# Non-local interaction equations: Stationary states and stability analysis

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**Abstract:** In this paper, we are interested in the long-time behavior of solutions to a non-local interaction equation. We show that up to an extraction, the solution converges to a steady-state. Then, we study the structure of stable steady-states.

## 1 Introduction

We are interested in the asymptotic behaviour of a density $\rho(t,x)$ of particles or individuals at position $x \in \mathbb{R}^d$ ($d \geq 1$) and at time $t \geq 0$, which evolves according to the nonlocal aggregation equation:

$$\partial_t \rho = \nabla_x \cdot (\rho \, \nabla_x [W * \rho + V]), \text{ for } (t,x) \in \mathbb{R}_+ \times \mathbb{R}^d . \tag{1}$$

This equation can be seen as a many particles limit of discret processes where particles (or individuals) can interact at a large distance, through an interaction potential $W$ (see [20, 15]). Such equations appear in various biological phenomenons like swarming (see [5, 11]), distribution of actin-filament networks (see [12, 14]), as well as in physical problems, for example in the field of granular media (see [1, 26]).

Many of the above models couple the long-range interaction between particles with a diffusive term. Nevertheless, in this paper we shall not consider a diffusion term, and focus our study on the effect of a long-range interaction.

Let us now describe typical interaction potentials $W$ which appear in the models quoted above:

- In [16, 22], interaction potentials are regular, repulsive at short range and attractive when particles are far apart, typically $W(x) = -x^2 + x^4$. In this case, the solution typically concentrates and tends to a finite number of Dirac masses, when time goes to infinity. This type of potentials have been studied in [9, 7], but we don't know any general study of the case of regular interaction potentials so far.

- In Chemotaxis models (see [21, 17]), interaction potentials are singular at $x = 0$ and attractive, typically, in dimension 2, $W(x) := -\frac{1}{2\pi} \log |x|$. In this case, the solution usually (if there is no diffusion) blows-up in finite time. Potentials singular at $x = 0$ and attractive have been widely studied both with a diffusion term (see [4, 6]), or without diffusion (see [8, 18, 10, 3, 2]), for various types of attractive singularities.

- In swarming models (see [11, 19, 25]), interaction potentials are usually singular at $x = 0$ and repulsive, typical examples are the repulsive Morse Potential $W(x) = -e^{-|x|}$, or the attractive-repulsive Morse potentials $W(x) = -C_a\, e^{-|x|/l_a} + C_r\, e^{-|x|/l_r}$ and $W(x) = -C_a\, e^{-|x|^2/l_a} + C_r\, e^{-|x|^2/l_r}$. Related interpolation potentials in physics are, for instance, the Lennard-Jones potential [24]. We don't know any qualitative study of such models.

We will show in this article that the asymptotic behaviour of the solution of (1) highly depends on the type of singukarity of $W$ at point $x = 0$.

In the present article, we shall focus on the one-dimensional case. We aim at understanding the dynamical behavior presented by a non-local interaction operator with even potential:

**Assumption 1:**

$$\forall x \in \mathbb{R}, W(x) = W(-x). \tag{2}$$

In this study, we shall focus on compactly supported densities, we shall thus only consider situations where a confinement exists, either from the external potential, or from the interaction potential itself. We shall assume that:

**Assumption 2:** One of the two following conditions is satisfied:
There exists $C > 0$ such that

$$\|W'\|_{L^\infty([-2C,2C])} < \min\left(V'(C),\, -V'(-C)\right), \tag{3}$$

or

$$V = 0, \quad \exists C_1,\, C_2 > 0, \forall x \geq C_1 : \quad W'(x) \geq C_2\, x,\, W'(-x) \leq -C_2\, x. \tag{4}$$

**Assumption 3:**

$$\rho^0 \in M^1(\mathbb{R}),\ \text{supp}\ \rho^0 \subset [-C, C]. \tag{5}$$

where $C < \infty$. If $V \neq 0$, $C$ must satisfy (3).

Assumption 2 together with Assumption 3 ensure that the support of $\rho(t, \cdot)$ is (uniformly w.r.t. time) bounded (see Prop. 1).

Note that (1) formally conserves the total mass $\int \rho(t,x)\,dx$, which w.l.o.g. we shall assume to be normalized $\int_{\mathbb{R}} \rho(x)\,dx = 1$. The quantity $\rho(t,\cdot)$ is then interpreted as a probability density. In particular in the one-dimensional case, this enables a change of variables in which one introduces the pseudo-inverse of the distribution function $\int_{-\infty}^{x} d\rho$, i.e.

$$u(t,z) = \inf\left\{x \in \mathbb{R} : \int_{(-\infty,x]} \rho(t,y)\,dy > z\right\} \qquad z \in [0,1], \tag{6}$$

which transforms the evolution equations (1) for measure solutions $\rho(t,\cdot)$ into an integral equation for the non-decreasing pseudo-inverse $u(t,z)$ satisfying (see, e.g. [7])

$$\partial_t u(t,z) = \int W'\left(u(t,\xi) - u(t,z)\right)\,d\xi - V'(u(t,z)), \qquad \forall z \in [0,1]. \tag{7}$$

Since eq. (7) is much more convenient than eq. (1) for stability analysis, we shall often use it in this paper. In particular, atomic parts of measure solutions $\rho(x)$ correspond to constant parts of the pseudo-inverse $u(z)$. Notice also the useful change of variable $\int g(x)\rho(x)\,dx = \int_0^1 g(u(\xi))\,d\xi$, which holds for any $g \in L^1(\text{supp }\rho)$.

In the absence of a confining potential $V$ (and if $W$ is symmetric), the center of mass $\int_{\mathbb{R}} x\,\rho(t,x)\,dx$ is conserved by eq. (1), or equivalently, $\int_0^1 u$ is preserved by (7):

$$\frac{d}{dt}\int_{\mathbb{R}} x\rho(t,x)\,dx = 0, \quad \frac{d}{dt}\int_0^1 u(t,z)\,dz = 0. \tag{8}$$

Note that eq. (1) can be seen as a gradient-flow equation for the following energy (see [8]):

$$E(t) := \frac{1}{2}\int\int \rho(t,x)\rho(t,y)W(x-y)\,dx\,dy + \int_{\mathbb{R}} \rho(t,x)V(x)\,dx. \tag{9}$$

In section 2, we shall consider regular interaction potentials $W$. We first prove the technical result Prop. 1, which shows that Assumptions 2 and 3 are sufficient to ensure that the support of $\rho(t,\cdot)$ remains uniformly bounded.

Then, Prop. 2 shows that $\rho(t,\cdot)$ converges (in a sense to be precised then) to a set of steady-states, as time goes to infinity. This result emphasizes the importance of steady-states, when one wishes to understand the long-time behavior of solutions to (1).

In subsection 2.3, we show that stable steady-states of (2) are generically sums of Dirac masses. More precisely, we show in Prop. 3 that for analytic $V$, $W$, the steady-states of (1) are necessarily finite sums of Dirac masses. If $V$, $W$ are only $C^2$, continuous steady-states may exist, but they cannot be linearly stable.

In Section 3, we consider interaction potentials having a singularity at $x = 0$.

In Subsection 3.1, we consider the steady-states of (1) for an interaction potential $W$ having an attractive singularity at $x = 0$. Since (1) may develop blow-ups in $L^\infty$ in finite time (see [3, 2]), we consider (following [8]), the extension (24) of (1) to measure-valued solutions. In Prop. 3.1, we show that a steady-state $\bar{\rho}$ of (24) such that supp $\bar{\rho}$ has an accumulation point (and a bit more, see (26)) is nonlinearly unstable.

In Subsection 3.2, we consider the steady-states of (1) for an interaction potential $W$ having a repulsive singularity at $x = 0$. In Prop. 6, we provide an existence proof for (1) with a regular initial condition (until now, no existence result had been written down for such interaction potentials). In particular, Prop. 6 provides a uniform bound on the solution in $L^\infty(\mathbb{R})$. The situation is therefore completely different from the two other cases: no blow-up can occur.

# 2 Regular interaction potentials

In this first section, we make the following regularity assumptions on $V$ and $W$:
**Assumption 4:**

$$V \in C^2(\mathbb{R}),\, W \in C^2(\mathbb{R}), \tag{10}$$

$$W \in W^{2,\infty}(\mathbb{R}). \tag{11}$$

We shall use in the following the Measure Space

$$\mathcal{P}_\infty(\mathbb{R}) := \{\rho \in M^1(\mathbb{R});\ \text{supp}\ \rho \text{ is bounded}\},$$

together with the Wasserstein distance

$$W_\infty(\rho_1, \rho_2) := \|u_1 - u_2\|_\infty, \tag{12}$$

where $u_1$, $u_2$ are the pseudo-inverses of $\rho_1$, $\rho_2$.

Under Assumption 1 to 4, it has been proven in [7] that a unique solution $\rho \in Lip_{loc}([0,\infty), \mathcal{P}_\infty(\mathbb{R}))$ to (1) exists.

