

SOME ASPECTS OF THE ASYMPTOTICS LEADING FROM GAS-PARTICLES EQUATIONS TOWARDS MULTIPHASE FLOWS EQUATIONS

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ABSTRACT. This paper is devoted to the establishment at the formal level of an expansion leading from gas-particles (also called Eulerian-Lagrangian) models for monodisperse thick sprays towards multiphase flows models (also called Eulerian-Eulerian). This passage to the limit is done through a Hilbert type expansion involving an inelastic collision operator whose kernel only consists of distributions in velocities and internal energies which are Dirac masses.

1. INTRODUCTION

Sprays are flows involving a continuous gaseous phase and a disperse phase (typically liquid) whose volume fraction is not too big. We consider in this paper only monodisperse sprays (that is, the disperse phase is constituted of droplets which all have the same radius r). Moreover, we shall also suppose that all droplets are incompressible and that no evaporation occurs, so that r will be in the sequel an absolute constant.

We denote by $\alpha := \alpha(t, x) \in [0, 1]$ the volume fraction of gas at time $t \in \mathbb{R}_+$ and point $x \in \Omega$ (Ω being a subset of $\mathbb{R}^3$). The underlying assumption when one writes such a quantity is that the volume $\frac{4}{3}\pi r^3$ of a typical droplet is much smaller than a small (but macroscopic) elementary volume of fluid. We say that the spray is thick (Cf. [O'R81]) when $1 - \alpha(t, x)$ is not negligible in at least part of $\mathbb{R}_+ \times \Omega$ (typically $1 - \alpha(t, x) \gg 10^{-3}$) but not too big either (typically, $1 - \alpha(t, x) \leq 0.2$ at worst). In such a situation (that we shall assume from now on), one can write a set of equations in which α is one of the unknowns (this is not the case in so-called thin sprays, where the approximation $\alpha \sim 1$ is performed).

We denote by $\rho_g := \rho_g(t, x) \in \mathbb{R}_+$, $p := p(t, x) \in \mathbb{R}_+$, $u_g := u_g(t, x) \in \mathbb{R}^3$, $e_g := e_g(t, x) \in \mathbb{R}_+$, $E_g := E_g(t, x) = e_g(t, x) + \frac{1}{2}|u_g(t, x)|^2 \in \mathbb{R}_+$ and $T_g := T_g(t, x) \in \mathbb{R}_+$ the respective density (of mass), pressure, velocity, internal energy (per unit of mass), total (internal + kinetic) energy (per unit of mass), and temperature of the gas. When the gas is inviscid, the following set of balance laws is written down for the quantities ρ_g , u_g and E_g ([O'R81]):

$$\begin{aligned} (1) \quad & \partial_t(\alpha \rho_g) + \nabla_x \cdot (\alpha \rho_g u_g) = 0, \\ (2) \quad & \partial_t(\alpha \rho_g u_g) + \nabla_x \cdot (\alpha \rho_g u_g \otimes u_g) + \nabla_x p = -A, \\ (3) \quad & \partial_t(\alpha \rho_g E_g) + \nabla_x \cdot \left(\alpha \rho_g \left(E_g + \frac{p}{\rho_g} \right) u_g \right) + p \partial_t \alpha = -B_1 - B_2, \end{aligned}$$

where A is the momentum transfered to the (elementary volume at time t and point x of) gas by the dispersed phase and B_1 , B_2 constitute the corresponding (resp. mechanical and thermal) transfer.

The density in the phase space $f := f(t, x, u_p, e_p) \geq 0$ of droplets which at time t and point x have velocity $u_p \in \mathbb{R}^3$ and internal energy $e_p \in \mathbb{R}_+$ satisfies the following Vlasov-Boltzmann equation:

$$(4) \quad \partial_t f + u_p \cdot \nabla_x f + \nabla_{u_p} \cdot (f \Gamma) + \partial_{e_p} (f \phi) = Q(f, f),$$

where Γ and ϕ represent the transfer of momentum and energy of the gaseous phase on a given droplet (which at time t and point x has velocity $u_p \in \mathbb{R}^3$ and internal energy $e_p \in \mathbb{R}_+$). Accordingly,

$$(5) \quad m_p \Gamma = -\frac{m_p}{\rho_p} \nabla_x p - D(u_p - u_g); \quad m_p \phi = \Phi(T_g - T_p),$$

$$(6) \quad A = \iint_{u_p, e_p} m_p \Gamma f du_p de_p,$$

$$(7) \quad B_1 = \iint_{u_p, e_p} m_p \left(\Gamma + \frac{\nabla_x p}{\rho_p} \right) \cdot u_p f du_p de_p,$$

$$(8) \quad B_2 = \iint_{u_p, e_p} m_p \phi f du_p de_p,$$

where m_p is the mass of one droplet, ρ_p is the density of the liquid constituting the droplets ($m_p = \frac{4}{3} \pi r^3 \rho_p$, and m_p, ρ_p, r are absolute constants), and T_p is the temperature of the droplet. In (5), the term $D(u_p - u_g)$ models the drag. The drag coefficient D is in general a function of $\rho_g, |u_g - u_p|$ (and also r, ρ_p and the molecular viscosity of the gas [this last quantity being neglected in the equation of momentum of the gas]).

Also in (5), the term $\Phi(T_g - T_p)$ models the thermal exchanges between the droplets and the gas. The coefficient Φ in general depends upon the thermal viscosity of the particle and the Nusselt number (and therefore upon $r, |u_g - u_p|$, etc.).

The system is closed thanks to the constitutive equations of the gas and the liquid:

$$(9) \quad p(t, x) = P_1(\rho_g(t, x), e_g(t, x)), \quad T_g(t, x) = T_1(\rho_g(t, x), e_g(t, x)),$$

$$(10) \quad T_p = T_2(e_p),$$

and the identity for the volume fraction of droplets:

$$(11) \quad 1 - \alpha(t, x) = \frac{4}{3} \pi r^3 \iint_{u_p, e_p} f(t, x, u_p, e_p) du_p de_p.$$

The set of equations (1) –(11) is sometimes called “Gas-particles” or “Eulerian-Lagrangian”. From the mathematical viewpoint, it consists in coupling the compressible Euler equation of fluid dynamics with the Vlasov-Boltzmann equation of kinetic theory, through:

- i) the exchange of momentum and energy between the phases,
- ii) the volume fraction of gas α .

A different kind of modeling is possible for sprays, in which the density in the phase space f of droplets is replaced by macroscopic quantities, namely: the density (of mass) $\rho := \rho(t, x) \in \mathbb{R}_+$ of liquid, its velocity $v := v(t, x) \in \mathbb{R}^3$, its internal energy (per unit of mass) $e := e(t, x) \in \mathbb{R}_+$, its total (internal + kinetic) energy (per unit of mass) $E := E(t, x) = e(t, x) + \frac{1}{2} |v(t, x)|^2 \in \mathbb{R}_+$ and its temperature $T := T(t, x) \in \mathbb{R}_+$. The set of equations for those quantities write ([Ish75])

$$(12) \quad \partial_t(\alpha \rho_g) + \nabla_x \cdot (\alpha \rho_g u_g) = 0,$$

$$(13) \quad \partial_t(\alpha \rho_g u_g) + \nabla_x \cdot (\alpha \rho_g u_g \otimes u_g) + \alpha \nabla_x p = -\tilde{A},$$

$$(14) \quad \partial_t(\alpha \rho_g E_g) + \nabla_x \cdot \left(\alpha \rho_g \left(E_g + \frac{p}{\rho_g} \right) u_g \right) + p \partial_t \alpha = -\tilde{B}_1 - \tilde{B}_2,$$

$$(15) \quad \partial_t((1 - \alpha)\rho) + \nabla_x \cdot ((1 - \alpha)\rho v) = 0,$$

$$(16) \quad \partial_t((1 - \alpha)\rho v) + \nabla_x \cdot ((1 - \alpha)\rho v \otimes v) + (1 - \alpha)\nabla_x p = \tilde{A},$$

$$(17) \quad \partial_t((1 - \alpha)\rho E) + \nabla_x \cdot \left((1 - \alpha)\rho \left(E + \frac{p}{\rho} \right) v \right) + p\partial_t(1 - \alpha) = \tilde{B}_1 + \tilde{B}_2.$$

Those balance laws are completed by the constitutive equations of the gas (similar to (9))

$$(18) \quad p(t, x) = P_1(\rho_g(t, x), e_g(t, x)); \quad T_g(t, x) = T_1(\rho_g(t, x), e_g(t, x)),$$

together with the constitutive equations of the liquid (incompressible) phase

$$(19) \quad T(t, x) = T_2(e(t, x)), \quad \rho(t, x) = \rho_p.$$

Finally, the transfer terms $\tilde{A}$, $\tilde{B}_1$, $\tilde{B}_2$ of momentum and energy write

$$\begin{aligned} \tilde{A} &= -(1 - \alpha) \frac{\rho}{m_p} \tilde{D} (v - u_g), & \tilde{B}_1 &= -(1 - \alpha) \frac{\rho}{m_p} \tilde{D} (v - u_g) \cdot v, \\ \tilde{B}_2 &= -(1 - \alpha) \frac{\rho}{m_p} \tilde{\Phi} (T - T_g). \end{aligned}$$

The terms $\tilde{A}$, $\tilde{B}_1$, $\tilde{B}_2$ respectively represent the drag force term, the deposit of the drag force term in energy and the thermal exchanges. The constants $\tilde{D}$, $\tilde{\Phi}$ respectively represent the drag force coefficient and the thermal conduction coefficient. They can be fitted using experimental data and in general depend upon α , $v - u_g$, etc. Note that systems like (12) – (19) appear not only in the theory of sprays, but also in many other kinds of multiphase flows (stratified, churning flows, etc.), the transfer terms (like $\tilde{A}$, etc.) depend in general of the type of flows which are considered.

