# Error estimate for the upwind finite volume method for nonlinear scalar conservation law


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**Abstract**

In this paper we estimate the error of upwind first order finite volume schemes applied to scalar conservation laws. As a first step we consider standard upwind and flux finite volume scheme discretization of a linear equation with space variable coefficients in conservation form. We prove that, in spite of their lack of consistency, both schemes lead to a first order error estimate. As a final step, we prove similar estimate for the nonlinear case. Our proofs rely on the notion of geometric corrector, introduced in our previous paper [2] in the context of constant coefficient linear advection equations.



*Key words:*
Finite Volume Method, Linear and nonlinear scalar problem, Stability and convergence of numerical methods, Geometric Corrector
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## 1. Introduction

As discussed in monograph of Eymard *et al* [12], Godlewski and Raviart [16], Kröner [18] and Leveque [20], Finite Volume Methods are well-adapted to the discretization of conservation laws for which solutions undergoes discontinuities. Moreover the theoretical study of convergence of these methods for nonlinear transport equations has been addressed in a large amount of papers, most of them based on the Kruzkov functional method (see for instance [19], [7], [8], [30], [5], [33], [32], [1], [26], [25], [10], [11]).

However, even for the scalar linear advection equation, the theoretical proof of an optimal a priori error estimates is still a challenging task. One of the main difficulties lies in the fact that the non uniformity of the mesh brings up an apparent loss of consistency in the finite differences sense. This loss of consistency is an artifact of the standard convergence proof ; Lax-Richtmyer theorem


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is not suitable. Actually, consistency is not necessary, the scheme maintains the accuracy and the global error behaves better than the local error would indicate. This property of enhancement of the truncation error is called supraconvergence and this phenomenon, discovered by Tikhonov and Samarskij [31], was widely analyzed in various cases by Manteuffel and his co-authors in [22], [23], [17], [24].

In a series of previous papers (Bouche *et al* [2], [3]) where we considered finite volume methods applied to the linear advection equation with a constant velocity, we introduced what we called the *geometric corrector* which is a sequence associated with every finite volume mesh. We proved that the error estimate for the scheme behaves like the norm of this corrector when the mesh size goes to zero and actually established that this norm is indeed bounded by the mesh size in several cases including the one where an arbitrary coarse mesh of triangles is uniformly refined.

Computing numerically this corrector (see Pascal [27]) allows us to state that this result might be extended to more general cases like the one with independent refined meshes. However in particular cases, we observe that the estimation of the norm of the geometric corrector (as well as the order of convergence) depends on the relative position of the advection vector with respect to the boundary. For instance, in case of a convection direction parallel to one side of the domain, the $l^\infty$ norm is only $\mathcal{O}(\sqrt{h})$ while the $l^1$ norm is $\mathcal{O}(h)$ where $h$ measures the mesh size. This behavior, similar to the loss of accuracy proved in Peterson [28], is widely analyzed in Bouche *et al* [4].

In the present paper, we extend the notion of geometric corrector to the non constant velocity case in one dimensional space. In section 3 and 4, we develop this concept, after having introduced the notations, and apply it to two types of explicit finite volume method: the linear standard upwind finite volume method and the flux finite volume method. We are able to prove that the $l^p$ norm of the error behaves like the mesh size.

In section 5, we study the flux finite volume method for a nonlinear conservation law. With a simple adaptation of the geometric corrector to this case, we can prove that, as long as the solution remains smooth, the scheme is first order accurate. For technical reasons, in the case of the $l^1$ norm, the local quasi-uniformity condition on the mesh has to be replaced by a less general global quasi-uniformity condition.

## 2. The continuous problem and notations

### 2.1. The continuous problem

Let $[\alpha, \beta]$ be an interval of $\mathbb{R}$. Let $a$ be a non zero function defined on $[\alpha, \beta]$ and assumed to be at least in $\mathcal{C}^1([\alpha, \beta])$. Given a function $\varphi$ defined on $[\alpha, \beta]$ and two functions $\psi_\alpha$ and $\psi_\beta$ defined on $[0, T]$, we consider the initial and boundary value problem for the linear convection equation in conservation form:

$$\begin{cases} \dfrac{\partial u(x,t)}{\partial t} + \dfrac{\partial(a(x)u(x,t))}{\partial x} = 0\,, \quad (x,t) \in ]\alpha,\beta[\times]0,T[\,, \\ u(x,0) = \varphi(x)\,, \quad x \in ]\alpha,\beta[\,, \\ a(\alpha)^+ \cdot (u(\alpha,t) - \psi_\alpha(t)) = 0\,, \quad t \in ]0,T[\,, \\ a(\beta)^- \cdot (u(\beta,t) - \psi_\beta(t)) = 0\,, \quad t \in ]0,T[\,, \end{cases} \tag{1}$$

where $z^+ = \frac{|z|+z}{2}$ (respectively $z^- = \frac{|z|-z}{2}$) is the positive (respectively negative) part of $z = z^+ - z^-$. We assume that the data $\varphi$, $\psi_\alpha$ and $\psi_\beta$ are smooth functions which satisfy the so called compatibility conditions (CCs) at $t = 0$ so that (1) has a unique smooth solution. These CCs are classical and can be found *e.g.* in Chazarin and Piriou [6]. For example the first CC, which corresponds to $C^0$ smoothness of the solution, reads

$$a^+(\alpha) \cdot (\varphi(0) - \psi_\alpha(0)) = 0\,, \quad a^-(\beta) \cdot (\varphi(0) - \psi_\beta(0)) = 0\,.$$