## 2.1 Support of $\rho(t, \cdot)$

In this subsection, we show that Assumptions 1 to 4 are sufficient to ensure that the support of $\rho$ is uniformly bounded w.r.t. time:

**Proposition 1.** *Let $\rho^0$, $V$, $W$ satisfy Assumption 1 to 4. Let $\rho \in Lip_{loc}([0,\infty), \mathcal{P}_\infty(\mathbb{R}))$ be the unique solution of* (1) *given by [7]. Then,*

$$\exists C > 0,\, \forall t \geq 0, \quad supp\, \rho(t, \cdot) \subset [-C, C]. \tag{13}$$

**Proof of Prop. 1**

We consider separately the case when (3) is satisfied, and the case when (4) is satisfied. We denote $u(t,\cdot)$ the pseudo-inverse of $\rho(t,\cdot)$.

*Step 1:* If $V$, $W$ satisfy (3).

Let $t = \inf\{\tau;\, \max(|u(\tau,0)|,\, |u(\tau,1)|) \geq C\}$. if $u(t,0) = -C$, then,

$$\begin{aligned}
\partial_t u(t,0) &= \int W'\left(u(t,\xi) - u(t,0)\right) d\xi - V'(u(t,0)) \\
&\geq -\|W'\|_{L^\infty([-2C,2C])} - V'(u(t,0)) \\
&\geq 0,
\end{aligned}$$

and similarly, if $u(t,1) = C$, then $\partial_t u(t,1) \leq 0$. Consequently, $t = \infty$, and at all times, $-C \leq u(0,\cdot) \leq u(1,\cdot) \leq C$, that is the support of $\rho(t,\cdot)$ is uniformly bounded.

*Step 2:* If $V$, $W$ satisfy (4).

Assume w.l.o.g. that the center of mass of $\rho^0$ (which is preserved by the equation, see (8)) is:

$$\int_{\mathbb{R}} x\rho^0(x)\, dx = \int_0^1 u^0(z)\, dz = 0.$$

We shall show that if $\|u(t,\cdot)\|_\infty \geq \max\left(2C_1, \frac{3}{C_2}\|W'\|_{L^\infty(-C_1,C_1)}\right)$, then $t \mapsto \|u(t,\cdot)\|_\infty$ is non increasing.

Assume w.l.o.g. that $|u(t,0)| \geq |u(t,1)|$. We define $\Lambda := \{\xi \in [0,1];\, u(t,\xi) \geq u(t,0) + C_1\}$. Then,

- We assumed that $|u(0)| \geq |u(1)|$, so that

  $$u(t,z) \leq |u(t,0)|$$

  on $[0,1]$, and in particular on $\Lambda$.

- On $\Lambda^c$,

  $$u(t,z) \leq u(t,0) + C_1 = C_1 - |u(t,0)|.$$

Since the center of mass of $\rho$ is 0,

$$\begin{aligned}
0 &= \int_\Lambda u + \int_{\Lambda^c} u \\
&\leq |u(0)|\,|\Lambda| - (|u(0)| - C_1)(1 - |\Lambda|) \\
&\leq (2|u(0)| - C_1)|\Lambda| - (|u(0)| - C_1).
\end{aligned}$$

Then, $|\Lambda| \geq \left(1 + \frac{1}{1 - \frac{C_1}{|u(0)|}}\right)^{-1}$, and since $|u(0)| \geq 2C_1$,

$$|\Lambda| \geq \frac{1}{3}. \tag{14}$$

We use (4) to estimate $\partial_t u(t,0)$:

$$\begin{aligned}
\partial_t u(t,0) &= \int W'\left(u(t,\xi) - u(t,0)\right) d\xi \\
&\geq -\|W'\|_{L^\infty(-C_1,C_1)} + C_2 \int_\Lambda [u(t,\xi) - u(t,0)] \, d\xi \\
&\geq -\|W'\|_{L^\infty(-C_1,C_1)} + C_2 \, |\Lambda| \int_\Lambda [u(t,\xi) - u(t,0)] \, \frac{d\xi}{|\Lambda|} \\
&\geq -\|W'\|_{L^\infty(-C_1,C_1)} + C_2 \, |\Lambda| \int_0^1 [u(t,\xi) - u(t,0)] \, d\xi,
\end{aligned}$$

since for $(\xi, \xi') \in \Lambda \times \Lambda^c$, $u(t,\xi) - u(t,0) \geq u(t,\xi') - u(t,0)$. Since $\int_0^1 u(t,\xi)\, d\xi = 0$, we get:

$$\begin{aligned}
\partial_t u(t,0) &\geq -\|W'\|_{L^\infty(-C_1,C_1)} - C_2 \, |\Lambda| \, u(t,0) \\
&\geq -\|W'\|_{L^\infty(-C_1,C_1)} + \frac{1}{3} C_2 |u(t,0)| \\
&\geq 0,
\end{aligned}$$

thanks to the assumption that $\|u\|_\infty \geq \max\left(2C_1, \frac{3}{C_2}\|W'\|_{L^\infty(-C_1,C_1)}\right)$. Then, $\|u\|_\infty$ is non increasing, which implies as in the previous case, that the support of $\rho(t,\cdot)$ is uniformly bounded.

□

## 2.2 Asymptotic behavior of the solution

In this subsection, we show that we cannot expect the solution to converge to anything else than a set of steady-states. In particular, no periodic limit cycles exist.

**Proposition 2.** *Let $\rho^0$, $V$, $W$ satisfy Assumptions 1 to 4. Let $\rho \in Lip_{loc}([0,\infty), \mathcal{P}_\infty(\mathbb{R}))$ be the unique solution of* (1) *given by [7]. Then,*

*1.*

$$\int \rho(t,x) \left( \int W'(x-y)\rho(t,y)\, dy + V'(x) \right)^2 dx \to 0 \text{ as } t \to \infty.$$

*2. For any sequence $t_k \to \infty$, there exists a subsequence, still denoted $(t_k)$, such that:*

$$W_1\left(\rho(t_k,\cdot),\bar\rho\right) \to 0 \quad \text{as } k \to \infty, \tag{15}$$

*where $W_1$ denotes the $1-$Wasserstein distance, and $\bar\rho$ is a steady-state of (1).*

**Remark 1.** *The limit $\bar\rho$ of $\rho(t_k,\cdot)$ in (15) is not necessarily unique : it may depend both on the sequence $(t_k)$ and the extracted sequence.*

**Proof of Prop. 2**

*Step 1:* Proof of 1.

We first show that the energy (9) is non-increasing in time, using integrations by parts:

$$\begin{aligned}
\frac{dE}{dt}(t) &= \int_{\mathbb{R}}\int_{\mathbb{R}} \partial_x\left(\rho(t,x)\left(\int W'(x-z)\rho(t,z)\,dz + V'(x)\right)(t,x)\right) \\
&\quad \rho(t,y)W(x-y)\,dx\,dy \\
&\quad + \int_{\mathbb{R}} \partial_x\left(\rho(t,x)\left(\int W'(x-y)\rho(t,y)\,dy + V'(x)\right)\right)V(x)\,dx \\
&= -\int \rho(t,x)\left(\int W'(x-y)\rho(t,y)\,dy + V'(x)\right)^2 dx \\
&\leq 0.
\end{aligned} \tag{16}$$

Next, we have the following estimate on the regularity of the energy dissipation:

$$\begin{aligned}
\frac{d^2E}{dt^2} =& -\int \partial_x\left(\rho(t,x)\left(\int W'(x-y)\rho(t,y)\,dy + V'(x)\right)\right) \\
&\left(\int W'(x-y)\rho(t,y)\,dy + V'(x)\right)^2 dx \\
&-2\int \rho(t,x)\left(\int W'(x-y)\rho(t,y)\,dy + V'(x)\right)\int W'(x-y) \\
&\partial_y\left(\rho(t,y)\left(\int W'(y-z)\rho(t,z)\,dz + V'(y)\right)\right) dy\,dx \\
=&2\int \rho(t,x)\left(\int W'(x-y)\rho(t,y)\,dy + V'(x)\right)^2 \\
&\partial_x\left(\int W'(x-y)\rho(t,y)\,dy + V'(x)\right) dx \\
&+2\int \rho(t,x)\left(\int W'(x-y)\rho(t,y)\,dy + V'(x)\right)\int \partial_y(W'(x-y)) \\
&\left(\rho(t,y)\left(\int W'(y-z)\rho(t,z)\,dz + V'(y)\right)\right) dy\,dx.
\end{aligned}$$

Since $V$, $W \in C^2(\mathbb{R})$, we can estimate $\frac{d^2E}{dt^2}$ as follows:

$$
\begin{aligned}
\left|\frac{d^2E}{dt^2}\right| &\leq 2\left(\|V\|_{W^{2,\infty}(-C,C)} + \|W\|_{W^{2,\infty}(-2C,2C)}\right)\left(\|W\|_{W^{2,\infty}(-2C,2C)} + \|V\|_{W^{2,\infty}(-C,C)}\right)^2 \\
&\leq C, \qquad (17)
\end{aligned}
$$

where $C < +\infty$ is a constant.