Our goal in this work is to provide a clear scaling which enables to derive “rigorously at the formal level” macroscopic equations such as (12) – (19) from “gas-particles” equations such as (1) – (11). This is done after having obtained a non-dimensional version of eq. (1) – (11) under suitable assumptions on the flow: basically, the collision term Q appearing in (4) must be dominant. This exactly corresponds in the context of standard kinetic theory to the Hilbert expansion, in which $\frac{1}{\varepsilon}$ is put in front of the collision kernel, and which leads from the Boltzmann equation of rarefied gases towards the compressible Euler equations of fluid dynamics (Cf. [KMN79] for a rigorous proof in the context of very smooth solutions on a small time interval, and [Gol05] for a general survey on the question).

Our paper is structured as follows: in section 2, the gas-particles equations are specified in detail, including the collision kernel Q (representing elastic or inelastic collisions). Then, a non-dimensional version of those equations is provided in section 3. The distributions which cancel Q are described in section 4. Finally, equations for the macroscopic quantities (for both phases) are written down, first in a non closed form in section 5, and then in closed form in section 6. A typical example of modeling of an experiment with realistic data is presented in section 7. Some conclusions and perspectives are presented at the end of the paper (section 8).

2. PRESENTATION OF THE GAS-PARTICLES EQUATIONS AND THE ELASTIC OR INELASTIC ASSOCIATED COLLISION KERNELS

We recall here the main assumptions that we presented in the introduction of this work about the flow we consider.

We assume that the flow is constituted of a surrounding gas and of a dispersed liquid phase. This phase is itself assumed to be of relatively small volume fraction (typically between 10^{-3} and 0.2),

and to be constituted of very tiny spherical incompressible droplets having all the same radius r (that is, the spray is monodisperse). The flow inside the droplets is not modeled.

As stated in the introduction, a system which models the spray under this assumption can be devised by considering the unknown $f := f(t, x, u_p, e_p) \geq 0$ for the droplets and $\rho_g := \rho_g(t, x) \in \mathbb{R}_+$, $u_g := u_g(t, x) \in \mathbb{R}^3$, $p := p(t, x) \in \mathbb{R}_+$, $E_g := E_g(t, x) \in \mathbb{R}_+$ for the gas. The set of equations is then (1) – (11), and it remains to precisely define the collision operator Q .

2.1. Formula for the collision kernel. The assumptions that underly the establishment of this operator are the following: First, since the spray is monodisperse, no complex phenomena of coalescence or breakup of droplets are considered. For the same reason, all collisions are supposed to be binary (that is, two droplets are present at the beginning of the collision and produce two droplets at the end of the collision).

Then, since droplets are macroscopic objects, the cross section will be that of hard spheres. For the same reason, kinetic energy conservation during the process of collision is not expected in general. As a consequence, one needs to write down a model in which part of the kinetic energy is lost: models of granular media (Cf. [Vil02]) provide a good solution for that (with small modifications due to the fact that here droplets are liquid and not solid).

Moreover, since the internal energy of the droplets is one of the variables in f , one needs a rule to exchange internal energy during the process of collision: models for polyatomic gases (Cf. [Des97]) provide a simple solution for this physical phenomenon.

Finally, the kinetic energy which is lost has to be converted in internal energy, and to be distributed between the two outgoing droplets. Since those droplets have the same volume, we choose to divide it equally.

Collecting all those ideas, we end up with a collision kernel which writes

$$(20) \quad Q(f, f)(t, x, u_p, e_p) = \iiint_{\substack{\sigma \in \mathbb{S}^2, u_{p*} \in \mathbb{R}^3 \\ e_{p*} \in \mathbb{R}^+}} \left(\frac{1}{1-a} \frac{1}{\beta^4} f(t, x, 'u_{p*}, 'e_{p*}) f(t, x, 'u_p, 'e_p) \right. \\ \left. - f(t, x, u_{p*}, e_{p*}) f(t, x, u_p, e_p) \right) \times 1_{\{'e_p, 'e_{p*} \geq 0\}} r^2 |u_p - u_{p*}| d\sigma du_{p*} de_{p*},$$

where the pre-collisional velocities $'u_{p*}$ and $'u_p$ are defined as

$$'u_p = \frac{u_p + u_{p*}}{2} + \frac{1}{\beta} \frac{|u_p - u_{p*}|}{2} \sigma, \\ 'u_{p*} = \frac{u_p + u_{p*}}{2} - \frac{1}{\beta} \frac{|u_p - u_{p*}|}{2} \sigma,$$

where σ belongs to the unit sphere $\mathbb{S}^2$, and $\int_{\sigma \in \mathbb{S}^2} d\sigma = 4\pi$. The pre-collisional internal energies $'e_{p*}$ and $'e_p$ are defined as

$$'e_p = \frac{2-a}{2-2a} e_p - \frac{a}{2-2a} e_{p*} - \frac{1}{2} \Delta E, \\ 'e_{p*} = -\frac{a}{2-2a} e_p + \frac{2-a}{2-2a} e_{p*} - \frac{1}{2} \Delta E,$$

where

$$(21) \quad \Delta E = \frac{1}{2} ('u_p^2 + 'u_{p*}^2 - u_{p*}^2 - u_p^2) = \left(\frac{1-\beta^2}{4\beta^2} \right) |u_p - u_{p*}|^2$$

is the loss of kinetic energy (or gain of internal energy) [divided by mass].

In those formulas, $\beta := \beta(|u_p - u_{p*}|)$ is a measure of the inelasticity of the collision (the collision is elastic when $\beta = 1$), and $a := a(|u_p - u_{p*}|)$ is the parameter which measures what part of the internal energy is exchanged during a collision (no internal energy is exchanged when $a = 0$).

Note that the prefactor $\frac{1}{1-a} \frac{1}{\beta^4}$ is related to the Jacobian of the pre-collisional transform $(u_p, e_p, u_{p*}, e_{p*}) \mapsto (u_p', e_p', u_{p*}', e_{p*}')$, and to the cross section of hard spheres. The model presented here is reminiscent of models appearing in granular gases, where the collisions concern solid particles. The main difference is that here, the loss of kinetic energy, due to viscosity effects in the liquid constituting the droplets, is the same in all directions.

2.2. Weak form of the collision kernels. The following weak form of the collision kernel can be obtained (at the formal level) (see [Bar04b] and [Des93]):

For all test function $\Psi := \Psi(u_p, e_p)$,

$$\begin{aligned}
& \iint_{u_p, e_p} Q(f, f) \Psi(u_p, e_p) du_p de_p \\
&= \iiint_{u_p, e_p, u_{p*}, e_{p*}, \sigma} f f_* [\Psi' - \Psi] r^2 |u_p - u_p^*| du_p de_p du_{p*} de_{p*} \\
&= \frac{1}{2} \iiint_{u_p, e_p, u_{p*}, e_{p*}, \sigma} f f_* [\Psi'_* + \Psi' - \Psi_* - \Psi] r^2 |u_p - u_p^*| du_p de_p du_{p*} de_{p*},
\end{aligned} \tag{22}$$

with the following formulas for the post-collisional velocities and internal energies:

$$\begin{aligned}
u_p' &= \frac{u_p + u_{p*}}{2} + \beta \frac{|u_p - u_{p*}|}{2} \sigma, & u_{p*}' &= \frac{u_p + u_{p*}}{2} - \beta \frac{|u_p - u_{p*}|}{2} \sigma, \\
e_p' &= \frac{2-a}{2} e_p + \frac{a}{2} e_{p*} + \frac{1}{2} \Delta E, & e_{p*}' &= \frac{a}{2} e_p + \frac{2-a}{2} e_{p*} + \frac{1}{2} \Delta E
\end{aligned}$$

(remember that ΔE is defined in (21) and that $\Psi'_* := \Psi(t, x, u_{p*}', e_{p*}')$, etc.).

Specializing the weak formulation to the functions $\Psi(u_p, e_p) = m_p$,

$\Psi(u_p, e_p) = m_p u_p$ and $\Psi(u_p, e_p) = \frac{1}{2} m_p u_p^2 + m_p e_p$, we get the conservations of mass, momentum and total energy :

$$\iint_{u_p, e_p} Q(f, f)(u_p, e_p) m_p du_p de_p = 0, \tag{23}$$

$$\iint_{u_p, e_p} Q(f, f)(u_p, e_p) m_p u_p du_p de_p = 0, \tag{24}$$

$$\iint_{u_p, e_p} Q(f, f)(u_p, e_p) \left[\frac{1}{2} m_p u_p^2 + m_p e_p \right] du_p de_p = 0. \tag{25}$$

Moreover, for elastic collisions (that is, in the case when $\beta = 1$) -in which both kinetic and internal energy are conserved-, we have

$$(26) \quad \iint_{u_p, e_p} Q(f, f)(u_p, e_p) \left[\frac{1}{2} m_p u_p^2 \right] du_p de_p = 0,$$

$$(27) \quad \iint_{u_p, e_p} Q(f, f)(u_p, e_p) [m_p e_p] du_p de_p = 0.$$

3. NON DIMENSIONAL FORM OF THE VLASOV-BOLTZMANN EQUATION

We write down in this short section the dimensional analysis which enables to obtain the Hilbert-like expansion for the Vlasov-Boltzmann equation (4). In order to do so, we first introduce the following time/space typical quantities:

- t_g : typical time of the experiment,
- L : typical length of the experiment.

