# Existence of nontrivial steady states for populations structured with respect to space and a continuous trait

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**Abstract**

*We prove the existence of nontrivial steady states to reaction-diffusion equations with a continuous parameter appearing in selection/mutation/competition/migration models for populations, which are structured both with respect to space and a continuous trait.*

**Key words:** Structured populations, infinite-dimensional reaction-diffusion equations, selection/mutation/competition/migration models

**AMS (2000) classification:** 45K05, 92D15, 92D25

version: August 20, 2009

## 1 Introduction

We are interested in this work in the steady solutions to models of population dynamics in which the population is structured both w.r.t. the space variable

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$x$ (here, $x \in \Omega$, a bounded regular open set of $I\!R^N$) and a trait variable denoted by $v$ (here, $v \in [0,1]$ for the sake of simplicity).

The distribution function $f := f(t,x,v) \geq 0$ shall then denote the number density of individuals at time $t \in I\!R_+$, position $x \in \Omega$, and whose trait is $v \in [0,1]$. We also denote by $\rho(t,x) = \int_0^1 f(t,x,v)\, dv$ the total number of individuals at time $t$ and position $x$.

This paper is concerned with an integro-PDE model of reaction-diffusion type in infinite (continuous) dimension in which selection, mutations, competition, and migrations are taken into account.

Our modeling assumptions are the following: migration is described by a diffusion (w.r.t. $x$) operator with a rate $\nu := \nu(x,v)$ [that is, individuals with different traits or at different positions can have a different migration rate]; mutations are described by a linear kernel $K^* := K^*(x,v,v') \geq 0$ which is related to the probability at point $x$ that individuals with trait $v'$ have offsprings with trait $v$; selection is implemented in the model thanks to a fitness function $k^* := k^*(x,v) \geq 0$ which may depend on both point $x$ and trait $v$; finally a logistic term involving a kernel $C^* := C^*(x,v,v') \geq 0$ models the competition (felt by individuals of trait $v$) at point $x$ due to individuals of trait $v'$.

Under those assumptions, the evolution of the population is governed by the following integro-PDE:

$$\begin{aligned} \frac{\partial f}{\partial t}(t,x,v) - \nu(x,v)\Delta_x f &= k^*(x,v)\, f(t,x,v) + \int_0^1 K^*(x,v,v')\, f(t,x,v')\, dv' \\ &\quad - f(t,x,v) \int_0^1 C^*(x,v,v')\, f(t,x,v')\, dv'. \qquad (1.1) \end{aligned}$$

For a mathematical study of eq. (1.1), we refer to [DFP].

Our goal in this paper is to investigate the existence of (non-trivial) steady states for eq. (1.1), that is (non-zero) solutions to the following non-linear, elliptic integro-PDE for $f := f(x,v)$ (where $x \in \Omega$, $v \in [0,1]$):

$$\begin{aligned} -\Delta_x f(x,v) &= k(x,v)\, f(x,v) + \int_0^1 K(x,v,v')\, f(x,v')\, dv' \\ &\quad - f(x,v) \int_0^1 C(x,v,v')\, f(x,v')\, dv', \end{aligned} \tag{1.2}$$

where $k(x,v) := k^*(x,v)/\nu(x,v)$, $K(x,v,v') := K^*(x,v,v')/\nu(x,v)$, and $C(x,v,v') := C^*(x,v,v')/\nu(x,v)$. This study will be carried out assuming moreover that the population is confined to the region $\Omega$, that is $f := f(x,v)$ satisfies the homogeneous Neumann boundary condition:

$$\forall x \in \partial\Omega, \qquad \nabla_x f(x,v) \cdot n(x) = 0, \tag{1.3}$$

where $n(x)$ is the outward unit normal vector to $\partial\Omega$ at point $x$.

We use in the sequel coefficients which satisfy the following assumption:

**Assumption A**

- The selection and mutation parameters $k := k(x,v)$ and $K := K(x,v,v')$ satisfy

$$\exists \kappa_+, \kappa_- > 0 : \quad \forall x \in \Omega, v \in [0,1], \quad \kappa_- \le k(x,v) + \int_0^1 K(x,w,v)\, dw \le \kappa_+; \tag{1.4}$$

- The competition kernel $C := C(x,v,v')$ satisfies

$$\exists C_-, C_+ > 0 : \quad \forall x \in \Omega, v, v' \in [0,1], \quad C_- \le C(x,v,v') \le C_+. \tag{1.5}$$

Our main result reads

**Theorem 1** *Let $\Omega$ be a smooth bounded subset of $I\!R^N$ ($N \in I\!N$), and $K, C, k$ satisfy Assumption A. We suppose moreover that $K, C, k$ are continuous on $\bar{\Omega}_x \times [0,1]_v (\times [0,1]_{v'}$ for $K, C)$.*
*Then there exists a function $x \mapsto \mu_x$ in $L^\infty(\Omega_x)$ with values in the set of bounded nonnegative measures on $[0,1]_v$ such that:*

1.

$$\text{For a.e. } x \in \Omega, \qquad a := \frac{\kappa_-}{C_+} \leq \langle \mu_x, v \mapsto 1 \rangle_v \leq \frac{\kappa_+}{C_-} := b. \tag{1.6}$$

2.

$$\text{For all } \xi \in C_n^2([0,1]), \quad (x \mapsto \langle \mu_x, \xi \rangle_v) \in H^1(\Omega), \tag{1.7}$$

*where $C_n^2([0,1])$ is the space of functions $\xi \in C^2([0,1])$ such that $\xi'(0) = \xi'(1) = 0$.*