Finally, notice that the energy is bounded from below:

$$
E \geq -\left(\frac{1}{2}\|W\|_{L^\infty(-2C,2C)} + \|V\|_{L^\infty(-C,C)}\right). \qquad (18)
$$

To prove that $\frac{dE}{dt}(t) \to 0$, we use an interpolation between $E(t) \to \bar{E}$ and $\frac{d^2}{dt^2}E(t)$ bounded :

Let $\varepsilon > 0$. Since the energy $E$ is non increasing (16) and bounded from below (18), $E$ has a limit $\bar{E}$ when $t \to \infty$. Let $t > 0$ and $\tau \in (0, \frac{t}{2}]$. Then,

$$
\begin{aligned}
\left|\frac{dE}{dt}(t)\right| &= \left|\frac{1}{\tau}\int_{t-\tau}^{t}\left[\frac{dE}{dt}(s) + \int_s^t \frac{d^2E}{dt^2}(\sigma)\,d\sigma\right]\,ds\right| \\
&= \left|\frac{1}{\tau}\left[E(t) - E(t-\tau)\right] + \frac{1}{\tau}\int_{t-\tau}^{t}\int_s^t \frac{d^2E}{dt^2}(\sigma)\,d\sigma\,ds\right| \\
&\leq \frac{2}{\tau}\|E - \bar{E}\|_{L^\infty([\frac{t}{2},\infty))} + \tau\left\|\frac{d^2E}{dt^2}\right\|_{L^\infty([0,\infty))}.
\end{aligned}
$$

For $t > 0$ large enough, $\tau := \frac{\|E-\bar{E}\|^{\frac{1}{2}}_{L^\infty([\frac{t}{2},\infty))}}{\left\|\frac{d^2E}{dt^2}\right\|^{\frac{1}{2}}_{L^\infty([0,\infty))}} < \frac{t}{2}$, and then,

$$
\left|\frac{dE}{dt}(t)\right| \leq 3\|E - \bar{E}\|^{\frac{1}{2}}_{L^\infty([\frac{t}{2},\infty))}\left\|\frac{d^2E}{dt^2}\right\|^{\frac{1}{2}}_{L^\infty([0,\infty))},
$$

which implies $\frac{dE}{dt}(t) \to 0$ as $t \to \infty$.

*Step 2:* Proof of 2.

The pseudo-inverse $u(t,\cdot)$ of $\rho(t,\cdot)$ is an increasing function, and is uniformly bounded thanks to Prop. 1. The sequence $u(t_k,\cdot)$ is then a uniformly bounded sequence of $BV([0,1])$. There exists then a subsequence, still denoted $u(t_k,\cdot)$, that converges in $L^1$ to a limit denoted by $\bar{u}$:

$$
\|u(t_k,\cdot) - \bar{u}\|_{L^1} \to 0.
$$

Our aim is to prove that $\bar{u}$ is a steady-state of (7). In order to prove that, we shall use the estimate obtained above, $\frac{dE}{dt}(t_k) \to 0$. Let us write this estimate in the pseudo-inverse setting:

$$\begin{aligned}
\frac{dE}{dt}(t_k) &= -\int \rho(t_k,x)\left(\int W'(x-y)\rho(t_k,y)\,dy + V'(x)\right)^2 dx \\
&= -\int_0^1 \left(\int W'(u(t_k,z)-u(t_k,\xi))\,d\xi + V'(u(t_k,z))\right)^2 dz.
\end{aligned}$$

We define $\bar{F} := -\int_0^1 \left(\int W'(\bar{u}(z)-\bar{u}(\xi))\,d\xi + V'(\bar{u}(z))\right)^2 dz$. Then,

$$\begin{aligned}
\bar{F} - \frac{dE}{dt} &= \int_0^1 \left(\int W'(u(z)-u(\xi))\,d\xi + V'(u(z))\right)^2 \\
&\quad - \left(\int W'(\bar{u}(z)-\bar{u}(\xi))\,d\xi + V'(\bar{u}(z))\right)^2 dz \\
&= \int_0^1 \Bigg(\int W'(u(z)-u(\xi))\,d\xi + V'(u(z)) \\
&\quad + \int W'(\bar{u}(z)-\bar{u}(\xi))\,d\xi + V'(\bar{u}(z))\Bigg) \\
&\quad \cdot\Bigg(\int W'(u(z)-u(\xi))\,d\xi - \int W'(\bar{u}(z)-\bar{u}(\xi))\,d\xi \\
&\quad + V'(u(z)) - V'(\bar{u}(z))\Bigg)\,dz \\
&\le C\left\|\int W'(u(z)-u(\xi))\,d\xi - \int W'(\bar{u}(z)-\bar{u}(\xi))\,d\xi\right\|_{L^1} \\
&\quad + C\|u-\bar{u}\|_{L^1} \\
&\le C\|W'\|_{L^\infty(-2C,2C)}\,\|u-\bar{u}\|_{L^1} + C\|u-\bar{u}\|_{L^1}.
\end{aligned}$$

Finally,

$$\begin{aligned}
\bar{F} &\le \frac{dE}{dt}(t_k) + C\|u(t_k,\cdot)-\bar{u}\|_{L^1} \\
&\to 0 \text{ as } k \to \infty.
\end{aligned}$$

Then, $\bar{F} = 0$, that is:

$$\operatorname{supp}\bar{\rho} \subset \left\{x \in \mathbb{R};\ \int W'(x-y)\bar{\rho}(y)\,dy + V'(x) = 0\right\},$$

and $\bar{\rho}$ is a steady-state of (1).

$\square$

## 2.3 Study of the steady states.

In the previous subsection, we showed that for any regular potential $W$ satisfying Assumption 4, the sequence $\rho(t_k,\cdot)$ converges, up to an extraction, to a steady solution of (1). In this subsection, we shall try to characterize the steady-states of (1).

The following proposition characterizes the steady-states for analytical interaction potentials $W$:

**Proposition 3.** *Assume $W$ and $V$ are analytical. Then, every steady state $\bar\rho \in M^1(\mathbb{R})$ of (1) with bounded support is a finite sum of Dirac masses:*

$$\bar\rho = \sum_{i=1}^{N} \bar\rho_i \delta_{\bar u_i},$$

*with $\bar\rho_1,\ldots,\bar\rho_N > 0$, $\bar u_1,\ldots,\bar u_N \in \mathbb{R}$.*

**Proof of Prop. 3**

Let us consider a steady solution $\bar\rho$ of (1), and the associated steady solution $\bar u$ of (7). For $z \in [0,1]$,

$$\begin{aligned} 0 &= \int W'(\bar u(z) - \bar u(\xi))\,d\xi - V'(\bar u(z)) \\ &= \int_0^1 W'(\bar u(z) - \bar u(\xi))\,d\xi - V'(\bar u(z)) \\ &= -(W' * \bar\rho + V')(\bar u(z)). \end{aligned}$$

Since $u([0,1]) = \mathrm{supp}(\bar\rho)$, for any $x \in \mathrm{supp}(\bar\rho)$,

$$0 = (W' * \bar\rho)(x) + V'(x).$$

Since $W$ and $V$ are analytic, so is $W' * \bar\rho + V'$, and if $\mathrm{supp}(\bar\rho)$ has an accumulation point, then

$$\forall x \in \mathbb{R}, \quad (W' * \bar\rho)(x) + V'(x) = 0,$$

which is not possible since $V, W$ satisfy (3) or (4). Then, $\mathrm{supp}(\bar\rho)$ cannot have any accumulation point, and is thus a finite set of points.

$\square$

For less regular potentials, for instance when $W$ is only $C^2$, the same result cannot be expected to hold anymore, as the following example shows.

**Example 1.** *Consider the interaction potential* $W(x) := (\mathrm{dist}(x, [-1, 1]))^3$*, where* $\mathrm{dist}(x, y) := |x - y|$*, and* $V = 0$*.* $W$ *is* $C^2$ *(one could even consider a smoothed (*$C^\infty$*) version of the potential), but* (1) *admits the* $L^1(\mathbb{R})$ *steady state:*

$$\bar{\rho} = \mathbb{I}_{[-\frac{1}{2},\frac{1}{2}]}.$$

Nevertheless, the following proposition shows that steady states which are linearly stable (in a sense made clear in the following Proposition) have to be sums of Dirac masses:

**Proposition 4.** *Let* $V, W$ *satisfy Assumptions 1 and 4. Let* $\bar{\rho} \in M^1(\mathbb{R})$ *be a compactly supported steady state of* (1)*, and* $\bar{u}$ *be its pseudo-inverse. If* $\bar{\rho}$ *is such that* $\mathrm{supp}(\bar{\rho})$ *has an accumulation point* $x_0$*, then the pseudo-inverse equation* (7) *linearized around* $\bar{u}$ *in* $L^1$ *has no spectral gap.*

**Remark 2.** *Since the perturbations* $u^\varepsilon$ *of* $\bar{u}$ *used in the proof of Prop. 4 satisfy* $\int_0^1 u^\varepsilon = \int_0^1 \bar{u}$*, Prop. 4 remains true if we only consider perturbations preserving the center of mass* $\int x\bar{\rho}(x)\,dx$ *of* $\bar{\rho}$ *(this is important since* (1) *is invariant w.r.t. translations along* $x$*).*

**Remark 3.** *For a stability analysis of steady-states* $\bar{\rho}$ *that are sums of Dirac masses, see [13, 23]. In [13], we exhibit necessary and sufficient condition for local stability of such steady-states with respect to perturbations* $\rho$ *of* $\bar{\rho}$ *such that* $W_\infty(\bar{\rho}, \rho)$ *is small (where* $W_\infty$ *denotes the* $\infty-$*Wasserstein distance). In [23] we show the orbital stability of* $\bar{\rho}(\mathbb{R})$ *in* $M^1$ *for the usual topology of* $M^1(\mathbb{R})$*.*