Next, we introduce quantities related to the gas and the droplets (remember that r , m_p , ρ_p are the radius, mass and density of droplets, and that D , Φ are the coefficients for drag force and thermal exchanges)

- N : typical number of droplets of the experiment,
- V : typical mean velocity of the droplets. We shall assume that it is also the typical thermal velocity of the droplets [that is, the square root of the variance of the velocity distribution], and the typical velocity of the gas. One has $V t_g = L$.
- I_p : typical internal energy of the droplets per mass unit,
- I_g : typical internal energy of the gas per mass unit,
- T_T : typical temperature of the droplets. We shall assume that it is also the typical temperature of the gas.
- P : Typical pressure of the gas
- $P' = \rho_p V^2$: this quantity has the dimension of a pressure

It is customary to introduce at this level the mean free path $\sigma = \frac{L^3}{r^2 N}$. Finally, we denote by ε the Knudsen number $\varepsilon = \frac{\sigma}{L}$. This quantity is at the basis of the Hilbert expansion: it is supposed to tend to zero in the hydrodynamic limit (see [Cer88] and [CC70]).

The orders of magnitude of the terms appearing in (4) are then given by the following formulas:

- Time derivative term:

$$(28) \quad \partial_t f \sim \frac{N}{I_p V^3 L^3} \frac{1}{t_g}.$$

- Transport term:

$$(29) \quad \begin{aligned} \nabla_x \cdot (f u_p) &\sim \frac{N}{I_p V^3 L^3} \frac{V}{L} \\ &\sim \partial_t f. \end{aligned}$$

- Pressure term:

$$(30) \quad \begin{aligned} \nabla_{u_p} \cdot \left(\frac{\nabla_x p}{\rho_p} f \right) &\sim \frac{N}{I_p V^3 L^3} \frac{V}{L} \frac{P}{P'} \\ &\sim \frac{P}{P'} \partial_t f. \end{aligned}$$

- Drag force term:

$$(31) \quad \begin{aligned} \nabla_{u_p} \cdot \left(\frac{D}{m_p} (u_p - u_g) f \right) &\sim \frac{N}{I_p V^3 L^3} \frac{D}{r^3 \rho_p} \\ &\sim \left(\frac{D t_g}{r^3 \rho_p} \right) \partial_t f. \end{aligned}$$

- Energy exchange term:

$$(32) \quad \begin{aligned} \partial_{e_p} \left(\frac{\Phi}{m_p} (T_g - T_p) f \right) &\sim \frac{N}{I_p V^3 L^3} \frac{\Phi T_T}{r^3 \rho_p I_p} \\ &\sim \frac{\Phi T_T t_g}{r^3 \rho_p I_p} \partial_t f. \end{aligned}$$

- Collisional term:

$$(33) \quad \begin{aligned} Q(f, f) &\sim r^2 V I_p V^3 \left(\frac{N}{I_p V^3 L^3} \right)^2 \\ &\quad \text{(we recall that we use the hard sphere cross section)} \\ &\sim \frac{1}{\varepsilon} \partial_t f. \end{aligned}$$

We now introduce non-dimensional quantities (denoted with a tilde) for the unknowns and parameters entering eq. (4). That is, we consider

$$\tilde{t} = \frac{t}{t_g}, \quad \tilde{x} = \frac{x}{L}, \quad \tilde{u}_p = \frac{u_p}{V}, \quad \tilde{e}_p = \frac{e_p}{I_p}, \quad \tilde{T}_p = \frac{T_p}{T_T},$$

$$\tilde{f}(\tilde{t}, \tilde{x}, \tilde{u}_p, \tilde{e}_p) = \frac{I_p L^3 V^3}{N} f(t_g \tilde{t}, L \tilde{x}, V \tilde{u}_p, I_p \tilde{e}_p),$$

for the particles and

$$\tilde{u}_g(\tilde{t}, \tilde{x}) = \frac{u_g(t_g \tilde{t}, L \tilde{x})}{V}, \quad \tilde{T}_g(\tilde{t}, \tilde{x}) = \frac{T_g(t_g \tilde{t}, L \tilde{x})}{T_T}, \quad \tilde{e}_g(\tilde{t}, \tilde{x}) = \frac{e_g(t_g \tilde{t}, L \tilde{x})}{I_g}, \quad \tilde{P}(\tilde{t}, \tilde{x}) = \frac{p(t_g \tilde{t}, L \tilde{x})}{P}$$

for the gas.

The equation satisfied by $\tilde{f}$ then becomes

$$(34) \quad \partial_{\tilde{t}} \tilde{f} + \tilde{u}_p \cdot \nabla_{\tilde{x}} \tilde{f} + \nabla_{\tilde{u}_p} \cdot (\tilde{f} \tilde{\Gamma}) + \partial_{\tilde{e}_p} (\tilde{f} \tilde{\phi}) = \frac{1}{\varepsilon} Q(\tilde{f}, \tilde{f}),$$

where

$$\begin{aligned} \tilde{\Gamma} &= \frac{P}{P'} \nabla_{\tilde{x}} \tilde{p} + C_2 (\tilde{u}_p - \tilde{u}_g), \\ \tilde{\phi} &= C_3 (\tilde{T}_g - \tilde{T}_p), \\ C_2 &= \frac{D}{m_p} t_g, \\ C_3 &= \frac{\Phi T_T t_g}{m_p I_p}. \end{aligned}$$

We shall now study the limit of eq. (34) when $\varepsilon \rightarrow 0$. We see that this limit makes sense when the typical parameters of the experiment under study are such that

(35a) from (28) and (29), $1 \gg \varepsilon$,

(35b) from (30), $\frac{P'}{P} \gg \varepsilon$,

(35c) from (31), $\frac{m_p}{D t_g} \gg \varepsilon$,

(35d) from (32), $\frac{m_p I_p}{\Phi T t_g} \gg \varepsilon$.

A typical situation appearing in the nuclear industry where those assumptions are fulfilled is described in [Mat06].

4. SOLUTIONS OF $Q(f, f) = 0$

In order to pass to the limit (at the formal level) in eq. (34) when $\varepsilon \rightarrow 0$, we study the solutions of the functional equation $Q(f, f) = 0$.

4.1. Elastic collisions and Maxwellian distribution. We suppose in this subsection that all collisions are elastic (that is, we assume that $\beta = 1$) so that $Q(f, f)$ takes the simpler form (w.r.t. (20))

$$Q(f, f)(u_p, e_p) = \iiint_{\substack{\sigma \in \mathbb{S}^2, u_{p*} \in \mathbb{R}^3 \\ e_{p*} \in \mathbb{R}^+}} \left(\frac{1}{|1-a|} {}'f_* {}'f - f_* f \right) \chi_{e_p, e_{p*} \geq 0}(e_p, e_{p*}) r^2 |u_p - u_{p*}| d\sigma du_{p*} de_{p*}$$

with the pre-collisional velocities

$${}'u_p = \frac{u_p + u_{p*}}{2} + \frac{|u_p - u_{p*}|}{2} \sigma, \quad {}'u_{p*} = \frac{u_p + u_{p*}}{2} - \frac{|u_p - u_{p*}|}{2} \sigma,$$

and the pre-collisional internal energies

$${}'e_p = \frac{2-a}{2-2a} e_p - \frac{a}{2-2a} e_{p*}, \quad {}'e_{p*} = -\frac{a}{2-2a} e_p + \frac{2-a}{2-2a} e_{p*}.$$

Here, we use the standard notations $'f_* = f(t, x, {}'u_{p*}, {}'e_{p*})$, etc.

In this model, both kinetic and internal energy are conserved during the collisions (that is, (26) and (27) hold). Note that the collision kernel presented here differs from the standard Boltzmann operator (of rarefied monoatomic gases, with hard spheres cross section) only by the taking into account of the exchange of internal energy between the droplets. Boltzmann's H-theorem takes in this case the following form:

Lemma 4.1 (H-theorem for the elastic kernel). *If $f := f(u_p, e_p)$ is a positive measure with compact support in e_p (and a function of u_p such that all the integrals below make sense) and the thermal exchange parameter $a := a(|u_p - u_{p*}|)$ satisfies $0 < a \leq 1$, then the following properties are equivalent:*

(1) *There exists a strictly convex function Ψ in $\mathcal{C}^\infty(\mathbb{R}^+)$ such that*

$$(36) \quad \iint_{u_p, e_p} Q(f, f)(u_p, e_p) \Psi(e_p) du_p de_p = 0,$$

and the following inequality holds:

$$(37) \quad \iint_{u_p, e_p} Q(f, f)(u_p, e_p) \log(f(u_p, e_p)) du_p de_p = 0;$$

$$(2) \quad Q(f, f) = 0;$$

$$(3) \quad \text{One can find } Z, T > 0, v \in \mathbb{R}^3 \text{ and } e > 0 \text{ such that}$$

$$(38) \quad f(u_p, e_p) = \frac{Z}{(2\pi T)^{\frac{3}{2}}} \exp\left(-\frac{(u_p - v)^2}{2T}\right) \delta_{e_p=e}(e_p).$$

The parameters Z, T, v, e represent the respective number of droplets, mean velocity of droplets, mean internal energy of the droplets, collective temperature of the droplets (with no direct link with the temperature T_p of each individual droplet).