3. *The function*

$$v \in [0,1] \mapsto \langle \mu_x, C(x,v,\cdot) \rangle_{v'} \tag{1.8}$$

*is continuous uniformly w.r.t. $x \in \Omega$.*

4. *The measure-valued function $\mu$ is a weak solution of eq. (1.2), (1.3) in the following sense: for all $\varphi(x,v) := \psi(x)\xi(v)$ with $\psi \in H^1(\Omega_x)$ and $\xi \in C_c^2(]0,1[)$,*

$$\begin{aligned} \int_\Omega \nabla_x \psi(x) \cdot \nabla_x \left\{ \langle \mu_x, \xi \rangle_v \right\} \, dx \\ = \int_\Omega \langle \mu_x, k(x,\cdot)\, \xi \rangle_v \, \psi(x) \, dx \\ + \int_\Omega \int_0^1 \langle \mu_x, K(x,v,\cdot) \rangle_{v'} \, \xi(v) \, dv \, \psi(x) \, dx \\ - \int_\Omega \left\langle \mu_x, \; v \mapsto \xi(v) \langle \mu_x, C(x,v,\cdot) \rangle_{v'} \right\rangle_v \psi(x) \, dx. \end{aligned} \tag{1.9}$$

Note that all terms are well-defined thanks to estimates (1.6), (1.7), and (1.8).

This result can be improved in situations when mutations are somehow predominant, as shown by the following

**Theorem 2** *Let $\Omega$ be a smooth bounded subset of $\mathbb{R}^N$ ($N \in \mathbb{N}$), and $K, C, k$ satisfy Assumption A. We suppose moreover that $K \in L^\infty(\Omega \times [0,1] \times [0,1])$ and that*

$$\kappa_- > \frac{C_+}{C_-} \, ||k||_{L^\infty}. \tag{1.10}$$

*Then, there exists $f := f(x,v) \in L^\infty(\Omega \times [0,1])$ such that $\Delta_x f \in L^\infty(\Omega \times [0,1])$, $f$ is nonnegative, and $f$ is a weak solution to eq. (1.2), (1.3) in the following sense: for all $\phi := \phi(x,v)$ in $H^1(\Omega_x; L^2([0,1]_v))$ (i.e. $\phi, \nabla_x \phi \in L^2(\Omega \times [0,1])$),*

$$\begin{aligned}\int_\Omega \int_0^1 \nabla_x \phi \cdot \nabla_x f \, dv dx &= \int_\Omega \int_0^1 k\, \phi\, f \, dv dx + \int_\Omega \int_0^1 \int_0^1 K\, \phi\, f(x,v')\, dv' dv dx \\ &\quad - \int_\Omega \int_0^1 \int_0^1 C\, \phi\, f(x,v)\, f(x,v')\, dv' dv dx.\end{aligned}$$

We now put these results in perspective. On one hand, models of selection/mutation/competition for populations structured w.r.t. a continuous trait were studied (especially from the point of view of their large time behavior) in [DP], [DJMR], [R], [Ca], etc. On the other hand, the very rich subject of reaction–diffusion equations with a finite number of equations has been the subject of innumerable studies (cf. [Ro] and [S] and the references therein). Extensions to an infinite (enumerable) number of such equations in the context of coagulation-fragmentation models can be found for example in [LM1].

Models involving infinite "continuous" dimensional reaction-diffusion equations were studied in [LM2, CDF1, CDF2], and the large time behavior of such equations in the presence of a Lyapunov functional was established (cf. [CDF1]).

The present work is a first step towards the extension [to infinite-dimensional (continuous) reaction-diffusion equations modeling selection/mutation/competition/migration] of two of the above research directions: of results on the large time behavior of the spatially homogeneous models [DP], [DJMR], [R], [Ca] on one hand, and of models with Lyapunov functionals, as in [CDF1], on the other hand. The absence of a Lyapunov functional in the considered model (1.1) makes our analysis much more difficult than the one performed in [CDF1]. Hence, we present here only existence results for steady states of equation (1.1) (but we shall not study their stability).

As when looking for nontrivial steady states of finite-dimensional reaction-diffusion equations, one needs topological tools. Here, we shall use Schauder's fixed point theorem (cf. [S]) for an approximate problem, together with a (weak) compactness method for removing the approximation.

The paper is organized as follows: In §2 we prove existence of nontrivial solutions to problem (1.2), (1.3) in the space of measures (that is, Theorem 1). In §3 we prove Theorem 2, that is, when mutations are predominant, the solutions to (1.2), (1.3) obtained in §2 lie in the Sobolev space $L^2([0,1]_v; H^2(\Omega_x))$. In the numerical examples of §4 we illustrate the effect of the diffusion strength on the steady states.

# 2 Stationary Solutions

The proof of Theorem 1 uses a compactness method based on the following regularized equations (with $\varepsilon > 0$):

$$-(\Delta_x f^\varepsilon + \varepsilon \partial_v^2 f^\varepsilon) = k f^\varepsilon + \int_0^1 K\, f^\varepsilon(v')\, dv' - f^\varepsilon \int_0^1 C\, f^\varepsilon(v')\, dv', \tag{2.1}$$

$$\forall x \in \partial\Omega, v \in [0,1], \qquad \nabla_x f^\varepsilon(x,v) \cdot n(x) = 0, \tag{2.2}$$

$$\forall x \in \Omega, \qquad \partial_v f^\varepsilon(x,0) = \partial_v f^\varepsilon(x,1) = 0. \tag{2.3}$$

These boundary value problems have solutions thanks to the following

**Proposition 1** *Let $\Omega$ be a smooth ($C^2$) bounded domain of $\mathbb{R}^N$ ($N \in \mathbb{N}$), and $K, C, k$ satisfy Assumption A. Then for all $\varepsilon > 0$, there exists a strong solution $f^\varepsilon := f^\varepsilon(x,v) \in \bigcap_{\eta>0} W^{2-\eta,1}(\Omega \times ]0,1[)$ of (2.1)-(2.3).*

