aligned}
I_j^{e,h} &= a_j^{e,h}(\sum_{\epsilon=-1,1}\frac{e^{\epsilon ik(z+h)}E_1(ik(R(-z)+\epsilon(z+h)))}{(g_j^{e,h}-\epsilon)} + \frac{2e^{-ik(z+h)}\ln\rho}{(g_j^{e,h}+1)} \\
&+\sum_{\epsilon=-1,1}\frac{\epsilon e^{ikg_j^{e,h}(z+h)}}{(g_j^{e,h}-\epsilon)}\int_{-i\tau_j\infty}^{0} e^{-ikg_j^{e,h}(z_j+z+h)}\,\frac{e^{-ikR(-z-z_j)}}{kR(-z-z_j)}kd(z_j)) \\
&= a_j^{e,h}\sum_{\epsilon=-1,1}(\frac{(e^{\epsilon ik(z+h)}(E_1(ik(R(-z)+\epsilon(z+h)))+(1-\epsilon)e^{-ik(z+h)}\ln\rho))}{(g_j^{e,h}-\epsilon)} \\
&+\frac{\epsilon e^{ikg_j^{e,h}(z+h)}\mathcal{J}_{g_j^{e,h}}(\rho,\,-z-h)}{(g_j^{e,h}-\epsilon)})
\end{aligned}
\tag{98}
$$

The function $\mathcal{J}_g$ has been studied in section 3 for arbitrary $g$ and $k$, and the expression of $I_j^{e,h}$ in (98) with $\mathcal{J}_{g_j^{e,h}}$ is general. It is regular when $g_j^{e,h} \neq -1$ and is valid as $|\arg(ik)| \leq \pi/2$. Letting $|r| = \sqrt{\rho^2 + z^2}$, and using (98) in (96), we obtain (93) and (94), for $z \geq 0$, where we have used (95). These formulas can be used for arbitrary bounded sources $J$ and $M$ above the plane. In other respects, using the properties of $\mathcal{J}_g$, the reader can verify by inspection that each component of the scattered field given by these potentials satisfes the condition (a)-(b).

Moreover, we can add the condition (71), and thus consider in previous expressions,

$$
\prod_{j=1}^{N}\frac{\epsilon + g_j^e}{\epsilon - g_j^e} = -\prod_{j=1}^{P}\frac{\epsilon + g_j^h}{\epsilon - g_j^h}.
\tag{99}
$$

Let us remark that $(\mathcal{E}_{inc}(-z), -\mathcal{H}_{inc}(-z))$ are the potentials attached to the field scattered by a perfectly conducting plane with $g^e = 1/g^h \to 0,\ N = P = 1.$ Besides, since derivatives of potentials are contained in the expression (66) of the field, it is worth noticing that, from the Helmhotz equation satisfied by the potentials and by

$\mathcal{I}_g = e^{ikg(z+h)}\mathcal{J}_g(\rho, -z-h)$, and from (11), $\partial_z \mathcal{I}_g = \frac{e^{-ikR(-z)}}{R(-z)} + ikg\mathcal{I}_g$, the functions to calculate are only $\mathcal{J}_{g_j^{e,h}}$ and $\partial_\rho \mathcal{J}_{g_j^{e,h}}$.

## ***6) Conclusion***

We have developed simple exact expressions of the field scattered by a multilayered medium modelled by a boundary condition of high order with active and passive modes, for general sources, in acoustics and in electromagnetism. We begin by studying the scattering for a point source illumination in acoustic scalar problem for a one mode plane with arbitrary impedance, passive or active, then the total field for a multimode plane, which is a combination of contributions of each modes. Different difficulties restrict the use of expressions known for a one mode passive plane, and we give here a novel exact expression for arbitrary case that we develop. Our analysis is then applied in electromagnetism, where we give a new expression for the scalar potentials in electromagnetism without diverging terms, for arbitrary sources in free space, then the one for the scattering by an imperfectly reflective plane with passive and active modes.