Proof of Lemma 4.1: As usually, (3) $\Rightarrow$ (2) $\Rightarrow$ (1) is a consequence of direct computations. It remains to prove that (1) $\Rightarrow$ (3).

In order to do so, we consider f a positive measure with compact support in e_p which satisfies relations (36) and (37). Then the tensor product $f \otimes f$ is also a positive measure with compact support in e_p and e_{p*} . Furthermore, the support of $f \otimes f$ [w.r.t. e_p, e_{p*}] is linked to the support of f [w.r.t. e_p] through (see [Hor90] or [Sch66])

$$(39) \quad \text{supp}(f \otimes f) = \text{supp}(f) \times \text{supp}(f).$$

Let Ψ be the strictly convex function in $\mathcal{C}^\infty(\mathbb{R}^+)$ which satisfies (36), that is, using a transparent notation,

$$\langle Q(f, f)(u_p, e_p), \Psi(e_p) \rangle = 0.$$

Using the properties of the collision kernel (see subsection 2.2), one gets (with $\Psi'_* := \Psi(e'_{p*})$, etc.):

$$\langle r^2 |u_p - u_{p*}| f \otimes f, \Psi + \Psi_* - \Psi' - \Psi'_* \rangle = 0.$$

The function Ψ is strictly convex and $0 < \frac{a}{2} < 1$, so that the following relations are satisfied for $e_p \neq e_{p*}$:

$$\begin{aligned} \Psi\left(\left(1 - \frac{a}{2}\right)e_p + \frac{a}{2}e_{p*}\right) &< \left(1 - \frac{a}{2}\right)\Psi(e_p) + \frac{a}{2}\Psi(e_{p*}), \\ \Psi\left(\left(1 - \frac{a}{2}\right)e_{p*} + \frac{a}{2}e_p\right) &< \left(1 - \frac{a}{2}\right)\Psi(e_{p*}) + \frac{a}{2}\Psi(e_p). \end{aligned}$$

As a consequence, the function

$$K(e_p, e_{p*}) := \Psi(e_p) + \Psi(e_{p*}) - \Psi\left(\left(1 - \frac{a}{2}\right)e_p + \frac{a}{2}e_{p*}\right) - \Psi\left(\left(1 - \frac{a}{2}\right)e_{p*} + \frac{a}{2}e_p\right)$$

is strictly positive except when $e_p = e_{p*}$.

We now prove that the support of $f \otimes f$ w.r.t. (e_p, e_{p*}) is restricted to a single point or empty.

Note first of all that if the support of $f \otimes f$ is empty, then the measure $r^2 |u_p - u_{p*}| f \otimes f$ is equal to zero and (38) is satisfied with $Z = 0$.

We now suppose that the support of $f \otimes f$ is not empty. Let $\chi := \chi(u_p, e_p, u_{p*}, e_{p*})$ be a test function in $\mathcal{D}(\mathbb{R}^3 \times \mathbb{R}^+ \times \mathbb{R}^3 \times \mathbb{R}^+)$ which vanishes in a neighborhood of the set A defined by

$$A = \left\{ (u_1, e_1, u_2, e_2) \in \mathbb{R}^3 \times \mathbb{R}^+ \times \mathbb{R}^3 \times \mathbb{R}^+ / e_1 = e_2 \right\}.$$

Since χ vanishes on a neighborhood of A , we can define the function $K_1 := \frac{\chi}{K}$ (with K_1 extended by 0 on A) in such a way that K_1 belongs to $\mathcal{D}(\mathbb{R}^3 \times \mathbb{R}^+ \times \mathbb{R}^3 \times \mathbb{R}^+)$, and

$$(40) \quad |u_p - u_{p*}| f f_* \chi = (|u_p - u_{p*}| f f_* K) K_1 = 0.$$

Since (40) holds for all χ vanishing in a neighborhood of A , the support of $f \otimes f$ is contained in A .

We conclude that the support of f w.r.t. e_p is restricted to one point thanks to (39). Since we now know that f is a positive measure in e_p whose support in e_p is restricted to a point, we end up with f having the following shape (see [Hor90]):

$$f(u_p, e_p) = \mu(u_p) \otimes \delta_{e_p=e}(e_p),$$

where μ is a positive measure.

It remains to find the shape of μ . Since the measure f is a Dirac mass w.r.t. e_p , the expression of the collision kernel is exactly similar to the formula for the Boltzmann collision kernel of monoatomic rarefied gases in the case of hard spheres. It is well-known that only Maxwellian functions of u_p (see [Cer88]) are such that

$$\langle Q(f, f), \log(f) \rangle = 0.$$

This ends the proof of Lemma 4.1. $\square$

4.2. Inelastic collisions. We now study the case of inelastic collisions, that is when $\beta := \beta(|u_p - u_{p*}|) \in [0, 1[$.

We recall that the kernel under study is given by formula (20). The computation of the exchange of kinetic energy leads to

$$(41) \quad \iint_{u_p, e_p} Q(f, f) \frac{1}{2} m_p u_p^2 du_p de_p = - \iiint \iiint_{u_p, e_p, u_{p*}, e_{p*}, \sigma} \frac{1 - \beta^2}{8} f f_* \\ \times r^2 m_p |u_p - u_{p*}|^3 d\sigma du_{p*} de_{p*} du_p de_p,$$

so that the effect of inelastic collisions is to concentrate the velocities of the droplets.

We begin with a lemma showing that $Q(f, f) = 0$ only when f is a Dirac mass w.r.t. the velocity. Note that in absence of internal energy exchange, the convergence towards a Dirac mass is rigorously proven in [Vil02], [BGP04] or [FM05] for solutions of the spatially homogeneous Boltzmann equation $\partial_t f = Q(f, f)$, for models of granular media which are close in spirit to the model studied here.

Lemma 4.2 (Equivalent of the H-theorem in the inelastic case). *Let f be a positive measure. Then, the following properties are equivalent:*

- (1) $\langle Q(f, f), u_p \mapsto u_p^2 \rangle = 0$ (with obvious notations);
- (2) $Q(f, f) = 0$;
- (3) there exists some positive measure μ and $v \in \mathbb{R}^3$ such that,

$$(42) \quad f(u_p, e_p) = \delta_{u_p=v}(u_p) \otimes \mu(e_p).$$

Proof of lemma 4.2: It is clear that (3) $\Rightarrow$ (2) $\Rightarrow$ (1). We now prove that (1) $\Rightarrow$ (3).

Note first that $r^2|u_p - u_{p_*}|f \otimes f$ is a positive measure. We assume that

$$\langle Q(f, f), u_p \mapsto u_p^2 \rangle = 0,$$

so that

$$(43) \quad \iint_{u_p, e_p} Q(f, f) \frac{1}{2} m_p u_p^2 du_p de_p = - \iiint_{u_p, e_p, u_{p_*}, e_{p_*}, \sigma} \frac{1 - \beta^2}{8} f f_* \\ \times r^2 m_p |u_p - u_{p_*}|^3 d\sigma du_{p_*} de_{p_*} du_p de_p = 0.$$

Using eq. (43), we prove that the support of f (w.r.t. velocities) is restricted to one point at most. In order to do so, we assume that the support of the measure is not empty. Let $\chi := \chi(u_p, e_p, u_{p_*}, e_{p_*})$ be a test function in $\mathcal{D}(\mathbb{R}^3 \times \mathbb{R}^+ \times \mathbb{R}^3 \times \mathbb{R}^+)$ which vanishes in a neighborhood of the set $A := \{(u_1, e_1, u_2, e_2) \in \mathbb{R}^3 \times \mathbb{R}^+ \times \mathbb{R}^3 \times \mathbb{R}^+ / u_1 = u_2\}$.

Since χ vanishes in a neighborhood of A , we can define the smooth function $K := \frac{\chi}{|u_p - u_{p_*}|^3}$ (extended by 0 on A). Then,

$$f \otimes f \chi = (f \otimes f |u_p - u_{p_*}|^3) K = 0.$$

Since this is true for every χ which vanishes in a neighborhood of A , the support of the distribution $f \otimes f$ is contained in A .

Finally, we use the property (39): if (u_1, e_1) and (u_2, e_2) are in the support of f , then (u_1, e_1, u_2, e_2) is in the support of $f \otimes f$. Moreover (u_1, e_1, u_2, e_2) is included in A since the support of $f \otimes f$ is in A so that necessarily $u_1 = u_2$. We see therefore that the support of f w.r.t. velocity is restricted to one point. The proof of Lemma 4.2 is then concluded by classical arguments (see [Hor90]). $\square$

We now wish to show, at the formal level, that among the equilibria of the form

$$f(u_p, e_p) = \delta_{u_p=v}(u_p) \otimes \mu(e_p),$$

only those given by the formula

$$(44) \quad f(t, u_p, e_p) = G \delta_{u_p=v}(u_p) \otimes \delta_{e_p=e}(e_p)$$

(with $v \in \mathbb{R}^3, G \geq 0, e > 0$) are stable. We assume in the following computation that a and β are constant, for the sake of simplicity. An extension of this computation in a case in which a and β are not constant is given in subsection 7.3.