*Moreover, $f^\varepsilon$ is nonnegative and satisfies (for a.e. $x \in \Omega$)*

$$a := \frac{\kappa_-}{C_+} \le \int_0^1 f^\varepsilon(x,v)\, dv \le \frac{\kappa_+}{C_-} := b. \tag{2.4}$$

**Proof of Proposition 1:** We establish this result thanks to Schauder's fixed point theorem. In order to do so, we introduce (for $\delta > 0$) on one hand the linear operator $L_\delta$ defined by the elliptic Neumann problem (with constant coefficients)

$$(Id - \delta\Delta_x - \delta\varepsilon\partial_v^2)(L_\delta f)(x,v) = f(x,v), \tag{2.5}$$

$$\forall x \in \partial\Omega, v \in [0,1], \qquad \nabla_x (L_\delta f)(x,v) \cdot n(x) = 0, \tag{2.6}$$

$$\forall x \in \Omega, \qquad \partial_v (L_\delta f)(x,0) = \partial_v (L_\delta f)(x,1) = 0. \tag{2.7}$$

And on the other hand we define the (nonlinear) operator $N_\delta$, $\delta > 0$, by

$$
\begin{aligned}
N_\delta(f)(x,v) \quad &:= \quad f(x,v) + \delta\, k(x,v)\, f(x,v) + \delta \int_0^1 K(x,v,v')\, f(x,v') dv' \\
&\qquad -\delta f(x,v) \int_0^1 C(x,v,v') f(x,v')\, dv'. \qquad (2.8)
\end{aligned}
$$

Then the boundary value problem (2.1)-(2.3) is equivalent to the fixed point problem

$$L_\delta N_\delta(f^\varepsilon) = f^\varepsilon\,. \qquad (2.9)$$

Next, we introduce the bounded convex (nonempty) closed subset

$$Y := \left\{ f \in L^1(\Omega \times [0,1]) \Big| f \geq 0, \quad a.e.\ x \in \Omega, \quad a \leq \int_0^1 f(x,v)\, dv \leq b \right\} \qquad (2.10)$$

(for $a, b$ defined in (1.6)) of the Banach space $L^1(\Omega \times [0,1])$.

We now prove the

**Lemma 1** *The operator $L_\delta N_\delta$ maps $Y$ into itself as soon as $\delta > 0$ is small enough.*

**Proof of Lemma 1:** Note first that (for $f \geq 0$)

$$\int N_\delta(f)\, dv \leq (1 + \delta\, \kappa_+) \int f\, dv - \delta C_- \Big( \int f\, dv \Big)^2, \qquad (2.11)$$

and

$$\int N_\delta(f)\, dv \geq (1 + \delta\, \kappa_-) \int f\, dv - \delta C_+ \Big( \int f\, dv \Big)^2. \qquad (2.12)$$

We now choose $\delta > 0$ sufficiently small for the following inequality to hold:

$$b \leq \frac{1 + \delta \kappa_-}{2\delta C_+}. \qquad (2.13)$$

Then, the functions $h_1$ and $h_2$: $\mathbb{R}_+ \to \mathbb{R}$ defined by

$$h_1(y) := (1 + \delta\,\kappa_+)y - \delta C_- y^2 \tag{2.14}$$

and

$$h_2(y) := (1 + \delta\kappa_-)y - \delta C_+ y^2 \tag{2.15}$$

are monotonously increasing on $[0, b]$ (note that $\frac{1+\delta\,\kappa_-}{2\delta\,C_+} \le \frac{1+\delta\kappa_+}{2\delta\,C_-}$), and $h_1(b) = b,\ h_2(a) = a$.

Now, if $f \in Y$, we know that $a \le \int f\,dv \le b$. According to (2.11) and (2.12),

$$h_2\Big(\int f\,dv\Big) \le \int N_\delta(f)\,dv \le h_1\Big(\int f\,dv\Big), \tag{2.16}$$

so that (using the monotonicity of $h_1$ and $h_2$)

$$a = h_2(a) \le \int N_\delta(f)\,dv \le h_1(b) = b. \tag{2.17}$$

But

$$\begin{aligned}
&\int N_\delta(f)\,dv \\
&= \int (Id - \delta\Delta_x - \delta\varepsilon\partial_v^2)(L_\delta N_\delta(f))\,dv \\
&= (Id - \delta\Delta_x)\int L_\delta N_\delta(f)\,dv,
\end{aligned}$$

and for $x \in \partial\Omega$,

$$\begin{aligned}
&\nabla_x\Big[\int L_\delta N_\delta(f)(x,v)\,dv\Big]\cdot n(x) \\
&= \int \nabla_x[L_\delta N_\delta(f)](x,v)\cdot n(x)\,dv \\
&= 0.
\end{aligned}$$

Then, by a standard maximum principle applied to the solution $\int L_\delta N_\delta(f)(x,v)dv$ of a Neumann elliptic problem, we see that

$$a \le \int L_\delta N_\delta(f)(x,v)\,dv \le b. \tag{2.18}$$

In order to conclude the proof of Lemma 1, it remains to show that $L_\delta N_\delta(f) \geq 0$ for $\delta > 0$ small enough.
Using again the maximum principle (but this time on the domain $\Omega\times]0,1[$) for the elliptic operator $Id - \delta\Delta_x - \delta\varepsilon\partial_v^2$, we just have to show that $N_\delta(f) \geq 0$ (for $\delta > 0$ small enough).