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**Appendix A : On some properties of the reflection coefficient and its zeros $g_j$**

In section 2, we consider a plane, composed of several homogenous layers on a perfectly reflecting plane, and its reflection coefficient, related to a component of an incident plane wave [1],

$$R(\beta) = \frac{\cos\beta - Z_r(\beta)}{\cos\beta + Z_r(\beta)}. \qquad (A.1)$$

In (A.1), $Z_r(\beta)$ is the ratio of the normal derivative $\partial_n u$ with $iku$, where $u$ is the total field component and $\beta$ is the angle of incidence with the normal $\hat{z}$. Denoting $u_{(m-1)}$, $m = 1, 2, ..., M$, the field component $u$ at the top of layer $m$ of thickness $d_m$ and wavenumber $k_M$, we can generally write [1]

$$\begin{pmatrix} b_{m-1}\partial_n u_{m-1} \\ c_{m-1}u_{m-1} \end{pmatrix} = \begin{pmatrix} \cos(\eta_m d_m) & s_m \sin(\eta_m d_m) \\ -\sin(\eta_m d_m)/s_m & \cos(\eta_m d_m) \end{pmatrix} \begin{pmatrix} b_m \partial_n u_m \\ c_m u_m \end{pmatrix} \qquad (A.2)$$

where the coefficients of the matrix have no cuts and are meromorphic functions of cos$\beta$, even when $\eta_m$ and $s_m$ have cuts [1]-[3], and $b_m$, $c_m$ are constants. If, at the last interface denoted $M$, we have

$$\partial_n u_{(M)} - ik_M Z_M u_{(M)} = 0 \tag{A.3}$$

where $Z_M$ is a constant, we derive from (A.2) that $Z_r = \partial_n u/ik_M u$ is meromorphic. It is in particular the case with $Z_M = 0$ or $\infty$ which corresponds to a perfectly reflecting plane. Let us notice that if we have $d_M \to \infty$ and some loss in the last layer, $Z_r$ is no more meromorphic and has generally a cut in complex plane [3].
Letting $W_m = b_m \partial_n u_m/(c_m u_m)$ , tan$\alpha_m = W_m/s_m$, we have

$$\begin{aligned}
W_{m-1} &= s_m \frac{W_m/s_m + \tan(\eta_m d_m)}{1 - (W_m/s_m)\tan(\eta_m d_m)} = s_m \tan(\eta_m d_m + \alpha_m) \\
\tan\alpha_m &= W_m/s_m = \frac{s_{m+1}}{s_m}\tan(\eta_{m+1} d_{m+1} + \alpha_{m+1}) \\
\alpha_m &= \frac{i}{2}\ln\left(\frac{\mathcal{R}_{m+1} + e^{-2i(\eta_{m+1}d_{m+1}+\alpha_{m+1})}}{1 + \mathcal{R}_{m+1}e^{-2i(\eta_{m+1}d_{m+1}+\alpha_{m+1})}}\right), \mathcal{R}_{m+1} = \frac{s_m - s_{m+1}}{s_m + s_{m+1}},
\end{aligned} \tag{A.4}$$

with $Z_r = \partial_n u/iku = -ic_0 W_0/k_0 b_0$, and $Z_M = \partial_n u_M/ik_M u_M = c_M W_M/ik_M b_M$. The reader will notice that by iteration we can simply express $\alpha_1$ and thus the reflection coefficient $R$ in function of the elementary reflection coefficients $\mathcal{R}_1$, $\mathcal{R}_2$,...,$\mathcal{R}_M$.

In acoustics, we have

$$grad\, p = -i\omega\rho v, div\, v = -i\frac{\omega}{\mathcal{B}}p \tag{A.5}$$

where $p$ is the pressure and $v$ is the particle velocity, in a medium with density $\rho$ and bulk modulus $\mathcal{B}$, and thus, when $u \equiv p$,

$$\begin{aligned}
b_m &= \frac{1}{\omega\rho_m}, c_m = 1, s_m = -\eta_m/(\omega\rho_m),\ \eta_m = k_0\sqrt{\rho_{mr}/\mathcal{B}_{mr} - \sin^2\beta}, \\
Z_r &= \frac{i\eta_1}{k_0\rho_{1r}}\tan(\eta_1 d_1 + \alpha_1).
\end{aligned} \tag{A.6}$$