In order to do so, we first recall Haff's law ([Haf83]): For $f := f(t, u_p)$ satisfying the spatially homogeneous equation $\partial_t f = Q(f, f)$ (with $\beta \in [0, 1[$ and no exchange of energy involved), the following estimate holds:

$$(45) \quad \frac{m}{1+t^2} \leq T(t) \leq \frac{M}{1+t^2},$$

where $T(t) := \frac{\int_{u_p} f(t, u_p) \frac{1}{3} m_p (u_p - v)^2 du_p}{\int_{u_p} f(t, u_p) m_p du_p}$, and where m and M are constants depending on

initial data.

A rigorous proof of this result can be found in [MM06] and [MMR06], in the case of granular media (that is, with a collision kernel which is slightly different), with the hard-sphere cross-section.

though our model of inelastic collisions is not exactly the same, an easy computation shows that, at the formal level, Haff's law still holds.

Our goal here is to estimate the evolution of the mean internal energy along the solutions of the equation

$$(46) \quad \partial_t f(t, u_p, e_p) = Q(f, f)(t, u_p, e_p).$$

The computations provided from now on are only approximations. They give an idea of what should be the evolution of the quantity

$$g(t) := \frac{\iint_{u_p, e_p} f(t, u_p, e_p) m_p (e_p - e(t))^2 du_p de_p}{\iint_{u_p, e_p} f(t, u_p, e_p) m_p du_p de_p},$$

that is the variance of f w.r.t. e_p . They will be sustained in next subsection by numerical simulations. Note first that thanks to the conservation of mass,

$$(47) \quad \begin{aligned} g'(t) &= \frac{\iint Q(f, f)(t, u_p, e_p) m_p (e_p - e(t))^2 du_p de_p}{\iint f(t, u_p, e_p) m_p du_p de_p} \\ &= \left(-a(1 - \frac{a}{2}) \iiint \frac{1}{2} f f^* 4\pi r^2 (e_p - e_{p_*})^2 |u_p - u_{p_*}| du_p du_{p_*} de_p de_{p_*} \right. \\ &\quad \left. + \frac{1}{2} \iiint 4\pi r^2 f f^* \left[\frac{1}{2} \Delta E^2 + \Delta E (e_p + e_{p_*} - 2e) \right] |u_p - u_{p_*}| du_p du_{p_*} de_p de_{p_*} \right) \\ &\quad \left/ \iint f du_p de_p \right. . \end{aligned}$$

We use the following approximation based on Haff's law: in all computations we replace $|u_p - u_{p_*}|$ by $\sqrt{6T}$ (the 6 comes from the fact that we are in 3D).

Then,

$$(48) \quad \begin{aligned} g'(t) &\sim \left(-a(1 - \frac{a}{2}) \sqrt{6T} \iiint \frac{1}{2} f f^* 4\pi r^2 (e_p - e_{p_*})^2 du_p du_{p_*} de_p de_{p_*} \right. \\ &\quad \left. + \iiint 2\pi r^2 f f^* \left(\frac{1}{2} \left(\frac{1 - \beta^2}{4} 6T \right)^2 + \frac{1 - \beta^2}{4} 6T (e_p + e_{p_*} - 2e) \right) \sqrt{6T} du_p du_{p_*} de_p de_{p_*} \right) \\ &\quad \left/ \iint_{u_p, e_p} f du_p de_p \right. . \end{aligned}$$

We use the identities

$$(49) \quad \begin{aligned} &\iiint f f^* (e_p - e_{p_*})^2 du_p du_{p_*} de_p de_{p_*} \\ &= \iiint f f^* [(e_p - e)^2 + (e - e_{p_*})^2] du_p du_{p_*} de_p de_{p_*} \\ &= 2g(t) \left(\iint_{u_p, e_p} f du_p de_p \right)^2 , \end{aligned}$$

and

$$(50) \quad \iiint f f^*(e_p + e_{p_*} - 2e) du_p du_{p_*} de_p de_{p_*} = 0.$$

As a consequence, we obtain the following (approximate) ODE for the evolution of g :

$$(51) \quad g'(t) \sim (1 - \alpha) \left(-a(1 - \frac{a}{2})\sqrt{6}\frac{3}{r}T(t)^{1/2}g(t) + \frac{1}{4} \left(\frac{1 - \beta^2}{4} \right)^2 \frac{3}{r} (6T(t))^{5/2} \right),$$

where $1 - \alpha = \iint_{u_p, e_p} f \frac{4}{3} \pi r^3 du_p de_p$ is the volume occupied by the droplets (this quantity is a constant). Using now (according to Haff's law) the approximation $T(t) = \frac{c_1^2}{(1 + c_2 t)^2}$ where c_1 and $c_2 > 0$, we solve eq. (51) and obtain (except in the exceptional case when $\frac{3}{r}(1 - \alpha)\frac{c_1}{c_2}\sqrt{6}a(1 - a/2) = 4$):

$$(52) \quad g(t) \sim \frac{g(0)}{(1 + c_2 t)^{\frac{3}{r}(1 - \alpha)\frac{c_1}{c_2}\sqrt{6}a(1 - a/2)}} + \frac{1}{4} \left(\frac{1 - \beta^2}{4} \right)^2 \frac{(\sqrt{6}c_1)^5 / c_2}{\frac{3}{r}(1 - \alpha)\frac{c_1}{c_2}\sqrt{6}a(1 - a/2) - 4} \left[(1 + c_2 t)^{-4} - (1 + c_2 t)^{-\frac{3}{r}(1 - \alpha)\frac{c_1}{c_2}\sqrt{6}a(1 - a/2)} \right].$$

We now discuss the behavior of g according to the sign of $\frac{3}{r}(1 - \alpha)\frac{c_1}{c_2}\sqrt{6}a(1 - a/2) - 4$.

- When $4 < \frac{3}{r}(1 - \alpha)\frac{c_1}{c_2}\sqrt{6}a(1 - a/2)$: we get

$$(53) \quad g(t) \sim \frac{Cst}{(1 + c_2 t)^4}.$$

This is the situation when thermal exchanges are predominant: $\sqrt{g(t)}$ then converges to zero as rapidly as the temperature $T(t)$ (note that $\sqrt{g}$ has the same dimension as an energy).

- When $4 > \frac{3}{r}(1 - \alpha)\frac{c_1}{c_2}\sqrt{6}a(1 - a/2)$,
we get

$$g(t) \sim \frac{Cst}{(1 + c_2 t)^{\frac{3}{r}(1 - \alpha)\frac{c_1}{c_2}\sqrt{6}a(1 - a/2)}},$$

so that $\sqrt{g(t)}$ still converges towards 0, but this convergence is slower than that of the temperature $T(t)$. It can even be very slow when a is close to 0 (that is, when the exchanges of internal energy are of small amplitude).

Note finally that the exceptional case $\frac{3}{r}(1 - \alpha)\frac{c_1}{c_2}\sqrt{6}a(1 - a/2) = 4$ leads to a formula close to (53) [but with a logarithmic correction].

The previous computations show (though not rigorously) that the only stable equilibrium of $\partial_t f = Q(f, f)$ in the case of inelastic collisions ($\beta \in [0, 1]$) are functions defined by (44).

We now detail a numerical simulation which confirms the approximate computations presented above.

4.3. Numerical results for the convergence towards the Dirac mass w.r.t. internal energy and velocity.

We present some numerical tests for the spatially homogeneous Boltzmann equation $\partial_t f = Q(f, f)$, when Q is the inelastic collision kernel defined by (20), with a and β fixed constants.

The computations are performed thanks to a particle method (Cf. [Bar04a, PR04]), where the density $f := f(t, u_p, e_p)$ is approximated by a sum of Dirac masses with the same numerical weight (that is, $f(t, u_p, e_p) \sim w \sum_{i=1}^N \delta_{u_p=u_{pi}; e_p=e_{pi}}$). This set of numerical particles then evolves according to Bird's method (Cf. [Bir94]). The tests which are presented correspond to the following parameters:

$$r = 10^{-4}, \quad f(0, u_p, e_p) = Cst 1_{u_p \in [-10^4, 10^4]^3; e_p \in [5 \cdot 10^5, 5 \cdot 10^6]}.$$

About 10^4 numerical particles are used.

First test: *Convergence towards the Dirac mass w.r.t. velocity; Haff's law*

We check that Haff's law holds for $a = 1$ and $\beta = 0.99, 0.95, 0.8$: we plot the results in logarithm scale: we expect to get a (asymptotically) straight line whose slope is -2 (since Haff's law means that $T(t) \sim t^{-2}$).

It is indeed what we observe in the figure below. Note also that, as expected, the convergence is slower when β increases.