**Proof of Prop. 4**

We begin by linearizing (7) around $\bar{u}$, with $u = \bar{u} + \delta v$, $\delta > 0$:

$$\begin{aligned}
\partial_t u(t,z) &= \int_0^1 W'(u(t,\xi) - u(t,z))\,d\xi - V'(u(t,z)) \\
&= \int_0^1 W'(\bar{u}(t,\xi) - \bar{u}(t,z))\,d\xi - V'(\bar{u}(t,z)) \\
&\quad +\delta\left(\int_0^1 W''(\bar{u}(\xi) - \bar{u}(z))(v(t,\xi) - v(t,z))\,d\xi - V''(\bar{u}(z))v(t,z)\right) + o(\delta) \\
&= \delta\bigg(\int_0^1 W''(\bar{u}(\xi) - \bar{u}(z))v(t,\xi)\,d\xi - \int_0^1 W''(\bar{u}(\xi) - \bar{u}(z))\,d\xi v(t,z) \\
&\qquad -V''(\bar{u}(z))v(t,z)\bigg) + o(\delta),
\end{aligned}$$

so that the linearization of (7) around $\bar{u}$ yields the linear operator $L : L^1([0,1]) \to L^1([0,1])$:

$$L(v)(z) = \int_0^1 W''(\bar{u}(\xi) - \bar{u}(z))v(\xi)\,d\xi - \left[\int_0^1 W''(\bar{u}(\xi) - \bar{u}(z))\,d\xi + V''(\bar{u}(z))\right] v(z). \quad (19)$$

We now shall show that if supp $\bar\rho$ has an accumulation point $x_0$, then we can build a sequence $(v^\varepsilon)$ of perturbations of $u$ such that:

$$\frac{\|L(v^\varepsilon)\|_{L^1}}{\|v^\varepsilon\|_{L^1}} \to 0,$$

which shows that the linear operator $L$ does not have any spectral gap. Since we are dealing with pseudo-inverses, we must however restrict to perturbations $v$ such that for some $\alpha > 0$, $u = \bar u + \alpha v$ is non decreasing.

We assume without any loss of generality that $x_0$ is an accumulation point of $\mathrm{supp}(\bar\rho)\cap [x_0,\infty)$. Then, for any $\varepsilon > 0$,

$$\int_{(x_0,x_0+\varepsilon)} d\bar\rho > 0. \tag{20}$$

For a given $\varepsilon > 0$, we define

$$\begin{aligned} z_0 &:= \inf\left\{z \in (0,1); \bar u(z) > x_0\right\}, \\ z_1^\varepsilon &:= \sup\left\{z \in (0,1); \bar u(z) < x_0+\varepsilon\right\}, \\ Z^\varepsilon &:= [z_0, z_1^\varepsilon]. \end{aligned}$$

We define the following perturbation $u^\varepsilon$ of $\bar u$:

$$u^\varepsilon(z) := \left| \begin{array}{l} \bar u(z) \text{ on } (Z^\varepsilon)^c, \\ \frac{1}{|Z^\varepsilon|}\int_{Z^\varepsilon} \bar u(y)\,dy \text{ on } Z^\varepsilon, \end{array}\right.$$

and we write $v^\varepsilon := u^\varepsilon - \bar u$. The function $u^\varepsilon$ is then the pseudo-inverse of the measure:

$$\rho^\varepsilon = \bar\rho|_{[x_0,x_0+\varepsilon]^c} + \left(\int_{[x_0,x_0+\varepsilon]} \bar\rho(x)\,dx\right)\delta_{\tilde x},$$

where $\tilde x = \frac{1}{|Z^\varepsilon|}\int_{Z^\varepsilon}\bar u(y)\,dy = \int_{[x_0,x_0+\varepsilon]} x\bar\rho(x)\,\frac{dx}{\int_{[x_0,x_0+\varepsilon]}\bar\rho(x)\,dx}$.

- We estimate $\int_0^1 W''(\bar u(\xi)-\bar u(z))v^\varepsilon(\xi)\,d\xi$:

$$\begin{aligned} \int_0^1 W''(\bar u(\xi)-\bar u(z))v^\varepsilon(\xi)\,d\xi &= \int_0^1 W''(\bar u(\xi)-x_0)v^\varepsilon(\xi)\,d\xi + \int_0^1 o_\varepsilon(1)v^\varepsilon(\xi)\,d\xi \\ &= o_\varepsilon(1)\|v^\varepsilon\|_{L^1}. \end{aligned} \tag{21}$$

- We estimate $\left[\int_0^1 W''(\bar u(\xi)-\bar u(z))\,d\xi + V''(\bar u(z))\right]v^\varepsilon(z)$:

  Since $\bar u$ is a steady state of (7),

$$\forall x \in \text{supp } \bar\rho, \quad (W' * \bar\rho)(x) + V'(x) = 0.$$

Thanks to Assumption 4, $W' * \bar\rho + V' \in C^1(\mathbb{R})$ is differentiable at $x = x_0$. Since $x_0$ is an accumulation point of supp $\bar\rho$, there exists a sequence $(x^k)_k \in (\text{supp } \bar\rho)^{\mathbb{N}}$ such that $x^k \to x_0$. Then,

$$\begin{aligned}
(W'' * \bar\rho)(x_0) + V''(x_0) &= \lim_{k\to\infty} \frac{((W' * \bar\rho)(x_0) + V'(x_0)) - \big((W' * \bar\rho)(x^k) + V'(x^k)\big)}{x_0 - x_k} \\
&= \lim_{k\to\infty} 0 \\
&= 0.
\end{aligned}$$

Since $W'' * \rho + V''$ is continuous, and thanks to the definition of $z_0$, $z_1^\varepsilon$, for any $z \in \text{supp}(v) \subset [z_0, z_1^\varepsilon]$,

$$[(W'' *_x \bar\rho)(\bar u(z)) + V''(\bar u(z))]\, v^\varepsilon(z) = \big(0 + o_{\bar u(z)-x_0}(1)\big)\, v^\varepsilon(z) = o_\varepsilon(1) v^\varepsilon(z). \quad (22)$$

Finally, using (22) and (21) in (19), we get:

$$\|L(v^\varepsilon)\|_{L^1} = o_\varepsilon(1)\|v^\varepsilon\|_{L^1},$$

which proves the proposition.

□

# 3 singular interaction potentials

In this section, we shall consider interaction potentials having a singularity at $x = 0$:

- Interaction potentials having an attractive singularity at $x = 0$, satisfying Assumption 5 (see below),
- Interaction potentials having a repulsive singularity at $x = 0$, satisfying Assumption 6 (see below).

The proof of Prop. 1 extends to singular potentials satisfying either Assumption 5 or 6 instead of Assumption 4, we shall therefore only consider compactly supported solutions. We shall show that those two cases have a very different dynamics : If Assumption 5 is satisfied, every steady-state apart from sums of Dirac masses are nonlinearly unstable, whereas if Assumption 6 is satisfied, the solution (of the time-dependant equation) is uniformly bounded in $L^\infty(\mathbb{R})$.

## 3.1 interaction potentials having an attractive singularity at $x = 0$.

We shall consider in this section potentials having an attractive singularity at $x = 0$:
**Assumption 5**

$$V \in C^2(\mathbb{R}), \quad W \in C^0(\mathbb{R}),$$

and there exist $W'(0^+) > 0$ such that

$$x \mapsto \tilde{W}(x) := W(x) - W'(0^+)|x| \in C^2(\mathbb{R}). \tag{23}$$

It is well known that in this case, classical solutions of (1) may blow up in finite time (see [3, 2]). Following [8], we extend (1) to measure-valued solutions with the following equation:

$$\partial_t \rho(t,x) = \partial_x \left[ \rho(t,x) \left( \int_{y\neq x} W'(x-y)\rho(t,y)\,dy + V'(x) \right) \right], \tag{24}$$

where we write (with a slight abuse of notation) $\rho(t,y)\,dy$ instead of $d\rho(t,\cdot)(y)$. If Assumptions 1 to 3 and 5 are satisfied, then it has been proven in [8] that a unique solution $\rho \in \mathrm{AC}_{loc}([0,\infty),\mathcal{P}_2(\mathbb{R}))$ to (24) exist. Note that the energy (9) is also a Lyapounov functional for (24).