where $\rho_{mr} = \rho_m/\rho_0, \mathcal{B}_{mr} = \mathcal{B}_m/\mathcal{B}_0, k_0 = \omega\sqrt{\rho_0/\mathcal{B}_0}$, while, in electromagnetism,

$$rotH = i\omega\epsilon E,\ rotE = -i\omega\mu H \tag{A.7}$$

and thus, when $u \equiv H_x$(if $H = (H_x, 0, 0)$ is normal to the incidence plane), we have

$$b_m = \frac{1}{\omega\epsilon_m}, c_m = 1, s_m = -\eta_m/(\omega\epsilon_m),\ \eta_m = k_0\sqrt{\epsilon_{mr}\mu_{mr} - \sin^2\beta},$$
$$Z_r = \frac{i\eta_1}{k_0\epsilon_{1r}}\tan(\eta_1 d_1 + \alpha_1) \qquad (A.8)$$

where $\epsilon_{mr} = \epsilon_m/\epsilon_0$, $\mu_{mr} = \mu_m/\mu_0$, $k_0 = \omega\sqrt{\epsilon_0\mu_0}$. Since $E_z = \frac{i}{\omega\epsilon}\partial_y H_x$, we notice that, for $u \equiv E_z$, $b_m = 1$, $c_m = \omega\epsilon_m$.
Let us consider the solutions $g_j = \cos\beta_j$ of $\cos\beta_j - Z_r(\beta_j) = 0$, $j \geq 1$, with $|g_{j+1}| \geq |g_j|$. For $\epsilon_{1r}$ and $\rho_{1r} \neq 1$, $\mathrm{Im}g_j$ is bounded. In other respects, we notice that, when there is no loss in layers, $\omega$ is real and $Z_M$ is purely imaginary, we have $Z_r(\beta)^* = -Z_r(\beta^*)$ and $R(\beta) = (R(\beta^*)^{-1})^*$. In this case, if $g_j$ is a zero of $R$, then $(-g_j)^*$ is too. Besides, for a simple layer backed by a plane with $Z_1 = 0$ (resp. $Z_1 = \infty$), we notice that $\alpha_1 = 0$ (resp. $\alpha_1 = \pi/2$). In this case, we notice that $g_j \sim w/(k_0 d_1)$ for large $g_j$ where $w$ satisfies $-k_0 b_1/c_1 \tan(w + \alpha_1) = ik_0 b_0/c_0$, in lossless case, the number of $g_j$ purely imaginary that can be found graphically when $\rho_{1r} > 1$ or $\epsilon_{1r} > 1$ (usual case in electromagnetism) is odd.

**Appendix B : On the functions $v_n$**

The function $v_n$, used in section 3, is defined following

$$v_n(z) = z^{n-1}(n-1)!\int_z^\infty \frac{1}{z^2}\int_z^\infty ... \frac{1}{z^2}\int_z^\infty \frac{e^{-z}}{z}dz...dzdz$$
$$= z^{n-1}\int_z^\infty \frac{e^{-t}(\frac{1}{z} - \frac{1}{t})^{n-1}}{t}dt = \int_z^\infty \frac{e^{-t}}{t^n}(t-z)^{n-1}dt. \qquad (B.1)$$

where $|\arg z| < \pi$. We can develop $(t-z)^{n-1}$, and obtain

$$v_n(z) = \sum_{m=0}^{n-1} C_{n-1}^m(-z)^m \int_z^\infty \frac{e^{-t}}{t^{m+1}}dt = \sum_{m=0}^{n-1}\frac{(-1)^m(n-1)!}{m!(n-1-m)!}E_{m+1}(z) \qquad (B.2)$$