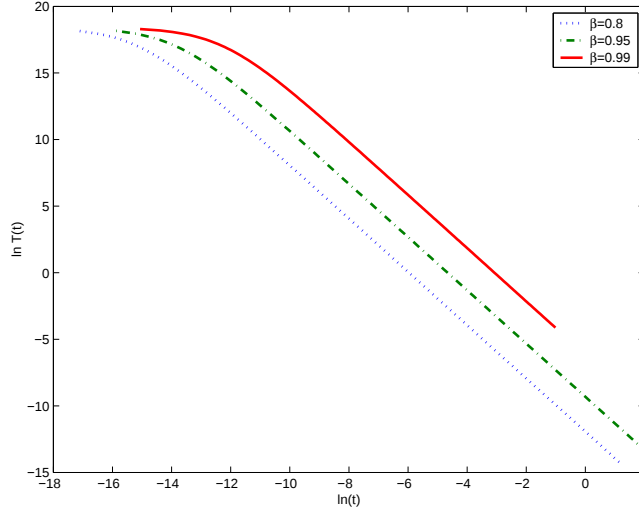


FIGURE 1. Behavior of kinetic temperature: $\ln T$ as a function of $\ln t$ for different β

Second test: *Convergence towards the Dirac mass w.r.t. the internal energy*

We now check the convergence towards the Dirac mass w.r.t internal energy. We fix $\beta = 0.99$ and let a vary between 0.01 and 1.0. We plot

$$W = \ln \left(\iint f(t, u_p, e_p) |e_p - e(t)| de_p du_p / \iint f(t, u_p, e_p) de_p du_p \right)$$

as a function of $\ln(t)$.

As can be seen in figure 2, the more a increases, the more the coefficients of the asymptotic straight line tend to -2 . More precisely (in accordance with the theoretical computation), we see that there exists a critical a (around 0.06) which separates a zone in which the behavior of W seems

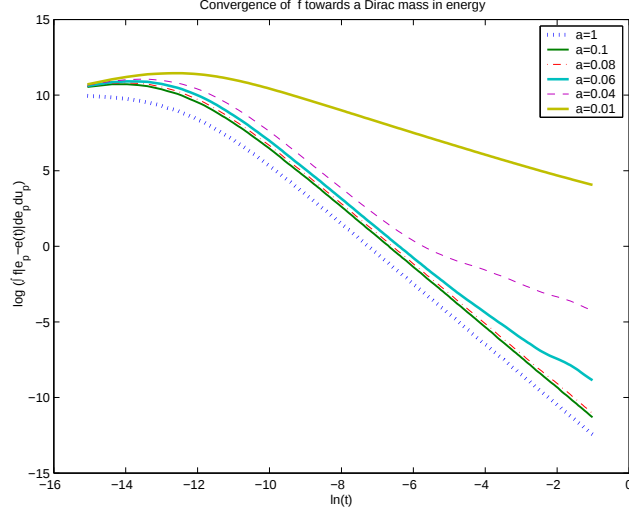


FIGURE 2. Convergence towards the Dirac mass w.r.t. the internal energy: $\ln \left(\iint f(t, u_p, e_p) |e_p - e(t)| de_p du_p / \iint f(t, u_p, e_p) de_p du_p \right)$ as a function of $\ln t$ for various values of parameters

to be in t^{-2} , and a zone in which it is rather in $t^{-\theta}$, with $\theta \in]0, 2[$ depending on a . Finally, we observe that for small a , the function W increases during a certain amount of time: thermal exchanges are then not significant enough to completely counterbalance the positive term in equation (51) (that is, the transfer of kinetic energy to internal energy) at all times.

5. FLUID OF PARTICLES: NON CLOSED FORM

According to the dimensional analysis of section 3, we end up with the following set of scaled equations for the spray:

$$(54) \quad \partial_t(\alpha^\varepsilon \rho_g^\varepsilon) + \nabla_x \cdot (\alpha^\varepsilon \rho_g^\varepsilon u_g^\varepsilon) = 0,$$

$$(55) \quad \partial_t(\alpha^\varepsilon \rho_g^\varepsilon u_g^\varepsilon) + \nabla_x \cdot (\alpha^\varepsilon \rho_g^\varepsilon u_g^\varepsilon \otimes u_g^\varepsilon) + \nabla_x p^\varepsilon = -A^\varepsilon,$$

$$(56) \quad \partial_t(\alpha \rho_g^\varepsilon E_g^\varepsilon) + \nabla_x \cdot \left(\alpha^\varepsilon \rho_g^\varepsilon \left(E_g^\varepsilon + \frac{p^\varepsilon}{\rho_g^\varepsilon} \right) u_g^\varepsilon \right) + p^\varepsilon \partial_t \alpha^\varepsilon = -B_1^\varepsilon - B_2^\varepsilon,$$

$$(57) \quad \partial_t f^\varepsilon + u_p^\varepsilon \cdot \nabla_x f^\varepsilon + \nabla_{u_p} \cdot (f^\varepsilon \Gamma^\varepsilon) + \partial_{e_p} (f^\varepsilon \phi^\varepsilon) = \frac{1}{\varepsilon} Q(f^\varepsilon, f^\varepsilon),$$

where

$$(58) \quad m_p \Gamma^\varepsilon = -\frac{m_p}{\rho_p} \nabla_x p^\varepsilon - D(u_p - u_g^\varepsilon); \quad m_p \phi^\varepsilon = \Phi(T_g^\varepsilon - T_p),$$

$$(59) \quad A^\varepsilon = \iint_{u_p, e_p} m_p \Gamma^\varepsilon f^\varepsilon du_p de_p,$$

$$(60) \quad B_1^\varepsilon = \iint_{u_p, e_p} m_p \left(\Gamma^\varepsilon + \frac{\nabla_x p^\varepsilon}{\rho_p} \right) \cdot u_p f^\varepsilon du_p de_p,$$

$$(61) \quad B_2^\varepsilon = \iint_{u_p, e_p} m_p \phi^\varepsilon f^\varepsilon du_p de_p.$$

In this section, we present the computations which enable to pass to the limit at the formal level in eq. (54) – (61), when $\varepsilon \rightarrow 0$. These formal computations are based on the same principle as the traditional Hilbert expansion for fluid mechanics: we first take moments of eq. (57) in this section, and then close the corresponding equations thanks to the study (in section 4) of the solutions of $Q(f, f) = 0$ in next section.

We define the following quantities associated with the moments of order zero (mass), one (momentum), two (energy, pressure (Reynolds') tensor) and three (flux of energy) of the fluid of particles (the notations used here are coherent with those of section 1):

$$\begin{aligned} (1 - \alpha)\rho &= \iint_{u_p, e_p} f m_p du_p de_p, & (1 - \alpha)\rho v &= \iint_{u_p, e_p} f m_p u_p du_p de_p, \\ (1 - \alpha)\rho e_c &= \iint_{u_p, e_p} \frac{1}{2} f m_p |u_p|^2 du_p de_p, & (1 - \alpha)\rho e &= \iint_{u_p, e_p} f m_p e_p du_p de_p, \\ (1 - \alpha)\rho E &= \iint_{u_p, e_p} f \left\{ \frac{1}{2} m_p |u_p|^2 + m_p e_p \right\} du_p de_p, \\ (1 - \alpha)P' &= \iint_{u_p, e_p} f m_p (v - u_p) \otimes (v - u_p) du_p de_p, \\ (1 - \alpha)q &= \iint_{u_p, e_p} f m_p (v - u_p)^2 (u_p - v) du_p de_p. \end{aligned}$$

Note that the pressure tensor P' will appear in our set of equations because the fluid of droplets does not "see" the same pressure as the gas. This extra term of pressure, sometimes called interfacial pressure, appears (usually in a non tensorial form) in many works concerned with the modeling of two-phase flows (see [Sai95] and [GHS04] for example). This pressure tensor vanishes when all the droplets have the same velocity.

We now integrate the Boltzmann equation against $m_p du_p de_p$ (mass conservation) and $m_p u_p du_p de_p$ (momentum conservation). We use properties (23) and (24) of the collision kernel. This leads to

$$\partial_t(1 - \alpha) + \nabla_x \cdot ((1 - \alpha)v) = 0.$$

$$\begin{aligned} \partial_t((1 - \alpha)\rho v) + \nabla_x \cdot ((1 - \alpha)\rho v \otimes v) + (1 - \alpha)\nabla_x p + \nabla_x \cdot ((1 - \alpha)P') &= \\ &= - \iint_{u_p, e_p} D(u_p - u_g) f du_p de_p. \end{aligned}$$

We next integrate the Boltzmann equation against $m_p e_p du_p de_p$. This leads to the balance equation for internal energy

$$\partial_t((1 - \alpha)\rho e) + \nabla_x \cdot ((1 - \alpha)\rho e v) = \iint_{u_p, e_p} \Phi(T_g - T_p) f du_p de_p$$

$$+ \iint_{u_p, e_p} Q(f, f) m_p e_p du_p de_p.$$

Finally, we integrate the Boltzmann equation against $\frac{1}{2} m_p u_p^2 du_p de_p$ and get the balance equation for kinetic energy:

$$(62) \quad \partial_t ((1 - \alpha) \rho e_c) + \nabla_x \cdot ((1 - \alpha) (\rho e_c + P') v) + (1 - \alpha) v \cdot \nabla_x p + \nabla_x \cdot ((1 - \alpha) q) = \\ - \iint_{u_p, e_p} D_p (u_p - u_g) \cdot u_p f du_p de_p + \iint_{u_p, e_p} Q(f, f) \frac{1}{2} m_p u_p^2 du_p de_p.$$

Adding the two last equations and using the conservation of mass, we get the equation of conservation of total energy:

$$(63) \quad \partial_t ((1 - \alpha) \rho E) + \nabla_x \cdot \left((1 - \alpha) \rho \left(E + \frac{p}{\rho} \right) v \right) + p \partial_t (1 - \alpha) + \nabla_x \cdot ((1 - \alpha) (P' v + q)) \\ = - \iint_{u_p, e_p} D (u_p - u_g) \cdot u_p f du_p de_p + \iint_{u_p, e_p} \Phi (T_g - T_p) f du_p de_p.$$

6. MACROSCOPIC EQUATIONS: CLOSED FORM

6.1. Macroscopic system in case of elastic collisions.

We now close the equations of the previous section by formally letting ε go to 0 in (54) – (61), first supposing that all collisions are elastic ($\beta = 1$).