One can check that the pseudo-inverse $u(t,z)$ of the solution $\rho(t,x)$ to (24) satisfies:

$$\partial_t u(t,z) = \int_{\{\xi\in[0,1];\, u(t,\xi)\neq u(t,z)\}} W'(u(t,\xi) - u(t,z))\,d\xi - V'(u(t,z)). \tag{25}$$

For regular potentials, we showed that if a (compactly supported) steady-state $\bar\rho \in M^1(\mathbb{R})$ of (1) is such that supp $\bar\rho$ has an accumulation point, then $\bar\rho$ cannot be linearly stable (in a sense defined in Prop. 4). In the case of interaction potentials having an attractive singularity at $x = 0$, we shall show that if a (compactly supported) steady-state $\bar\rho \in M^1(\mathbb{R})$ of (24) is such that supp $\bar\rho$ has an accumulation point (and a bit more, see (26)), then $\bar\rho$ is actually nonlinearly unstable in the sense of the Proposition below:

**Proposition 5.** *Let $V$, $W$ satisfy Assumptions 1 and 5. Let $\bar\rho$ be a compactly supported steady-state of* (24)*. If supp $\bar\rho$ has an accumulation point $x_0$ such that:*

$$\exists C > 0,\, \exists \eta > 0,\, \forall \gamma \in (0,\eta), \quad \frac{1}{\gamma}\int_{x_0}^{x_0+\gamma} \bar\rho(y)\,dy \geq C \tag{26}$$

*(or the same estimate with $-\eta < \varepsilon < 0$), then it is locally unstable: For any $\varepsilon > 0$, there exists $\rho^\varepsilon \in M^1(\mathbb{R})$, such that $W_1(\rho^\varepsilon,\bar\rho) \leq \varepsilon$ and*

$$E(\rho^\varepsilon) < E(\bar\rho), \tag{27}$$

*where $E$ is the energy defined by* (9)*.*

**Remark 4.** *As in the case of regular potentials, there may exist $L^1$ steady-states of* (24)*: For example, if $V(x) := \frac{-x^2}{2}$, $W(x) := |x|$,*

$$\bar{\rho} := \frac{1}{2}\mathbb{I}_{[-1,1]} \tag{28}$$

*is a steady-state of* (24)*. Prop. 5 shows that such steady-states are unstable.*

*Eq.* (25) *is not linearisable around steady-states (in $L^1$) in general. As a consequence, in order to define the nonlinear instability of steady-states like* (28)*, we use the energy $E$ (which is a Lyapounov functional of* (1)*), see* (27)*.*

**Proof of Prop. 5**

*Step 1 :* We define a sequence of measures $(\rho^\varepsilon)$ approaching $\bar{\rho}$.

We assume w.l.o.g. that $x_0$ is an accumulation point of $\operatorname{supp}\bar{\rho} \cap [x_0, \infty)$ such that (26) is satisfied. We define for $\varepsilon > 0$ such that $x_0 + \varepsilon \in \operatorname{supp}\bar{\rho}$:

$$\begin{gathered}
z_0 := \inf\left\{z \in (0,1); \bar{u}(z) \geq x_0\right\}, \\
z_1^\varepsilon := \sup\left\{z \in (0,1); \bar{u}(z) \leq x_0 + \varepsilon\right\}, \\
Z^\varepsilon := [z_0, z_1^\varepsilon].
\end{gathered}$$

Since $x_0$, $x_0 + \varepsilon \in \operatorname{supp}\bar{\rho}$ and $\bar{\rho}$ is a steady-state of (24),

$$\int_{\{y \notin [x_0, x_0+\varepsilon]\}} W'(x_0 - y)\bar{\rho}(y)\,dy + V'(x_0) = -\int_{y \in (x_0, x_0+\varepsilon]} W'(x_0 - y)\bar{\rho}(y)\,dy,$$
$$\int_{\{y \notin [x_0, x_0+\varepsilon]\}} W'(x_0 + \varepsilon - y)\bar{\rho}(y)\,dy + V'(x_0 + \varepsilon) = -\int_{y \in [x_0, x_0+\varepsilon)} W'(x_0 + \varepsilon - y)\bar{\rho}(y)\,dy.$$

If $\varepsilon > 0$ is small enough, then, $\operatorname{sign}(W'(x)) = \operatorname{sign}(x)$ for $x \in [-\varepsilon, \varepsilon]$. Then,

$$\int_{\{y \notin [x_0, x_0+\varepsilon]\}} W'(x_0 - y)\bar{\rho}(y)\,dy + V'(x_0) > 0 > \int_{\{y \notin [x_0, x_0+\varepsilon]\}} W'(x_0 + \varepsilon - y)\bar{\rho}(y)\,dy + V'(x_0 + \varepsilon).$$

On $[x_0, x_0 + \varepsilon]$,

$$\begin{aligned}
F(x) &= \int_{\{y \notin [x_0, x_0+\varepsilon]\}} W'(x - y)\bar{\rho}(y)\,dy + V'(x) \\
&= W'(0^+)\int_{(-\infty, x_0)} \bar{\rho}(y)\,dy - W'(0^+)\int_{(x_0, +\infty)} \bar{\rho}(y)\,dy \\
&\quad + \int_{\{y \notin [x_0, x_0+\varepsilon]\}} \tilde{W}'(x - y)\bar{\rho}(y)\,dy + V'(x),
\end{aligned}$$

where $\tilde{W}$ is defined in (23), and F is then continuous on $[x_0, x_0+\varepsilon]$. There exists then $\bar{x}^\varepsilon \in [x_0, x_0+\varepsilon]$ such that

$$\int_{\{y \notin [x_0, x_0+\varepsilon]\}} W'(\bar{x}^\varepsilon - y)\bar{\rho}(y)\, dy + V'(\bar{x}^\varepsilon) = 0. \tag{29}$$

We define the following perturbation $u^\varepsilon$ of $\bar{u}$:

$$u^\varepsilon(z) := \left| \begin{array}{l} \bar{u}(z) \text{ on } (Z^\varepsilon)^c, \\ \bar{x}^\varepsilon \text{ on } Z^\varepsilon, \end{array} \right.$$

and we write $v^\varepsilon := u^\varepsilon - \bar{u}$. $u^\varepsilon$ is then the pseudo-inverse of the measure:

$$\rho^\varepsilon = \bar{\rho}|_{[x_0,x_0+\varepsilon]^c} + \left( \int_{[x_0,x_0+\varepsilon]} \bar{\rho}(x)\, dx \right) \delta_{\bar{x}^\varepsilon}.$$

Notice that $W_1(\rho^\varepsilon, \bar{\rho}) \le \varepsilon$.

*Step 2:* We estimate $E(\rho^\varepsilon) - E(\bar{\rho})$.

We use the symmetry of $W$ and the fact that $u^\varepsilon = \bar{u}$ on $(Z^\varepsilon)^c$ to compute:

$$\begin{aligned} E(\rho^\varepsilon) - E(\bar{\rho}) &= \frac{1}{2}\int\int_{(Z^\varepsilon)^2} W(u^\varepsilon(\xi) - u^\varepsilon(z))\, d\xi\, dz - \frac{1}{2}\int\int_{(Z^\varepsilon)^2} W(\bar{u}(\xi) - \bar{u}(z))\, d\xi\, dz \\ &\quad + \int_{Z^\varepsilon}\int_{(Z^\varepsilon)^c} W(u^\varepsilon(z) - u^\varepsilon(\xi))\, d\xi\, dz - \int_{Z^\varepsilon}\int_{(Z^\varepsilon)^c} W(\bar{u}(z) - \bar{u}(\xi))\, d\xi\, dz \\ &\quad + \int_{Z^\varepsilon} V(u^\varepsilon(z))\, dz - \int_{Z^\varepsilon} V(\bar{u}(z))\, dz. \end{aligned}$$

Since $u^\varepsilon$ is constant on $Z^\varepsilon$, the first term can be computed. We estimate the second term using the expansion $W(x) = W(0) + W'(0)|x| + \tilde{W}'(0)x + O(x^2)$ (thanks to Assumption 5), where we notice that $\tilde{W}'(0) = 0$ thanks to Assumption 1. We use Taylor expansions on the fourth and sixth terms to get:

$$
\begin{aligned}
E(\rho^\varepsilon) - E(\bar\rho) \quad = \quad & W(0)\left(|Z^\varepsilon|^2 - |Z^\varepsilon|^2\right) \\
& - \frac{W'(0^+) + O(\varepsilon)}{2} \int\int_{(Z^\varepsilon)^2} |\bar u(\xi) - \bar u(z)| \, d\xi \, dz \\
& + \int_{Z^\varepsilon} \left\{ \left[ \int_{(Z^\varepsilon)^c} W(u^\varepsilon(z) - u^\varepsilon(\xi)) \, d\xi \right] + V(u^\varepsilon(z)) \right\} dz \\
& - \int_{Z^\varepsilon} \left\{ \left[ \int_{(Z^\varepsilon)^c} W(\bar x^\varepsilon - \bar u(\xi)) \, d\xi \right] + V(\bar x^\varepsilon) \right\} dz \\
& + \int_{Z^\varepsilon} \left\{ \left[ \int_{(Z^\varepsilon)^c} W'(\bar x^\varepsilon - \bar u(\xi)) \, d\xi \right] + V'(\bar x^\varepsilon) \right\} (\bar x^\varepsilon - \bar u(z)) \, dz \\
& - \frac{1}{2} \int_{Z^\varepsilon} \left\{ \left[ \int_{(Z^\varepsilon)^c} W''(\theta_1(\xi, z) - \bar u(\xi))) \, d\xi \right] + V''(\theta_2(z)) \right\} (\bar x^\varepsilon - \bar u(z))^2 \, dz,
\end{aligned}
$$

where $\theta_1(\xi, z)$, $\theta_2(z) \in [(\bar u(z), \bar x^\varepsilon)]$. Since $u^\varepsilon(z) = \bar x^\varepsilon$ on $Z^\varepsilon$, the third and fourth line cancel. The fifth line is equal to 0 thanks to the definition of $\bar x^\varepsilon$ (see (29)). Then,

$$
\begin{aligned}
E(\rho^\varepsilon) - E(\bar\rho) \quad = \quad & - \frac{W'(0^+) + O(\varepsilon)}{2} \int\int_{(Z^\varepsilon)^2} |\bar u(\xi) - \bar u^\varepsilon(z)| \, d\xi \, dz \\
& - \frac{1}{2} \int_{Z^\varepsilon} \left\{ \left[ \int_{(Z^\varepsilon)^c} W''(\theta_1(\xi, z) - \bar u(\xi)) \, d\xi \right] + V''(\theta_2(z)) \right\} (\bar x^\varepsilon - \bar u(z))^2 \, dz.
\end{aligned}
$$