For small $z$, this gives us,

$$v_n(z) = -\ln z - \gamma + \sum_{m=1}^{n-1}\frac{(-1)^m(n-1)!}{m!(n-1-m)!}\frac{1}{m} + O(z\ln z)$$
$$= -\ln z - \gamma + \int_0^{-1}\frac{(1+x)^{n-1} - 1}{x}dx + O(z\ln z)$$
$$= -\ln z - \Psi(n) - 2\gamma + O(z\ln z) \qquad (B.3)$$

where $\Psi(n)$ is the digamma function, $\Psi(1) = -\gamma$, and $\Psi(n) \sim \ln n$ when $n$ is large, with $\gamma = .57721...$ (Euler's constant).
For large $z$, we use an integration by parts,

$$v_n(z) = \int_z^\infty \frac{e^{-t}}{t^n}(t-z)^{n-1}dt = \int_z^\infty (-1)^{n-1}(e^{-t})^{(n-1)}\frac{(t-z)^{n-1}}{t^n}dt$$
$$= \frac{(n-1)!}{z^{n-1}}\sum_{m=0}^{n-1}\frac{(2n-2-m)!}{m!(n-1-m)!}\sum_{p=0}^{n-m-1}\frac{(-1)^{n-m-1-p}}{(n-m-1-p)!p!}E_{n+p}(z) \qquad (B.4)$$

which gives us

$$v_n(z) \sim \frac{e^{-z}(n-1)!}{z^n}\,(1+O(\frac{1}{z})) \qquad (B.5)$$

In other respects, from [25, p. 505], $v_n$ is related to the Whittaker function $W_{1/2-n,0}(z)$ and thus to the confluent hypergeometric function $\Psi(n,1,z)$, also denoted $U(n,1,z)$,

$$v_n(z) = \int_z^\infty \frac{(t-z)^{n-1}}{t^n}e^{-t}dt = e^{-z}(n-1)!U(n,1,z) \qquad (B.6)$$

In this respect, we can also use [25]

$$(n-1)^2U(n,1,z) = -U(n-2,1,z) + (2n+z-3)U(n-1,1,z) \qquad (B.7)$$

so that, for $n \geq 3$,

$$\begin{aligned}(n-1)v_n(z) &= (2n-3+z)v_{n-1}(z) - (n-2)v_{n-2}(z) \qquad (B.8)\\ (n-1)w_n(z) &= (2n-3+z)w_{n-1}(z) - (n-2)w_{n-2}(z) + zE_1(z)\end{aligned}$$

where $v_1(z) = E_1(z),\ v_2(z) = E_1(z) - E_2(z),\ w_n(z) = v_n(z) - E_1(z).$
From Temme [26, p.65], we have

$$v_n(z) \sim 2e^{-z}(e^{z/2}K_0(2\sqrt{nz})) \text{ as } n \text{ is large, } z \text{ fixed} \qquad (B.9)$$

remark :
We notice that,

$$s_n(z) = \frac{v_n(z)}{z^{n-1}(n-1)!} = -\sum_{p=1}^{n-1}\frac{s_{n-p}(z)}{p!(-z)^p} + \frac{E_n(z)}{(n-1)!(-z)^{n-1}}, \qquad (B.10)$$

where, since $\sum_{p=0}^{n-1}\frac{(-1)^p}{(n-p-1)!p!} = 0$ for $n \geq 2,$ we have $h_n(z) = -\sum_{p=1}^{n-1}\frac{h_p(z)}{(n-p)!} + \frac{E_n(z)}{(n-1)!}$ and $h_1(z) = 0$, when $h_n(z) = (-z)^{n-1}s_n(z) - \frac{(-1)^{n-1}}{(n-1)!}E_1(z)$.

Concerning general properties of $E_p$, we have $E_p(z) = \frac{1}{n}(e^{-z} - zE_{p-1}(z))$, and [25]

$$E_p(z) = (-z)^{p-1}\frac{E_1(z)}{(p-1)!} - \frac{e^{-z}}{z}\sum_{m=1}^{p-1}\frac{(m-1)!}{(p-1)!}(-z)^{p-m},$$
$$E_1(z) = \frac{e^{-z}}{z}(1 + O(1/z)) \text{ for } z \gg 1,$$
$$E_1(z) = -\gamma - \ln z - \sum_{n=1}^{\infty}\frac{(-1)^n z^n}{n\,n!} \text{ for } z \text{ small.} \qquad (B.11)$$