According to the results of subsection 4, we end up with $f^\varepsilon \rightarrow f$, where

$$(64) \quad f(t, x, u_p, e_p) = \frac{Z(t, x)}{(2\pi T(t, x))^{\frac{3}{2}}} \exp \left(-\frac{(u_p - v(t, x))^2}{2T(t, x)} \right) \delta_{e_p = e(t, x)}(e_p).$$

We can now compute the moments and close the system. One can note that the set of independent variables for the fluid of particles is now $(1 - \alpha) \rho$, v , e and T . Therefore, we need a set of seven equations. Using (64) in the computations of the moments, we see that

$$Z = (1 - \alpha) \frac{\rho}{m_p},$$

$$(1 - \alpha) P' = (1 - \alpha) \rho T \mathbf{I},$$

$$(1 - \alpha) \rho e_c = \frac{1}{2} (1 - \alpha) \rho v^2 + \frac{3}{2} (1 - \alpha) \rho T, \quad (1 - \alpha) q = 0.$$

We now write down the expressions of the transfer of momentum and energy between the two phases. We obtain integrals with parameters which depend on u_g , v , ρ , $(1 - \alpha)$ and T :

- transfer of momentum:

$$\begin{aligned}
& \mathcal{M}(\rho, 1 - \alpha, v, T, u_g, D) \\
&:= \iint_{u_p, e_p} D(u_p - u_g) \frac{Z}{(2\pi T)^{\frac{3}{2}}} \exp\left(-\frac{(u_p - v)^2}{2T}\right) \delta_{e_p=e}(e_p) du_p de_p \\
&= \frac{(1 - \alpha)\rho}{m_p (2\pi T)^{\frac{3}{2}}} \int_{u_p} D(u_p - u_g) \exp\left(-\frac{(u_p - v)^2}{2T}\right) du_p.
\end{aligned}
\tag{65}$$

This expression cannot in general (that is, when D depends on $|u_p - u_g|$) be simplified.

- transfer of energy due to drag force :

$$\begin{aligned}
& \mathcal{I}(\rho, 1 - \alpha, v, T, u_g, D) \\
&:= \iint_{u_p, e_p} D(u_p - u_g) \cdot u_p \frac{Z}{(2\pi T)^{\frac{3}{2}}} \exp\left(-\frac{(u_p - v)^2}{2T}\right) \delta_{e_p=e}(e_p) du_p de_p \\
&= \frac{(1 - \alpha)\rho}{m_p (2\pi T)^{\frac{3}{2}}} \int_{u_p} D(u_p - u_g) \cdot u_p \exp\left(-\frac{(u_p - v)^2}{2T}\right) du_p.
\end{aligned}
\tag{66}$$

Thanks to the computations above, we eventually obtain a closed set of seven equations for seven unknowns, that is

$$\partial_t(\alpha\rho_g) + \nabla_x \cdot (\alpha\rho_g u_g) = 0,
\tag{67}$$

$$\partial_t((1 - \alpha)\rho) + \nabla_x \cdot ((1 - \alpha)\rho v) = 0,
\tag{68}$$

$$\partial_t(\alpha\rho_g u_g) + \nabla_x \cdot (\alpha\rho_g u_g \otimes u_g) + \alpha \nabla_x p = \mathcal{M}(\rho, 1 - \alpha, v, T, u_g, D),
\tag{69}$$

$$\partial_t((1 - \alpha)\rho v) + \nabla_x \cdot ((1 - \alpha)\rho v \otimes v) + (1 - \alpha) \nabla_x p + \nabla_x ((1 - \alpha)\rho T) = -\mathcal{M}(\rho, 1 - \alpha, v, T, u_g, D),
\tag{70}$$

$$\begin{aligned}
(71) \quad \partial_t(\alpha\rho_g E_g) + \nabla_x \cdot \left(\alpha\rho_g \left(E_g + \frac{p}{\rho_g} \right) u_g \right) + p \partial_t \alpha = \\
\mathcal{I}(\rho, 1 - \alpha, v, T, u_g, D) - \Phi(T_g - T) \frac{(1 - \alpha)\rho}{m_p},
\end{aligned}$$

$$(72) \quad \partial_t((1 - \alpha)\rho e_c) + \nabla_x \cdot ((1 - \alpha)\rho(e_c + T)v) + (1 - \alpha)v \cdot \nabla_x p = -\mathcal{I}(\rho, 1 - \alpha, v, T, u_g, D),
\tag{72}$$

$$(73) \quad \partial_t((1-\alpha)\rho e) + \nabla_x \cdot ((1-\alpha)\rho e v) + p(\partial_t(1-\alpha) + \nabla_x \cdot ((1-\alpha)v)) = \Phi(T_g - T) \frac{(1-\alpha)\rho}{m_p},$$

where $E_g = e_g + \frac{1}{2}|u_g|^2$, $e_c = \frac{3}{2}T + \frac{1}{2}|v|^2$.

Remember also that the following equations of state are added to (67) – (73).

$$(74) \quad p = P_1(\rho_g, e_g), \quad T_g = T_1(\rho_g, e_g), \quad T_p = T_2(e), \quad \rho = \rho_p.$$

This system is not found in textbooks about multiphase flows since usually the collisions between droplets are considered to be inelastic, so that their kinetic and internal energy are not separately conserved.

6.2. Macroscopic system in case of inelastic equations.

We now close the equations when the collision kernel is inelastic (that is, $\beta \in [0, 1]$). We formally let ε go to 0 in (54) – (61). According to the results of subsection 4, we know (at the formal level) that $f^\varepsilon \rightarrow f$, with

$$(75) \quad f(t, x, u_p, e_p) = G(t, x) \delta_{u_p=v(t,x)}(u_p) \delta_{e_p=e(t,x)}(e_p).$$

We end up with a system of 6 equations which write (remember that $e_g = E_g - \frac{1}{2}u_g^2$ and $e = E - \frac{1}{2}v^2$).

$$(76) \quad \partial_t(\alpha\rho_g) + \nabla_x \cdot (\alpha\rho_g u_g) = 0,$$

$$(77) \quad \partial_t((1-\alpha)\rho) + \nabla_x \cdot ((1-\alpha)\rho v) = 0,$$

$$(78) \quad \partial_t(\alpha\rho_g u_g) + \nabla_x \cdot (\alpha\rho_g u_g \otimes u_g) + \alpha \nabla_x p = -\tilde{A},$$

$$(79) \quad \partial_t((1-\alpha)\rho v) + \nabla_x \cdot ((1-\alpha)\rho v \otimes v) + (1-\alpha)\nabla_x p =$$

$$(80) \quad \tilde{A},$$

$$(81) \quad \partial_t(\alpha\rho_g E_g) + \nabla_x \cdot \left(\alpha\rho_g \left(E_g + \frac{p}{\rho_g} \right) u_g \right) + p\partial_t\alpha = -\tilde{B}_1 - \tilde{B}_2,$$

$$(82) \quad \partial_t((1-\alpha)\rho E) + \nabla_x \cdot \left((1-\alpha)\rho \left(E + \frac{p}{\rho} \right) v \right) + p\partial_t(1-\alpha) = \tilde{B}_1 + \tilde{B}_2,$$

where $\tilde{A}$, $\tilde{B}_1$ and $\tilde{B}_2$ are defined in the introduction, the functions $\tilde{D}$ and $\tilde{\Phi}$ being the same as D , Φ , but taken at points v, e instead of u_p, e_p .

We recall the equations of state which complete this system:

$$(83) \quad p = P_1(\rho_g, e_g), \quad T_g = T_1(\rho_g, e_g),$$

$$(84) \quad \rho = \rho_p, \quad T = T_2(e).$$

Finally, the assumption that the collisions are inelastic leads to the derivation of the model for multiphase flows described in the introduction (see [GHS04, Bou98, Rov06] for more details about two-phase flow models and [Duf05, Lau02] for other ways to obtain this system using the multi-fluid approach).

7. AN EXAMPLE OF EVALUATION OF THE PARAMETERS α AND β

We briefly explain in this section how it is possible to estimate the parameters a and β which appear in our model of collisions, under the assumption that viscosity is the main reason why kinetic energy is lost during collisions. A more complete description of the procedure together with numerical values in a typical experiment in the context of the nuclear industry can be found in [Mat06].

7.1. Probability laws of exchange of energy.

We first explain the process of transfer of energy during the collisions. We obtain a hint of the expression of a .

A collision of droplets can be considered as a coalescence of two particles which split after some time. The droplets remain stuck together for a while, and they exchange energy through thermal conduction.

The time $\Delta\tau_{coll}$ of collision between two droplets p (with velocity u_p , internal energy e_p and temperature T_p) and p_* (with velocity u_{p_*} , internal energy e_{p_*} and temperature T_{p_*}) is generally assessed as $\Delta\tau_{coll} = \frac{2r}{|u_p - u_{p_*}|}$ (see [Cer88, Hyl99]).