Since $\bar\rho$ is compactly supported, $W''$, $V''$ are continuous, and $\theta_1(\xi, z)$, $\theta_2(z) \in [(\bar u(z), \bar x^\varepsilon)]$, we have uniform estimates:

$$
\begin{aligned}
& \sup_{\{\xi \in (Z^\varepsilon)^c, \, z \in Z^\varepsilon\}} |W''(\theta_1(\xi, z) - \bar u(\xi)) - W''(\bar x^\varepsilon - \bar u(\xi))| = o_\varepsilon(1), \\
& \sup_{\{z \in Z^\varepsilon\}} |V''(\theta_2(z)) - V''(\bar x^\varepsilon)| = o_\varepsilon(1).
\end{aligned}
\tag{30}
$$

Then, if we define $\omega^\varepsilon := \int_{(Z^\varepsilon)^c} W''(\bar u(\xi) - \bar x^\varepsilon) \, d\xi + V''(\bar x^\varepsilon)$, we get:

$$
\begin{aligned}
E(\rho^\varepsilon) - E(\bar\rho) \quad = \quad & - \frac{W'(0^+) + O(\varepsilon)}{2} \int\int_{(Z^\varepsilon)^2} |\bar u(\xi) - \bar u^\varepsilon(z)| \, d\xi \, dz \\
& + \frac{1}{2} \left( -\omega^\varepsilon + o_\varepsilon(1) \right) \|v^\varepsilon\|_{L^2}^2.
\end{aligned}
\tag{31}
$$

In order to prove the proposition, we shall show that the first term of (31) is strictly negative and dominates the second term (which is strictly positive). Then, $E(\rho^\varepsilon) - E(\bar\rho) <$

0 if $\varepsilon > 0$ is small enough. However, the two terms of (31) are of the same order in $\varepsilon$, we shall thus need to estimate precisely the second term.

*Step 3:* We estimate $\|v^\varepsilon\|_{L^2}^2$.

Since $\bar{u}$ is a steady-state, for any $z \in Z^\varepsilon$,

$$\begin{aligned}
0 &= \int_{\{\xi;\, \bar{u}(\xi)\neq\bar{u}(z)\}} W'(\bar{u}(\xi)-\bar{u}(z))\, d\xi - V'(\bar{u}(z)) \\
&= \left[\int_{(Z^\varepsilon)^c} W'(\bar{u}(\xi)-\bar{u}(z))\, d\xi - V'(\bar{u}(z))\right] \\
&\quad + \int_{\{\xi\in Z^\varepsilon;\, \bar{u}(\xi)\neq\bar{u}(z)\}} W'(\bar{u}(\xi)-\bar{u}(z))\, d\xi.
\end{aligned}$$

We estimate the first term through Taylor expansions of $x \mapsto W'(\bar{u}(\xi)-x)$, $x \mapsto V'(x)$ around $\bar{x}^\varepsilon$ (the rest term is estimated as in (30)), and the second term using $W'(x) = W'(0^+)\text{sign}(x) + \tilde{W}'(x) = W'(0^+)\text{sign}(x) + \tilde{W}''(\theta)x$ and $\text{sign}(0) = 0$ to get:

$$\begin{aligned}
0 &= \left[\int_{(Z^\varepsilon)^c} W'(\bar{u}(\xi)-\bar{x}^\varepsilon)\, d\xi - V'(\bar{x}^\varepsilon)\right] \\
&\quad + \left[\int_{(Z^\varepsilon)^c} W''(\bar{u}(\xi)-\bar{x}^\varepsilon)\, d\xi + V''(\bar{x}^\varepsilon)\right](\bar{x}^\varepsilon-\bar{u}(z)) + o_\varepsilon(1)(\bar{x}^\varepsilon-\bar{u}(z)) \\
&\quad + W'(0^+)\int_{Z^\varepsilon} \text{sign}(\bar{u}(\xi)-\bar{u}(z))\, d\xi + \int_{Z^\varepsilon} W''(\theta)(\bar{u}(\xi)-\bar{u}(z))\, d\xi \\
&= 0 + \omega^\varepsilon\, v^\varepsilon(z) + W'(0^+)\int_{Z^\varepsilon} \text{sign}(\bar{u}(\xi)-\bar{u}(z))\, d\xi \\
&\quad + O(1)\int_{Z^\varepsilon} |\bar{u}(\xi)-\bar{u}(z)|\, d\xi + o_\varepsilon(1)v^\varepsilon(z),
\end{aligned}$$

thanks to the definition of $\bar{x}^\varepsilon$. Then,

$$\begin{aligned}
\|v^\varepsilon\|_{L^2}^2 &= \int_{Z^\varepsilon} v^\varepsilon(z)^2\, dz \\
&= \int_{Z^\varepsilon}\left[\frac{W'(0^+)}{-\omega^\varepsilon}\int_{z_0}^{z_1^\varepsilon} \text{sign}(\bar{u}(\xi)-\bar{u}(z))\, d\xi\right] v^\varepsilon(z)\, dz \\
&\quad + \frac{1}{-\omega^\varepsilon}O(1)\|v^\varepsilon\|_\infty \int\int_{(Z^\varepsilon)^2} |\bar{u}(\xi)-\bar{u}(z)|\, d\xi\, dz + \frac{o_\varepsilon(1)}{\omega^\varepsilon}\|v^\varepsilon\|_{L^2}^2. \qquad (32)
\end{aligned}$$

Let $z \in [0,1]$, and $\zeta := \inf\{\xi \in [z_0, z_1^\varepsilon];\, \bar{u}(\xi) = \bar{u}(z)\}$, $\zeta' := \sup\{\xi \in [z_0, z_1^\varepsilon];\, \bar{u}(\xi) = \bar{u}(z)\}$. Then,

$$\begin{aligned}\int_{z_0}^{z_1^\varepsilon} \operatorname{sign}(\bar{u}(\xi)-\bar{u}(z))\,d\xi &= \int_{[z_0,z_1^\varepsilon]\setminus(\zeta,\zeta')} \operatorname{sign}(\xi-z)\,d\xi + \int_\zeta^{\zeta'} 0\,d\xi \\ &= \int_{z_0}^{z_1^\varepsilon} \operatorname{sign}(\xi-z)\,d\xi - \int_\zeta^{\zeta'} \operatorname{sign}(\xi-z)\,d\xi \\ &= [(z_1^\varepsilon - z)-(z-z_0)] - [(\zeta'-z)-(z-\zeta)] \\ &= -2\left[z-\frac{z_0+z_1^\varepsilon}{2}\right] + 2\left[z-\frac{\zeta+\zeta'}{2}\right].\end{aligned}$$

Then, since $\bar{u}$ is constant on $(\zeta,\zeta')$, so is $z\mapsto v^\varepsilon(z)=\bar{x}^\varepsilon-\bar{u}(z)=v^\varepsilon\left(\frac{\zeta+\zeta'}{2}\right)$, and

$$\begin{aligned}&\int_\zeta^{\zeta'}\left[\int_{z_0}^{z_1^\varepsilon} \operatorname{sign}(\bar{u}(\xi)-\bar{u}(z))\,d\xi\right] v^\varepsilon(z)\,dz \\ &= -2\int_\zeta^{\zeta'}\left[z-\frac{z_0+z_1^\varepsilon}{2}\right] v^\varepsilon(z)\,dz + 2v^\varepsilon\left(\frac{\zeta+\zeta'}{2}\right)\int_\zeta^{\zeta'}\left[z-\frac{\zeta+\zeta'}{2}\right]dz \\ &= -2\int_\zeta^{\zeta'}\left[z-\frac{z_0+z_1^\varepsilon}{2}\right] v^\varepsilon(z)\,dz. \qquad (33)\end{aligned}$$

We consider

$$\Omega := \Big\{(\zeta,\zeta')\subset Z^\varepsilon;\ \bar{u} \text{ is constant on } (\zeta,\zeta'), \\ (\zeta,\zeta') \text{ being the maximal interval such that this is true}\Big\}.$$