But

$$\begin{aligned} N_\delta(f)(x,v) &\geq f(x,v)\left(1 - \delta \int_0^1 C(x,v,v')f(x,v')\,dv'\right) \\ &\geq f(x,v)(1 - \delta C_+ b), \end{aligned}$$

so it is enough to choose $\delta < \frac{1}{C_+ b}$. This concludes the proof of Lemma 1. □

We now turn to the proof of

**Lemma 2** *As soon as $\delta > 0$ is small enough, the operator $L_\delta N_\delta$ is continuous and compact on $Y$ (as a subset of the Banach space $L^1(\Omega \times [0,1])$).*

**Proof of Lemma 2:**
Note first that $L_\delta$ is a bounded operator on $L^1(\Omega \times [0,1])$ (cf. Prop. 13 of §II.8.4 in [DL], Comments on §IX in [Br]). Moreover, for $f_1,\ f_2 \in Y$,

$$\begin{aligned} \|N_\delta(f_1) - N_\delta(f_2)\|_{L^1} \leq & \int\int (1 + \delta k(x,v))\,|f_1(x,v) - f_2(x,v)|\,dvdx \\ & +\delta \int\int\int K(x,v,v')\ |f_1(x,v') - f_2(x,v')|\,dv\,dv'\,dx \\ & +\delta \int\int\int C(x,v,v')\ |f_1(x,v)f_1(x,v') - f_2(x,v)f_2(x,v')|\,dv\,dv'\,dx \\ & \leq [1 + \delta(\kappa_+ + 2bC_+)]\ \|f_1 - f_2\|_{L^1}. \end{aligned}$$

It remains to prove that $L_\delta N_\delta$ is compact. For this, we observe that according to the proof of Lemma 1, $N_\delta(Y) \subset Y$ when $\delta > 0$ is small enough, and that $L_\delta$ sends $L^1(\Omega \times [0,1])$ in $W^{2-\eta,1}(\Omega \times [0,1])$ for all $\eta > 0$ (cf. [BG]: Th. 3.1.5, Warning 3.1.9, Remark 3.1.11).
This concludes the proof of Lemma 2. □

**End of the Proof of Proposition 1**:

Then, the existence of a solution (in $\bigcap_{\eta>0} W^{2-\eta,1}(\Omega \times [0,1]) \cap Y$) to eq. (2.1) – (2.3) is a consequence of Lemmas 1, 2, and Schauder's fixed point theorem (cf. [S]) used for (2.9) and $\delta > 0$ small enough. This concludes the proof of Proposition 1. $\square$

**Remark 1**: Before turning to the proof of Theorem 1, we observe that solutions to (1.2), (1.3) are sometimes only measures: Consider indeed the coefficients $k := k_0 > 0$, $K := 0$, $C := C_0 > 0$. Then, all distributions of the form $f(x,v) = \frac{k_0}{C_0}\mu$, where $\mu$ is a nonnegative bounded measure on $C([0,1]_v)$ with $\langle \mu, 1 \rangle = 1$, are solutions of eq. (1.2), (1.3), that is

$$-\Delta_x f = k_0 f - C_0 f \int f \, dv',$$
$$\forall x \in \partial\Omega, \qquad \nabla_x f \cdot n(x) = 0.$$

We now turn to the

**Proof of Theorem 1:** We start from a sequence of solutions $f^\varepsilon = f^\varepsilon(x,v)$, $\varepsilon > 0$ to the regularized problem (2.1) – (2.3) given by Proposition 1. Hence (2.4) holds.

As a consequence, we can extract from $f^\varepsilon$ a subsequence (still denoted by $f^\varepsilon$) such that

$$f^\varepsilon \overset{\varepsilon \to 0}{\rightharpoonup} \mu \text{ in } L^\infty(\Omega_x; M^1([0,1]_v) \text{ weak } *, \tag{2.19}$$

where $\mu$ is some function $x \mapsto \mu_x$ of $L^\infty(\Omega_x)$ with values in the set of (nonnegative) bounded measures on $[0,1]$ (denoted here by $M_v^1$). This means that for all functions $\varphi \in L^1(\Omega_x; C([0,1]_v))$,

$$\int_\Omega \int_0^1 f^\varepsilon(x,v)\,\varphi(x,v)\,dv\,dx \to \int_\Omega \langle \mu_x, \varphi(x,\cdot)\rangle_v \,dx.$$

Note that $x \mapsto \mu_x$ clearly satisfies estimate (1.6).

Next, we prove property (1.7). Let $\xi$ lie in $C_n^2([0,1])$. Then, multiplying (2.1) by $\xi$ and integrating with respect to $v$, we get

$$-\Delta_x \int f^\varepsilon \xi \, dv - \varepsilon \int f^\varepsilon \xi'' \, dv$$
$$= \int k\xi f^\varepsilon \, dv + \iint K\,\xi(v)\,f^\varepsilon(v')\,dv\,dv' - \iint C\,\xi(v)\,f^\varepsilon(v)\,f^\varepsilon(v')\,dv\,dv'.$$

Multiplying now this equation by $\int f^\varepsilon \xi \, dv$ and integrating with respect to $x$, we obtain

$$
\begin{aligned}
&\int \left| \nabla_x \Big( \int f^\varepsilon \xi \, dv \Big) \right|^2 dx - \varepsilon \int \Big( \int f^\varepsilon \xi'' \, dv \Big) \Big( \int f^\varepsilon \xi \, dv \Big) \, dx \\
= \;& \int \Big( \int k \xi f^\varepsilon \, dv \Big) \Big( \int f^\varepsilon \xi \, dv \Big) \, dx \\
+ \;& \int \Big( \iint K \, \xi(v) \, dv \;\; f^\varepsilon(v') \, dv' \Big) \Big( \int f^\varepsilon \xi \, dv \Big) \, dx \\
- \;& \int \Big( \iint C \, \xi(v) \, f^\varepsilon(v) \, f^\varepsilon(v') \, dv \, dv' \Big) \Big( \int f^\varepsilon \xi \, dv \Big) \, dx.
\end{aligned}
$$