Because of heat transfers, an amount $4\pi r\lambda_p(T_p - T_{p_*})$ of internal energy is exchanged by unit of time (with λ_p the thermal conductivity of the droplets, supposed to be constant) from p towards p_* . The 4π factor might not be the good geometric factor because one particle does not surround the other during the collision as the gas surrounds a particle, but this quantity gives the right order of magnitude. Hence, we get

$$\frac{d}{dt}(e_{p_*} - e_p) = \frac{4\pi r\lambda_p}{m_p}(T_p - T_{p_*}).$$

Supposing that the specific heat C_p of the liquid is a constant, we end up with the following evolution for the difference of internal energy between the two droplets during the collision:

$$(e_{p_*} - e_p)(t) = \exp\left(-\frac{4\pi r\lambda_p}{C_p m_p}t\right) \times (e_{p_*} - e_p)(0).$$

Using the last equality for $t = \Delta\tau_{coll}$, one gets

$$(85) \quad a = 1 - \exp\left(-\frac{4\pi r\lambda_p}{C_p m_p} \frac{2r}{|u_p - u_{p_*}|}\right).$$

7.2. Estimate of the inelasticity.

We assess the effect of inelasticity through some computations using the T.A.B. model (Taylor Analogy Break-up model) used in the Kiva code (see [Bar04a] and [AO89]), under the assumption that viscosity is the main factor of loss of kinetic energy during collisions between liquid droplets ([JUL92] and [WO03]). More precisely, the distortion of sphericity y satisfies the following ordinary

differential equation (see [Bar04a]),

$$\ddot{y} = \frac{2}{3} \frac{\rho_g}{\rho_p} \frac{|u_p - u_g|^2}{r^2} - \frac{8\sigma_T}{\rho_p r^3} y - \frac{10\mu_p}{\rho_p r^2} \dot{y},$$

where σ_T is the surface tension of the liquid constituting the droplets, and μ_p is the dynamic viscosity of this same liquid. Under the approximation when the surface tension plays a negligible

role (w.r.t. the viscosity) in the loss of kinetic energy during a collision, we find a characteristic time τ_c (viscous time):

$$\tau_c = 1 \left/ \frac{10\mu_p}{\rho_p r^2} \right.$$

Since oscillatory energy is proportional to $\dot{y}^2$, we get that the kinetic energy (in the center of mass) of the two droplets $E(t) = \frac{1}{4} (u_p(t) - u_{p*}(t))^2$ is controlled through an exponential evolution of parameter $\frac{\tau_c}{2}$, that is

$$E(t) = E(0) \exp\left(-\frac{2t}{\tau_c}\right),$$

so that the loss of kinetic energy ΔE during a collision is

$$\Delta E = E(0) \left(1 - \exp\left(-\frac{2\Delta\tau_{coll}}{\tau_c}\right)\right).$$

Since we also know that

$$\Delta E = \left(\frac{1 - \beta^2}{4}\right) (u_p(0) - u_{p*}(0))^2,$$

we end up with

$$(86) \quad \beta = \exp\left(-\frac{\Delta\tau_{coll}}{\tau_c}\right) = \exp\left(-\frac{20\mu_p}{\rho_p r |u_p - u_{p*}|}\right).$$

This procedure of course only provides an order of magnitude for β . This is nevertheless enough to determine if the regime of collision is elastic or not.

7.3. Instability of equilibria of the form $\delta_v(u_p) \times \mu(e_p)$ when μ is not a Dirac mass with a, β defined above. We (briefly) present here a formal computation showing that the results of subsection 4.2 obtained when a and β are constant still hold when a and β are given by (85) and (86).

The computation performed in subsection 4.2 for the evolution of temperature (with the approximation $|u_p - u_{p*}| \sim \sqrt{6T}$) leads to the following ODE, when β is given by formula (86):

$$(87) \quad \begin{aligned} \frac{d}{dt}T &\sim -\frac{C}{r} \left(1 - \exp\left(-2\frac{\Delta\tau_{coll}}{\tau_c}\right)\right) (1 - \alpha)T^{\frac{3}{2}} \\ &\sim -\frac{C}{r} \left(1 - \exp\left(-\frac{4r}{\sqrt{6T}\tau_c}\right)\right) (1 - \alpha)T^{\frac{3}{2}}, \end{aligned}$$

for some C strictly positive that we do not compute.

When T is large, $\frac{d}{dt}T \sim -\frac{4C}{\tau_c}(1 - \alpha)T$ so that T decays exponentially.

When T becomes small enough, T satisfies $\frac{d}{dt}T \sim -\frac{C}{r}(1 - \alpha)T^{\frac{3}{2}}$ and we are back to the situation that we studied in subsection 4.2 with β constant (and close to 0).

We now present a numerical simulation which sustains those computations.

7.4. Convergence towards the Dirac masses for a, β defined above: Numerical test.

We use the expression of a and β found in subsections 7.1 and 7.2. The numerical code is similar to the code used previously.

As one can see in figure 3, the temperature $T(t)$ does converge to zero. At the beginning, this convergence is exponential, whereas as soon as T becomes small enough, the rate of convergence corresponds to Haff's law, i.e. is in $\frac{1}{t^2}$.

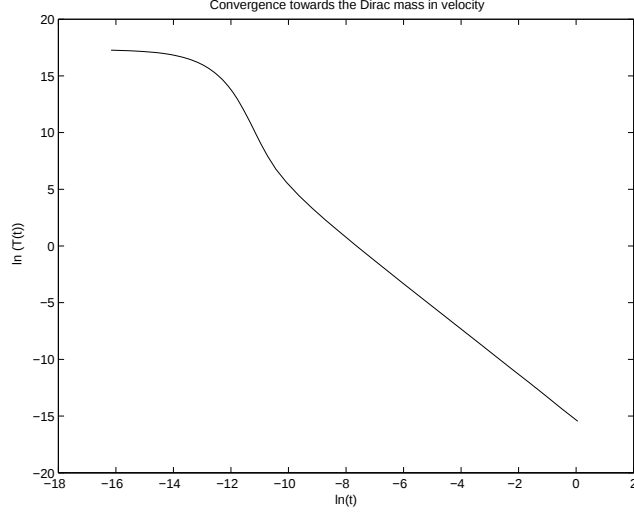


FIGURE 3. Convergence towards the Dirac mass in velocity: $\ln T$ as a function of $\ln t$

Next we observe the convergence of the internal energy in figure 4.

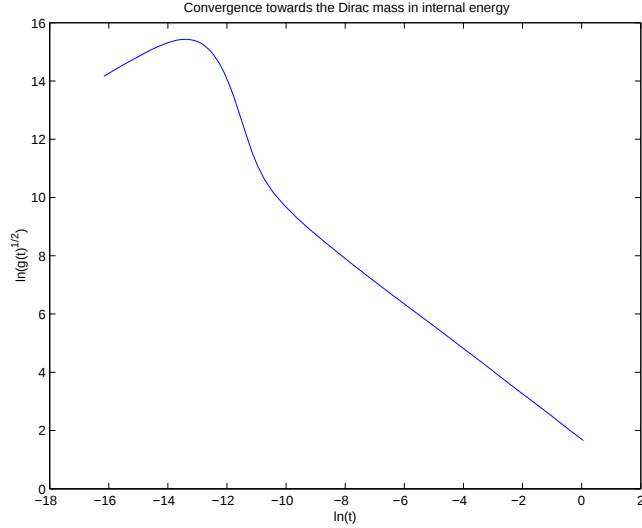


FIGURE 4. Convergence towards the Dirac mass in internal energy: $\ln \left(\iint f(t, u_p, e_p) |e_p - e(t)| de_p du_p / \iint f(t, u_p, e_p) de_p du_p \right)$ as a function of $\ln t$

We see that the internal energy indeed converges to a Dirac mass in this simulation.

8. CONCLUSION AND PERSPECTIVES

We have presented here a formalism analogous to the Hilbert expansion of rarefied gases which enables to link the equations of thick monodisperse sprays with the "one pressure" equations of multiphase flows. We now wish to briefly comment some of the issues related to this presentation.

Firstly, we wish to explain what can be the extensions of the asymptotics presented in this work: the presence of (molecular or turbulent) diffusion in the gas equations does not change the computations. Then, it is possible in principle to take into account chemistry terms (e.g. combustion terms) in the equations, this leads however to serious complications. Finally, it is known that polydispersion plays a decisive role in the construction of macroscopic models starting from spray equations (Cf. [DMV03]). In general, it is not possible to guess the evolution of droplets w.r.t. radius, and one has to cut into "sections" the various possible radii r . It however happens sometimes that processes of coagulation/breakup lead to such specific profiles (Cf. for example [AB79]). In such (unfortunately unrealistic when sprays are concerned) situations, 2-phase macroscopic equations can be obtained (at the formal level) by an asymptotics.

Secondly, we would like to emphasize the extreme difficulty of making rigorous the passage to the limit that we propose (even in a "small time" setting). This is related to the very bad mathematical behavior of the limiting eq. (12) – (19). Those equations are not written in conservative form and have a domain of non hyperbolicity (Cf. [Ram00]). Moreover, the set of eq. (1) – (11) has not been studied from the mathematical point of view. It might indeed present a behavior as bad as the limiting system [though this guess is not yet sustained by convincing arguments]. One possibility could be to try to pass to the limit in an analogous system, where the molecular viscosity of the gas is not neglected (then the limiting equations are better behaved, Cf. [Ram00]).

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