Since each element of $\Omega$ contains a rational number, $\Omega$ is at most countable, and then, thanks to (33),

$$
\begin{aligned}
&\int_{Z^\varepsilon}\left[\int_{z_0}^{z_1^\varepsilon}\operatorname{sign}(\bar u(\xi)-\bar u(z))\,d\xi\right]v^\varepsilon(z)\,dz\\
&=\int_{Z^\varepsilon\setminus\left(\cup_{(\zeta,\zeta')\in\Omega}(\zeta,\zeta')\right)}\left[\int_{z_0}^{z_1^\varepsilon}\operatorname{sign}(\bar u(\xi)-\bar u(z))\,d\xi\right]v^\varepsilon(z)\,dz\\
&\quad+\sum_{(\zeta,\zeta')\in\Omega}\int_\zeta^{\zeta'}\left[\int_{z_0}^{z_1^\varepsilon}\operatorname{sign}(\bar u(\xi)-\bar u(z))\,d\xi\right]v^\varepsilon(z)\,dz\\
&=\int_{Z^\varepsilon\setminus\left(\cup_{(\zeta,\zeta')\in\Omega}(\zeta,\zeta')\right)}\left[\int_{z_0}^{z_1^\varepsilon}\operatorname{sign}(\xi-z)\,d\xi\right]v^\varepsilon(z)\,dz\\
&\quad+\sum_{(\zeta,\zeta')\in\Omega}-2\int_\zeta^{\zeta'}\left[z-\frac{z_0+z_1^\varepsilon}{2}\right]v^\varepsilon(z)\,dz\\
&=-2\int_{Z^\varepsilon}\left[z-\frac{z_0+z_1^\varepsilon}{2}\right]v^\varepsilon(z)\,dz. \qquad (34)
\end{aligned}
$$

Thanks to (34), (32) becomes:

$$
\begin{aligned}
\left(1-\frac{o_\varepsilon(1)}{\omega^\varepsilon}\right)\|v^\varepsilon\|_{L^2}^2 &= -2\frac{W'(0^+)}{-\omega^\varepsilon}\int_{Z^\varepsilon}\left(z-\frac{z_0+z_1^\varepsilon}{2}\right)v^\varepsilon(z)\,dz\\
&\quad+\frac{1}{-\omega^\varepsilon}O(\varepsilon)\int\int_{(Z^\varepsilon)^2}|\bar u(\xi)-\bar u(z)|\,d\xi\,dz. \qquad (35)
\end{aligned}
$$

We notice that:

$$
\begin{aligned}
\int\int_{(Z^\varepsilon)^2}|\bar u(\xi)-\bar u(z)|\,d\xi\,dz &= 2\int\int_{(Z^\varepsilon)^2,\,\xi\geq z}\left[\bar u(\xi)-\bar u(z)\right]\,d\xi\,dz\\
&= 2\int_{Z^\varepsilon}\left[(z-z_0)\bar u(z)-(z_1^\varepsilon-z)\bar u(z)\right]\,dz\\
&= 4\int_{Z^\varepsilon}\left(z-\frac{z_0+z_1^\varepsilon}{2}\right)\bar u(z)\,dz,
\end{aligned}
$$

and since $\int_{Z^\varepsilon}(z-\frac{z_0+z_1^\varepsilon}{2})\,dz=0$, we have:

$$
\begin{aligned}
\int\int_{(Z^\varepsilon)^2}|\bar u(\xi)-\bar u(z)|\,d\xi\,dz &= 4\int_{Z^\varepsilon}\left(z-\frac{z_0+z_1^\varepsilon}{2}\right)\left(\bar u(z)-\bar x^\varepsilon\right)\,dz\\
&= -4\int_{Z^\varepsilon}\left(z-\frac{z_0+z_1^\varepsilon}{2}\right)v^\varepsilon(z)\,dz. \qquad (36)
\end{aligned}
$$

Finally, thanks to (36), (35) becomes:

$$\|v^\varepsilon\|_{L^2}^2 = \frac{W'(0^+) + O(\varepsilon)}{-2\omega^\varepsilon + o_\varepsilon(1)} \int\int_{(Z^\varepsilon)^2} |\bar{u}(\xi) - \bar{u}(z)|\, d\xi\, dz. \tag{37}$$

*Step 4:* We estimate $\omega^\varepsilon$.

Since $x_0$, $x_0 + \varepsilon \in \text{supp}\ \bar{\rho} = \overline{\bar{u}([0,1])}$ and $\bar{u}$ is a steady-state of (25),

$$\begin{aligned}
0 &= \left( \int_{\{\xi\in[0,1];\, \bar{u}(\xi)\neq x_0+\varepsilon\}} W'(\bar{u}(\xi) - (x_0+\varepsilon))\, d\xi - V'(x_0+\varepsilon) \right) \\
&\quad - \left( \int_{\{\xi\in[0,1];\, \bar{u}(\xi)\neq x_0\}} W'(\bar{u}(\xi) - x_0)\, d\xi - V'(x_0) \right) \\
&= \left( \int_0^1 \Big( W'(0^+)\text{sign}(\bar{u}(\xi) - (x_0+\varepsilon)) + \tilde{W}'(\bar{u}(\xi) - (x_0+\varepsilon)) \Big)\, d\xi - V'(x_0+\varepsilon) \right) \\
&\quad - \left( \int_0^1 \Big( W'(0^+)\text{sign}(\bar{u}(\xi) - x_0) + \tilde{W}'(\bar{u}(\xi) - x_0) \Big)\, d\xi - V'(x_0) \right) \\
&= \Bigg( W'(0^+)\Big( \bar{\rho}\big((x_0+\varepsilon, +\infty)\big) - \bar{\rho}\big((-\infty, x_0+\varepsilon)\big) \Big) \\
&\qquad + \int_0^1 \tilde{W}'(\bar{u}(\xi) - (x_0+\varepsilon))\, d\xi - V'(x_0+\varepsilon) \Bigg) \\
&\quad - \left( W'(0^+)\Big( \bar{\rho}\big((x_0, +\infty)\big) - \bar{\rho}\big((-\infty, x_0)\big) \Big) + \int_0^1 \tilde{W}'(\bar{u}(\xi) - x_0)\, d\xi - V'(x_0) \right) \\
&= -W'(0^+)\left[ \bar{\rho}(\{x_0, x_0+\varepsilon\}) + 2\bar{\rho}\big((x_0, x_0+\varepsilon)\big) \right] \\
&\quad - \left[ \int_0^1 \tilde{W}''(\bar{u}(\xi) - \bar{x}^\varepsilon)\, d\xi + V''(\bar{x}^\varepsilon) \right] \varepsilon + o(\varepsilon),
\end{aligned}$$

where we applied a Taylor expansion to the regular terms $x \mapsto \tilde{W}'(\bar{u}(\xi) - x)$ and $x \mapsto V'(x)$ at point $x = \bar{x}^\varepsilon$ (the rest term is estimated as in (30)). We notice that

$$\begin{aligned}
\int_0^1 \tilde{W}''(\bar{u}(\xi) - \bar{x}^\varepsilon)\, d\xi + V''(\bar{x}^\varepsilon) &= \omega^\varepsilon + \int_{Z^\varepsilon} \tilde{W}''(\bar{u}(\xi) - \bar{x}^\varepsilon)\, d\xi \\
&= \omega^\varepsilon + O(|Z^\varepsilon|),
\end{aligned}$$

and then,

$$-\varepsilon(\omega^\varepsilon + O(|Z^\varepsilon|)) = \left[ \bar{\rho}(\{x_0, x_0+\varepsilon\}) + 2\bar{\rho}\big((x_0, x_0+\varepsilon)\big) \right] + o(\varepsilon). \tag{38}$$

Since $|\omega^\varepsilon| \leq \|W''\|_{L^\infty(\text{supp }\bar\rho - \text{supp }\bar\rho)} + \|V''\|_{L^\infty(\text{supp }\bar\rho)}$, we have in particular that $|Z^\varepsilon|$ is of order $\varepsilon$:

$$|Z^\varepsilon| = \bar\rho([x_0, x_0+\varepsilon]) = O(\varepsilon), \tag{39}$$

and then, using again (38), we get that for $\varepsilon$ small enough,

$$\begin{aligned} -\omega^\varepsilon &= \frac{W'(0^+)}{\varepsilon}\left[\bar\rho(\{x_0, x_0+\varepsilon\}) + 2\bar\rho\big((x_0, x_0+\varepsilon)\big)\right] + o_\varepsilon(1) \\ &\geq W'(0^+)\,\frac{1}{\varepsilon}\bar\rho\big([x_0, x_0+\varepsilon]\big) + o_\varepsilon(1). \end{aligned}$$

We assumed (see (26)) that $\frac{1}{\varepsilon}\int_{[x_0,x_0+\varepsilon]} \bar\rho(x)\,dx > C > 0$ for $\varepsilon$ small enough. Then, for $\varepsilon > 0$ small enough,

$$-\omega^\varepsilon \geq Cst > 0. \tag{40}$$

*Step 5:* We conclude.