Then, remembering that $k, \int K(\cdot, w, \cdot) \, dw, C, \xi, \xi''$ are bounded and that $f^\varepsilon$ satisfies (2.4), we immediately see that

$$
\begin{aligned}
&\int \left| \nabla_x \Big( \int f^\varepsilon \xi \, dv \Big) \right|^2 dx \leq \varepsilon b^2 \|\xi\|_{L^\infty} \|\xi''\|_{L^\infty} |\Omega| \\
+ \;& b^2 \|k\|_{L^\infty} \|\xi\|^2_{L^\infty} |\Omega| + b^2 \left\| \int K(\cdot, w, \cdot) \, dw \right\|_{L^\infty} \|\xi\|^2_{L^\infty} |\Omega| \\
+ \;& b^3 \|C\|_{L^\infty} \|\xi\|^2_{L^\infty} |\Omega|.
\end{aligned}
\tag{2.20}
$$

Letting $\varepsilon$ go to 0, we recover the property (1.7). Note that thanks to (2.4), $\int f^\varepsilon \xi dv$ is bounded in $L^2(\Omega)$, so that $x \mapsto \langle \mu_x, \xi \rangle_v$ lies in $L^2(\Omega)$ (by (2.19) it is even in $L^\infty(\Omega)$).

We now introduce our test function $\varphi(x, v) = \psi(x)\xi(v)$ with $\psi \in H^1(\Omega)$, $\xi \in C_c^2(]0, 1[)$, and we rewrite problem (2.1) – (2.3) in the weak form:

$$
\begin{aligned}
&\int_\Omega \nabla_x \psi(x) \cdot \nabla_x \left\{ \int f^\varepsilon(x, v) \xi(v) \, dv \right\} dx - \varepsilon \iint_\Omega f^\varepsilon(x, v) \xi''(v) \psi(x) \, dv \, dx \\
&\quad = \iint_\Omega f^\varepsilon(x, v) k(x, v) \xi(v) \psi(x) \, dv \, dx \\
&\quad + \iiint_\Omega f^\varepsilon(x, v') K(x, v, v') \xi(v) \psi(x) \, dv \, dv' \, dx \\
&\quad - \iiint_\Omega f^\varepsilon(x, v) \, f^\varepsilon(x, v') \, \xi(v) \, C(x, v, v') \, \psi(x) \, dv \, dv' \, dx.
\end{aligned}
\tag{2.21}
$$

Then, it is easy to pass to the limit in the (linear) first four terms (using estimate (2.20) for the first one), and we only need to show that we can also pass to the limit in the (nonlinear) last one:

**Lemma 3** *The following result holds:*

$$\int_\Omega \int_0^1 f^\varepsilon(x,v) \left[ \xi(v) \int_0^1 f^\varepsilon(x,v')\, C(x,v,v')\, dv' \right] dv\, \psi(x)\, dx \tag{2.22}$$
$$- \int_\Omega \left\langle \mu_x, v \mapsto \xi(v) \, \langle \mu_x, C(x,v,\cdot)\rangle_{v'} \right\rangle_v \psi(x)\, dx \stackrel{\varepsilon \to 0}{\longrightarrow} 0$$

**Proof of Lemma 3**: Due to the uniform continuity of $C$ (and estimate (2.4)), the function $v \mapsto \int_0^1 f^\varepsilon(x,v')C(x,v,v')\, dv'$ is continuous, uniformly with respect to $\varepsilon > 0$ and $x \in \Omega$. Hence, $v \mapsto \langle \mu_x, C(x,v,\cdot)\rangle_{v'}$ is also continuous (uniformly w.r.t. $x \in \Omega$) that is, (1.8) holds. As a consequence, the second part of the l.h.s. of (2.22) is well-defined.

Now we estimate the l.h.s. of (2.22) by

$$\left| \int_\Omega \int_0^1 f^\varepsilon(x,v)\, \psi(x)\, \xi(v) \left[ \int_0^1 f^\varepsilon(x,v')\, C(x,v,v')\, dv' - \langle \mu_x, C(x,v,\cdot)\rangle_{v'} \right] dv\, dx \right|$$
$$+ \left| \int_\Omega \int_0^1 f^\varepsilon(x,v)\, \psi(x)\, \xi(v)\, \langle \mu_x, C(x,v,\cdot)\rangle_{v'}\, dv\, dx \right. \tag{2.23}$$
$$\left. - \int_\Omega \left\langle \mu_x, v \mapsto \xi(v)\, \langle \mu_x, C(x,v,\cdot)\rangle_{v'} \right\rangle_v \psi(x)\, dx \right|.$$

The second term tends to zero since $(x,v) \mapsto \psi(x)\xi(v)\langle \mu_x,\, C(x,v,\cdot)\rangle_{v'}$ is a test function in $L^1(\Omega_x; C([0,1]_v))$. Then, the first term of (2.23) is bounded by $b\|\psi\|_{L^2(\Omega)}\|\xi\|_{L^\infty}\|K_\varepsilon\|_{L^2(\Omega)}$, where

$$K_\varepsilon := \sup_{v \in [0,1]} \left| \int_0^1 f^\varepsilon(x,v')\, C(x,v,v')\, dv' - \langle \mu_x, C(x,v,\cdot)\rangle_{v'} \right| . \tag{2.24}$$