**Appendix C: another asymptotics when Re$g > 0$**

We give here some other asymptotic expression of

$$u_s = \frac{e^{-ikR(-z)}}{kR(-z)} + 2ig\int_{-i\tau\infty}^{0} e^{-ikgz_1}\frac{e^{-ikR(-z_1-z)}}{R(-z_1-z)}dz_1, \qquad (C.1)$$

when Re$g > 0$ and $R(-z) = \sqrt{r^2 + (z+h)^2}$ is large, to complete section 3. We notice that, $R(-z_1-z) - (R(-z) + z_1\cos\varphi) = \frac{z_1^2\sin^2\varphi}{R(-z_1-z)+R(-z)+z_1\cos\varphi} \ll 1$. In this case, if we let $f(-z_1-z) = \frac{e^{-ik(R(-z_1-z)-(R(-z)+z_1\cos\varphi))}}{kR(-z_1-z)}$ which is a smooth function, an integration by parts gives us,

$$u_s = \frac{e^{-ikR(-z)}}{kR(-z)}(\frac{\cos\varphi - g}{\cos\varphi + g} + \frac{2g}{ikR(-z)}(\frac{\cos\varphi}{(g+\cos\varphi)^2} + \frac{(\sin\varphi)^2}{(g+\cos\varphi)^3}) + \ldots$$
$$- \frac{e^{-ik(g+\cos\varphi)z_1}}{(i(g+\cos\varphi))^n}\frac{\partial^{n-1}f(-z_1-z)}{\partial(kz_1)^{n-1}}) +$$
$$+ 2ig_i e^{-ikR(-z)}\int_{-i\tau\infty}^{0}\frac{e^{-ik(g+\cos\varphi)z_1}}{(i(g+\cos\varphi))^n}\frac{\partial^n f(-z_1-z)}{\partial(kz_1)^n}kd(z_1) \qquad (C.2)$$

as given by Maliuzhinets in [7].

**Appendix D : expression of the field radiated by $J$ and $M$ sources**

Considering the Maxwell equations with electric and magnetic sources $J$ and $M$, rot$E = -i\omega\mu_0 H - M$, rot$H = J + i\omega\epsilon_0 E$, with $k = \omega\sqrt{\mu_0\epsilon_0}$, we can write [1]

$$E = \mathrm{rot}(G*M) + \frac{i}{\omega\epsilon_0}(\mathrm{grad}(\mathrm{div}(.)) + k^2)(G*J)$$
$$\sqrt{\frac{\mu_0}{\epsilon_0}}H = -\sqrt{\frac{\mu_0}{\epsilon_0}}\mathrm{rot}(G*J) + \frac{i}{k}(\mathrm{grad}(\mathrm{div}(.)) + k^2)(G*M) \qquad (D.1)$$

where $(\Delta + k^2)G(r) = \delta(r)$, $G(r) = -e^{-ik|r|}/4\pi|r|$ (time convention $e^{i\omega t}$), with $*$ the convolution product. This expression is used in section 5. Notice that $(\text{grad}(\text{div}(.)) + k^2)$ can be replaced by $\text{rot}(\text{rot}(.))$ outside the sources. For electrical or magnetic dipole, $J = J_0\delta(r - r')$ or $M = M_0\delta(r - r')$, we obtain for $r \neq r'$, using that $\text{rot}(\text{grad}(.)) = 0$ and $\text{grad}(u_0\text{grad}(G)) = (u_0\text{grad})(\text{grad}(G)))$,

$$E = \text{grad}_r(G(r - r')) \wedge M_0 + \frac{i}{\omega\epsilon_0}((J_0\text{grad}_r)(\text{grad}_r(.)) + k^2 J_0)G(r - r')$$
$$H = -\text{grad}_r(G(r - r')) \wedge J_0 + \frac{i}{\omega\mu_0}((M_0\text{grad}_r)(\text{grad}_r(.)) + k^2 M_0)G(r - r') \quad (D.2)$$

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