Thanks to (37), (31) becomes:

$$\begin{aligned} E(\rho^\varepsilon) - E(\bar\rho) &= -\frac{W'(0^+) + O(\varepsilon)}{2}\int\int_{(Z^\varepsilon)^2} |\bar u(\xi) - \bar u^\varepsilon(z)|\,d\xi\,dz \\ &\quad + \frac{1}{2}\left(-\omega^\varepsilon + o_\varepsilon(1)\right)\frac{W'(0^+) + O(\varepsilon)}{-2\omega^\varepsilon + o_\varepsilon(1)}\int\int_{(Z^\varepsilon)^2} |\bar u(\xi) - \bar u(z)|\,d\xi\,dz \\ &= -\left[\frac{W'(0^+)}{4} + o_\varepsilon(1)\right]\int\int_{(Z^\varepsilon)^2} |\bar u(\xi) - \bar u(z)|\,d\xi\,dz, \end{aligned}$$

thanks to (40). Finally, we assumed that $x_0$ is an accumulation point of supp $\rho^0 \cap [x_0, \infty)$, $\varepsilon$ can thus be chosen small enough for $o_\varepsilon(1) \leq \frac{W'(0^+)}{8}$ to hold, and then,

$$E(\rho^\varepsilon) - E(\bar\rho) \leq -\frac{W'(0^+)}{8}\int\int_{(Z^\varepsilon)^2} |\bar u(\xi) - \bar u(z)|\,d\xi\,dz. \tag{41}$$

Since $x_0$ is an accumulation point of supp $\bar\rho \cap [x_0, x_0+\varepsilon] = \bar u(Z^\varepsilon)$, $\bar u$ cannot be constant on $Z^\varepsilon$, and then:

$$E(\rho^\varepsilon) - E(\bar\rho) < 0. \tag{42}$$

$\square$

## 3.2 potentials having a repulsive singularity at $x = 0$.

In this section, we shall consider potentials having a repulsive singularity at $x = 0$:
**Assumption 6**

$$V \in C^2(\mathbb{R}), \quad W \in C^0(\mathbb{R}),$$

and there exists $W'(0^+) < 0$ such that

$$(x \mapsto \tilde{W}(x) := W(x) - W'(0^+)|x|) \in C^2(\mathbb{R}).$$

For such potentials, we don't know any existence theory, we thus prove in Prop. 6 that if Assumptions 1, 2, 3 and 6 are satisfied, and if $\rho^0 \in W^{2,\infty}(\mathbb{R})$, then there exists a unique solution $\rho \in L^\infty(\mathbb{R}_+ \times \mathbb{R}) \cap \mathrm{Lip}_{loc}(\mathbb{R}_+, W^{2,\infty}(\mathbb{R}))$.

**Proposition 6.** *Let $\rho^0$, $V$, $W$ satisfy Assumptions 1, 2, 3 and 6. Assume moreover that $\rho^0 \in W^{2,\infty}(\mathbb{R})$. Then there exists a unique solution*

$$\rho \in L^\infty(\mathbb{R}_+ \times \mathbb{R}) \cap Lip_{loc}(\mathbb{R}_+, W^{2,\infty}(\mathbb{R}))$$

*to* (1).
*If $\rho^0 \in W^{N,\infty}(\mathbb{R})$ and $V \in W^{N+2,\infty}(\mathbb{R})$ (for $N \in \mathbb{N}$), then $\rho \in Lip_{loc}(\mathbb{R}_+, W^{N,\infty}(\mathbb{R}))$*

**Remark 5.** *The uniform bound $\rho \in L^\infty(\mathbb{R}_+ \times \mathbb{R})$ ensures that the solution does not converge to any singular measure. The behavior of the solution in this case is then very different from the two other cases (Assumptions 4 or 5) studied in this paper, where the solution generically converges to a sum of Dirac masses. For a short investigation on the transition from the situation of regular kernels to the situation where $W$ has a singularity at $x = 0$ and is locally repulsive, see [13].*

**Proof of Prop. 6**

*Step 1:* We show some a priori estimates on $\rho$, using maximum principle arguments:

We consider first $x \in \mathbb{R}$ such that $\rho(t,x) = \|\rho(t,\cdot)\|_\infty$. Then $\partial_x \rho(t,x) = 0$, and

$$\begin{aligned}
\partial_t \rho(t,x) &= \partial_x \rho(t,x)(W' * \rho)(t,x) + \rho(t,x)\left((\tilde{W}'' * \rho)(t,x) + V''(x)\right) \\
&\quad -2W'(0^+)\rho(t,x)^2 \\
&= \left((\tilde{W}'' * \rho)(t,x) + V''(x) - 2W'(0^+)\rho(t,x)\right)\rho(t,x) \\
&\leq \left(\|\tilde{W}''\|_{L^\infty} + \|V''\|_\infty - 2W'(0^+)\|\rho(t,\cdot)\|_\infty\right)\|\rho(t,\cdot)\|_\infty.
\end{aligned}$$

Then,

$$\|\rho(t,\cdot)\|_\infty \leq \max\left(\|\rho^0\|_\infty, \frac{1}{2|W'(0^+)|}\left(\|\tilde{W}''\|_{L^\infty} + \|V''\|_\infty\right)\right). \tag{43}$$

Let now $N \in \mathbb{N}$ and $x \in \mathbb{R}$ be such that $|\partial_x^N \rho(t,x)| = \|\partial_x^N \rho(t,\cdot)\|_\infty$. W.l.o.g.,$\partial_x^N \rho(t,x) \geq 0$, then,

$$\begin{array}{rcl}
\partial_t \partial_x^N \rho(t,x) & = & \partial_x^{N+1}\left(\rho(W' * \rho + V')\right)(t,x) \\
& = & \displaystyle\sum_{n=0}^{N+1} \left( \begin{array}{c} N \\ n \end{array} \right) \partial_x^n \rho(t,x) \partial_x^{N+1-n}(W' * \rho + V')(t,x) \\
& = & \displaystyle\sum_{n=1}^{N} \left( \begin{array}{c} N \\ n \end{array} \right) \partial_x^n \rho(t,x)(\tilde{W}'' * \partial_x^{N-n}\rho - 2W'(0^+)\partial_x^{N-n}\rho + \partial_x^{N+2-n}V)(t,x) \\
& & +\partial_x(\partial_x^N \rho)(t,x)(W' * \rho + V')(t,x) \\
& & +\rho(t,x)\Big[ -2W'(0^+)\partial_x^N \rho(t,x) + \tilde{W}'' * \partial_x^N \rho + \partial_x^{N+2}V\Big] \\
& \leq & \displaystyle\sum_{n=1}^{N} \left( \begin{array}{c} N \\ n \end{array} \right) \Big[ \Big(\|\tilde{W}''\|_{L^1([-2C,2C])} + 2W'(0^+)\Big) \|\partial_x^n \rho(t,\cdot)\|_\infty \|\partial_x^{N-n}\rho(t,\cdot)\|_\infty \\
& & \quad +\|\rho(t,\cdot)\|_{W^{N,\infty}} \|V\|_{W^{N+2,\infty}([-C,C])}\Big] \\
& & +0 + \|\rho(t,\cdot)\|_\infty \Big[ \|\tilde{W}''\|_{L^1([-2C,2C])} \|\rho(t,\cdot)\|_{W^{N,\infty}} + \|V\|_{W^{N+2,\infty}([-C,C])}\Big] \\
& \leq & C\left(1 + \|\rho(t,\cdot)\|_{W^{N-1,\infty}}\right) \|\rho(t,\cdot)\|_{W^{N,\infty}},
\end{array}$$

where we used the assumption on $x$ to get $\partial_x(\partial_x^N \rho)(t,x) = 0$, the assumption $\partial_x^N \rho(t,x) \geq 0$ to get $\rho(t,x)\Big[ -2W'(0^+)\partial_x^N \rho(t,x)\Big] \leq 0$, and the estimate of Prop. 1 to get that $\operatorname{supp}\rho(t,\cdot) \subset [-C,C]$ (uniformly in time).

Since this inequality holds for any $N \geq 1$, and $\|\rho(t,\cdot)\|_{L^\infty} < Cst$ by (43), an induction argument shows that if $\rho^0 \in W^{N,\infty}$, there exists $C = C(N, \|\rho^0\|_{W^{N,\infty}})$ such that

$$\|\rho(t,\cdot)\|_{W^{N,\infty}} \leq \|\rho^0\|_{W^{N,\infty}} e^{Ct}. \tag{44}$$

*Step 2:* We build the solution using the above a priori estimates:

In order to prove the existence of a solution $\rho \in L^\infty(\mathbb{R}_+ \times \mathbb{R}) \cap \mathrm{Lip}_{loc}(\mathbb{R}_+, W^{2,\infty}(\mathbb{R}))$ to (1), we use the inductive scheme: $\rho_0(t,x) := \rho^0(x)$, and

$$\left\{ \begin{array}{l} \rho_{n+1}(0,\cdot) = \rho^0, \\ \partial_t \rho_{n+1}(t,x) = \partial_x\left(\rho_{n+1} W' * \rho_n + V'\right). \end{array} \right.$$

Thanks to estimates similar to the a priori estimates done in the first part of this proof, one gets the following (uniform in $n$) estimates:

$$\|\rho_{n+1}(t,\cdot)\|_\infty \leq \|\rho^0\|_\infty e^{C\,t},$$

and ther exist $C$, $T > 0$ such that $\forall t \leq T$,

$$\|\partial_x \rho_{n+1}(t,\cdot)\|_\infty \leq C\|\partial_x \rho^0\|_\infty, \quad \|\partial_t \rho_{n+1}(t,\cdot)\|_\infty \leq C\left(\|\partial_x \rho^0\|_\infty + \|\rho^0\|_\infty\right).$$

Those estimates show that $(\rho_n)$ converges in $L^\infty([0,T]\times\mathbb{R})$ up to an extraction. A further study of $(\rho_{n+1}-\rho_n)$ shows that the whole sequence converges to the unique strong solution $\rho$ of (1).

Finally, estimate (44) shows the propagation of regularity anounced in Prop. 6.

□

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