The property $\|K_\varepsilon\|_{L^2} \to 0$ when $\varepsilon \to 0$ can be shown by first approximating $C(x, v, v')$ by a sequence

$$C_n(x, v, v') = \sum_{j=1}^{R_n} a_j(x) b_j(v) c_j(v') \ \in \ \text{span}\left\{ C(\bar{\Omega}_x) \times C([0,1]_v) \times C_n^2([0,1]_{v'}) \right\},$$

such that $C_n \to C$ uniformly. Then we estimate

$$\begin{aligned} \|K_\varepsilon\|_{L^2(\Omega)} \ &\leq \ \sum_{j=1}^{R_n} \|a_j\|_{L^\infty} \|b_j\|_{L^\infty} \left\| \int_0^1 f^\varepsilon(x, v')\, c_j(v')\, dv' - \langle \mu_x, c_j \rangle_{v'} \right\|_{L^2(\Omega_x)} \\ &+ \ \|C_n - C\|_{L^\infty} \left\| \int_0^1 f^\varepsilon(x, v')\, dv' + \langle \mu_x, 1 \rangle_{v'} \right\|_{L^2(\Omega_x)}, \end{aligned}$$

where the last term is bounded by $2\|C_n - C\|_{L^\infty} b |\Omega|^{1/2}$. The first term tends to 0 as $\varepsilon \to 0$ for all fixed $n \in I\!N$ since $x \mapsto \int_0^1 f^\varepsilon(x, v')\, c_j(v')\, dv'$ is bounded in $H^1(\Omega)$ due to (2.20), and therefore compact in $L^2(\Omega)$ (strong). □

As a consequence of Lemma 3, the limit of (2.21) as $\varepsilon \to 0$ yields (1.9). And this finishes the proof of Theorem 1. □

# 3 The case when mutations are predominant

We develop in this section the

**Proof of Theorem 2**: We once again pass to the limit when $\varepsilon \to 0$ in the regularized problem (2.1) – (2.3) in order to get a solution to problem (1.2), (1.3). In order to do so, we multiply (2.1) by $(f^\varepsilon)^p$ ($p \geq 1$) and we integrate with respect to $x$ and $v$.

We get

$$p \int_0^1 \int_\Omega |\nabla_x f^\varepsilon(x,v)|^2 \, (f^\varepsilon(x,v))^{p-1} \, dx \, dv + \varepsilon p \int_0^1 \int_\Omega |\partial_v f^\varepsilon(x,v)|^2 \, (f^\varepsilon(x,v))^{p-1} \, dx \, dv$$

$$= \int_0^1 \int_\Omega k(x,v) \, (f^\varepsilon(x,v))^{p+1} \, dx \, dv$$

$$+ \int_0^1 \int_0^1 \int_\Omega K(x,v,v') \, (f^\varepsilon(x,v))^p \, f^\varepsilon(x,v') \, dx \, dv \, dv'$$

$$- \int_0^1 \int_0^1 \int_\Omega C(x,v,v') \, (f^\varepsilon(x,v))^{p+1} \, f^\varepsilon(x,v') \, dx \, dv \, dv' \, .$$

Due to Proposition 1, the solution $f^\varepsilon \geq 0$ satisfies (2.4), and hence:

$$p \iint |\nabla_x f^\varepsilon(x,v)|^2 \, (f^\varepsilon(x,v))^{p-1} dxdv \leq ||K||_{L^\infty} \, b \int \int |f^\varepsilon(x,v)|^p \, dxdv \quad (3.1)$$

$$+ \left(||k||_{L^\infty} - a\, C_-\right) \iint |f^\varepsilon(x,v)|^{p+1} \, dx \, dv.$$

So, under hypothesis (1.10), which can be rewritten as $a\, C_- > ||k||_{L^\infty}$, we see (using (3.1) for $p = 1$) that $f^\varepsilon$ is uniformly bounded (w.r.t. $\varepsilon$) in $H^1(\Omega_x;\, L^2([0,1]_v)) := \{f \in L^2(\Omega_x \times [0,1]_v) \,\big|\, \nabla_x f \in L^2(\Omega_x \times [0,1]_v)\}$.

Moreover, for any $p \geq 1$,

$$||f^\varepsilon||_{L^{p+1}}^{p+1} \leq \frac{||K||_{L^\infty} \, b}{a\, C_- - ||k||_{L^\infty}} \, ||f^\varepsilon||_{L^p}^p ,$$

so that letting $p \to +\infty$, we see that $f^\varepsilon$ is bounded in $L^\infty(\Omega_x \times [0,1]_v)$. Hence, we can extract from the family $(f^\varepsilon)_{\varepsilon>0}$ a subsequence still denoted by $(f^\varepsilon)_{\varepsilon>0}$ such that $f^\varepsilon \rightharpoonup f$ in $H^1(\Omega_x;\, L^2([0,1]_v))$ weak and $L^\infty(\Omega_x \times [0,1]_v)$ weak *.

As a consequence, for all $\varphi \in H^1(\Omega_x;\, L^2([0,1]_v))$ such that $\varphi, \partial_v\varphi, \partial_v^2\varphi \in C(\bar\Omega_x \times [0,1]_v)$ and (for all $x \in \Omega$) $\partial_v\varphi(x,0) = \partial_v\varphi(x,1) = 0$, we can write the weak form of (2.1) – (2.3):

$$\int_\Omega \int_0^1 \nabla_x f^\varepsilon \cdot \nabla_x \varphi \, dv dx - \varepsilon \int_\Omega \int_0^1 f^\varepsilon(x,v) \partial_v^2 \varphi \, dv \, dx \tag{3.2}$$
$$= \int_\Omega \int_0^1 k \, f^\varepsilon \, \varphi \, dv \, dx + \int_\Omega \int_0^1 \int_0^1 K \, f^\varepsilon(v') \, \varphi(v) \, dv' dv \, dx$$
$$- \int_\Omega \int_0^1 \int_0^1 C \, f^\varepsilon(v) \, f^\varepsilon(v') \, \varphi(v) \, dv' dv \, dx.$$

Then, we can pass to the limit (as $\varepsilon \to 0$) in the following terms:

$$\begin{aligned}
\iint \nabla_x f^\varepsilon \cdot \nabla_x \varphi \, dx \, dv \quad &\to \quad \iint \nabla_x f \cdot \nabla_x \varphi \, dx \, dv, \\
\varepsilon \iint f^\varepsilon \partial_{vv} \varphi \, dx \, dv \quad &\to \quad 0, \\
\iint k f^\varepsilon \varphi \, dx \, dv \quad &\to \quad \iint k f \varphi \, dx \, dv, \\
\iiint K \, f^\varepsilon(x,v') \, \varphi(x,v) \, dx \, dv \, dv' \quad &\to \quad \iiint K \, f(x,v') \, \varphi(x,v) \, dx \, dv \, dv'.
\end{aligned}$$

We now wish to prove that

$$\iiint C(x,v,v') \, f^\varepsilon(x,v') \, f^\varepsilon(x,v) \, \varphi(x,v) \, dx \, dv \, dv'$$
$$\to \quad \iiint C(x,v,v') \, f(x,v') \, f(x,v) \, \varphi(x,v) \, dx \, dv \, dv'.$$

We write therefore

$$\iiint C(x,v,v') \, f^\varepsilon(x,v') \, f^\varepsilon(x,v) \, \varphi(x,v) \, dx \, dv \, dv'$$
$$= \quad \iint f^\varepsilon(x,v') \left[ \int C(x,v,v') \, f^\varepsilon(x,v) \, \varphi(x,v) \, dv \right] dv' \, dx, \tag{3.3}$$

and observe that since $f^\varepsilon \rightharpoonup f$ in $L^\infty$ weak *, we only need to show that (up to a subsequence) $\int C f^\varepsilon \varphi dv$ strongly converges in $L^1(\Omega \times [0,1]_{v'})$ to $\int C f \varphi dv$. To this end we use the following lemma (with $s(x,v,v') := \varphi(x,v) C(x,v,v')$):

**Lemma 4** *Let $(f^\varepsilon)_{\varepsilon>0}$ be a sequence converging in $H^1(\Omega_x; L^2([0,1]_v))$ weak towards $f$, and $s \equiv s(x,v,v') \in L^2(\Omega_x \times [0,1]_v \times [0,1]_{v'})$.*

*Then, a subsequence (still denoted by $f^\varepsilon$) satisfies*

$$\int f^\varepsilon(x,v)\, s(x,v,v')dv \to \int f(x,v)\, s(x,v,v')dv \qquad \textit{strongly in } L^1(\Omega_x \times [0,1]_{v'}).$$

**Proof of Lemma 4:** We first observe that

$$\int f^\varepsilon(x,v)\xi(v)\, dv \to \int f(x,v)\xi(v)\, dv \qquad \text{in } L^1(\Omega_x). \tag{3.4}$$

Then, we approximate $s$ (in $L^2(\Omega_x \times [0,1]_v \times [0,1]_{v'})$) by a sequence

$$s_n(x,v,v') = \sum_{j=1}^{K_n} a_j^n(x) b_j^n(v) c_j^n(v') \quad \in \ \text{span}\left\{C(\bar{\Omega}_x) \times C([0,1]) \times C([0,1])\right\}.$$

We see that

$$\begin{aligned}
&\iint \left| \int [f^\varepsilon(x,v) - f(x,v)]\, s(x,v,v')\, dv \right| dx\, dv' \\
&\quad \leq \left[ \|f^\varepsilon\|_{L^2(\Omega_x \times [0,1]_v)} + \|f\|_{L^2(\Omega_x \times [0,1]_v)} \right] \|s_n - s\|_{L^2(\Omega_x \times [0,1]_v \times [0,1]_{v'})} \\
&\quad + \sum_{j=1}^{K_n} \|a_j^n c_j^n\|_{L^\infty} \int_{\Omega_x} \left| \int [f^\varepsilon(x,v) - f(x,v)]\, b_j^n(v)\, dv \right| dx.
\end{aligned}$$

Thanks to (3.4), the second term tends to 0, hence $\int f^\varepsilon s\, dv$ converges to $\int f s\, dv$ in $L^1(\Omega_x \times [0,1]_{v'})$. This ends the proof of Lemma 4. $\square$

Finally, we can pass to the limit when $\varepsilon \to 0$ for all the terms in (the weak form (3.2) of) problem (2.1) – (2.3), and get a weak solution $f$ to eq. (1.2), (1.3), for test functions $\varphi \in H^1(\Omega_x;\, L^2([0,1]_v))$ such that moreover $\varphi, \partial_v\varphi, \partial_v^2\varphi \in C(\bar{\Omega}_x \times [0,1]_v)$ and (for all $x \in \Omega$) $\partial_v\varphi(x,0) = \partial_v\varphi(x,1) = 0$. These last assumptions can easily be removed thanks to an approximation procedure. Note that these additional assumptions on $\varphi$ were only used to treat the term $\varepsilon\, \partial_v^2 f^\varepsilon$ which disappears in the limit. As a last remark, note that since $f \in L^\infty(\Omega \times [0,1])$, we also have $\Delta_x f \in L^\infty(\Omega \times [0,1])$. $\square$

**Remark 2**: Theorem 2 shows that smooth solutions of problem (1.2), (1.3) exist when mutations are predominant. Our feeling is that in many situations, all steady solutions of this equation are smooth when mutations are present (i.e. $K > 0$). This is supported by the study of the time-dependant equation (1.1) with suitable initial and boundary conditions, cf. [DFP].

# 4 Some numerical examples

We present here some computations which illustrate the results obtained in §2. In particular we show the influence of the diffusion strength on the shape of the steady states.

More precisely, we chose to use the following parameters for the selection, mutation, and competition:

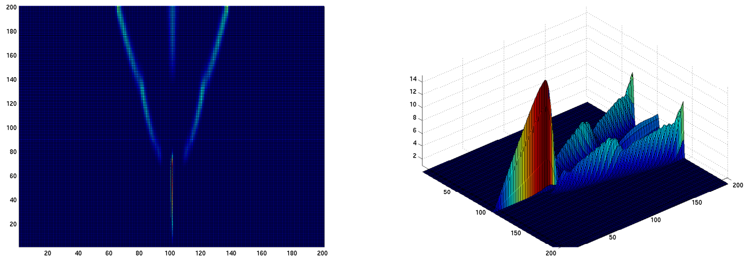
$$k(x,v) = A\,(3\,x - 12(v-1/2)^2), \qquad K(x,v,v') = 0,$$

$$C(x,v,v') = \frac{10\,A}{1+12\,(v-v')^2}.$$

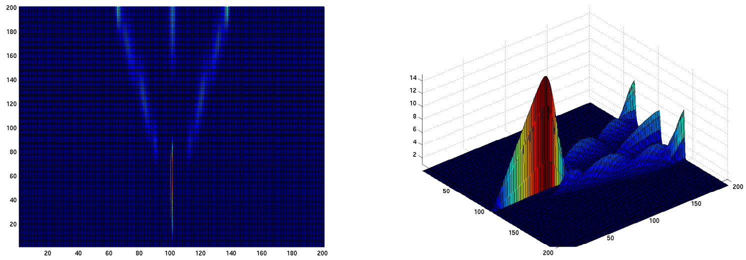
Note that this choice of $k$ does not satisfy the positivity assumption from (1.4) on the whole $(x,v)$–domain. It would, however, hold on an appropriate subdomain. Anyhow, the above fitness function $k$ prefers one single trait (at $v = \frac{1}{2}$). On the other hand, the competition kernel $C$ favors a clear splitting of population into well separated traits. This example hence illustrates the balance between these two opposing effects.

The computation is performed on the square $[0,1] \times [0,1]$ by letting $t \to \infty$ in the time-dependent equation (1.1), (1.3). We use 200 cells in the $x$-space and 200 cells in the $v$-space. The time step is adjusted in order to obtain a CFL parameter of 0.396. We use a semi-implicit finite difference scheme (the 0-th order part of the equation is discretized in an implicit way, but not the diffusion part).

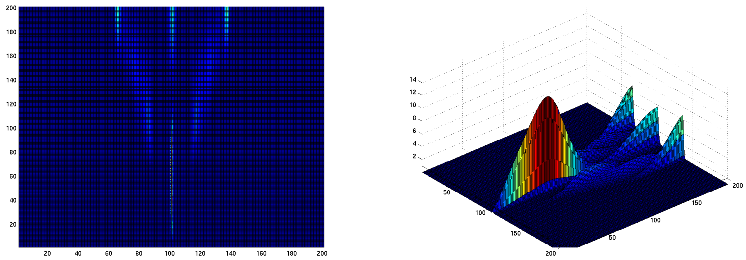
We present results obtained for $A = 10^7$, $10^6$, $10^5$, $10^4$ where $1/A$ plays the role of the diffusion coefficient in (1.2). The surface that defines the (quasi) stationary solution is presented at two different angles so that the shape of the solution is clearer. The coordinate $x$ corresponds to the vertical axis and the coordinate $v$ to the horizontal axis in the figures on the left. The graduation from 0 to 200 corresponds to the numbering of cells.



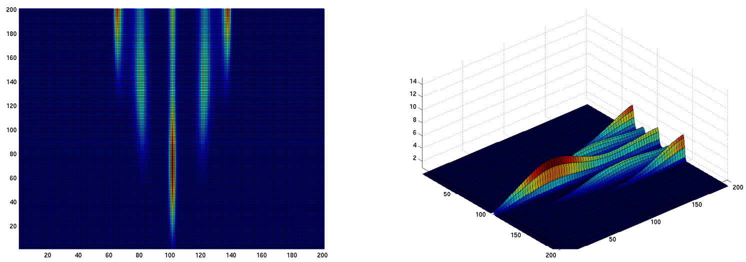
$A = 10^7$



$A = 10^6$



$A = 10^5$



$A = 10^4$

The first figure ($A = 10^7$) corresponds to a case in which the diffusion is very small, so that its solution is very close to the case without diffusion which can be computed explicitly (cf. [DJMR]): for $x$ small, the function of $v$ is a Dirac mass at $v = 0.5$, for $x$ bigger, it is the sum of two Dirac masses, and for $x$ large, it is the sum of three Dirac masses (one of them sits at $v = 0.5$). Note the quite sharpe transitions (in $x$) between the regions populated by individuals with one, two, or three traits.

In the other figures, the diffusion w.r.t. $x$ entails the presence of individuals with various $v$ in the whole domain $x \in [0, 1]$. This is particularly clear in the last figure, where the diffusion is strong enough to build a "five-modal" function of $v$ at point $x = 1$.

**Acknowledgement:**

This work was partly supported by an ÖAD-Amadeus project. The first author was partially supported by the FWF-Wissenschaftskolleg "Differential Equation Models in Science and Engineering". He is greatful to the hospitality of the ENS-Cachan where this research was carried out.

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