Let us introduce the flux function

$$f(x,t) = a(x)u(x,t)$$

and let us observe that it also satisfies a convection equation which, unlike the previous one, is not written in a conservative form and reads

$$\begin{cases} \dfrac{\partial f(x,t)}{\partial t} + a(x)\dfrac{\partial f(x,t)}{\partial x} = 0\,, \quad (x,t) \in ]\alpha,\beta[\times]0,T]\,, \\ f(x,0) = a(x)\varphi(x)\,, \quad x \in ]\alpha,\beta[\,, \\ f(\alpha,t) = a(\alpha)\psi_\alpha(t)\,, \quad \text{if } a(\alpha) > 0 \\ f(\beta,t) = a(\beta)\psi_\beta(t)\,, \quad \text{if } a(\beta) < 0\,. \end{cases} \tag{2}$$

*2.2. Notations*

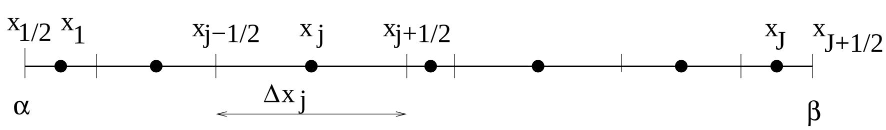


Figure 1: Discretization of the domain $[\alpha,\beta]$

Let $\mathcal{T} = \{K_j : \ j = 1,\ldots,J\}$ be a partition of the domain $]\alpha,\beta[$ in volumes $K_j = ]x_{j-\frac{1}{2}}, x_{j+\frac{1}{2}}[$ as presented in Figure 1. The centroid of $K_j$ is given by $x_j = (x_{j-\frac{1}{2}} + x_{j+\frac{1}{2}})/2$ and its measure by $|K_j| = \Delta x_j = x_{j+\frac{1}{2}} - x_{j-\frac{1}{2}}$. Since we are interested in convergence results, we consider families of partitions $\mathcal{T}^h$ indexed by the real number $h = \max\{\Delta x_j, K_j \in \mathcal{T}^h\}$, the size of the mesh. We assume that there exists $h_0 > 0$ and a positive constant $\kappa$ such that for every $h < h_0$ we have the following local quasi-uniformity relation

$$\frac{1}{\kappa}\,\Delta x_{j-1} \le \Delta x_j \le \kappa\,\Delta x_{j-1}\,, \quad \forall j = 1,\ldots,J\,. \tag{3}$$

We shall consider sequences $\xi = (\xi_j)_{j=1}^J$ in $\mathbb{R}^J$ and we shall estimate their norm induced by $\ell^p$, readily $||\xi||_\infty = \max_{1\le j\le J} |\xi_j|$, and for $p > 1$, $||\xi||_p^p = \sum_{j=1}^J |K_j||\xi_j|^p$. The value and the sign of the function $a$ at the centroid $x_j$ and at the interface $x_{j+\frac{1}{2}}$ are denoted by

$$a_j = a(x_j)\,, \quad a_{j+\frac{1}{2}} = a(x_{j+\frac{1}{2}})\,, \quad \sigma_j = \mathrm{sign}(a_j)\,, \quad \sigma_{j+\frac{1}{2}} = \mathrm{sign}(a_{j+\frac{1}{2}})\,.$$

Finally concerning the time discretization, we consider an increasing sequence $t_0 = 0 < t_1 < \ldots < t_n < \ldots \le T$ and set $\Delta t_n = t_{n+1} - t_n$.

## 3. A linear standard upwind finite volume method

The underlying philosophy of the finite volume scheme is to approximate on each control volume $K_j$ in $\mathcal{T}^h$ the mean value of the exact solution

$$u_j^n \approx \frac{1}{\Delta x_j} \int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}} u(x, t_n)\, dx\,, \tag{4}$$

by taking into account the direction where the information comes from. For explicit upwind scheme, the sequence $u^n = (u_j^n)_{j=1}^J$ satisfies for $j = 1$ to $J$

$$\frac{u_j^{n+1} - u_j^n}{\Delta t_n} + \frac{\Phi_{j+\frac{1}{2}}(u^n) - \Phi_{j-\frac{1}{2}}(u^n)}{\Delta x_j} = 0 \tag{5}$$

where the numerical flux $\Phi_{j+\frac{1}{2}}(u^n)$ approximates $f(x_{j+\frac{1}{2}}, t_n)$. A first standard way consists in evaluating the function $a$ at the point $x_{j+\frac{1}{2}}$, so that the numerical flux reads as follows for $j = 0$ to $J$

$$\begin{aligned} \Phi_{j+\frac{1}{2}}(u^n) &= \frac{a_{j+\frac{1}{2}}(u_j^n + u_{j+1}^n)}{2} - \sigma_{j+\frac{1}{2}} \frac{a_{j+\frac{1}{2}}(u_{j+1}^n - u_j^n)}{2} \\ &= a_{j+\frac{1}{2}}^+ u_j^n - a_{j+\frac{1}{2}}^- u_{j+1}^n \end{aligned} \tag{6}$$

where by convention $u_0^n = \psi_\alpha(t_n)$ and $u_{J+1}^n = \psi_\beta(t_n)$. Our goal is now to establish the following result.

**Theorem 1.** *The explicit linear standard upwind finite volume scheme (5)-(6) applied to the system* (1) *is first order accurate.*

### *3.1. Stability*

The classical way to check if the goal is achieved is to evaluate the truncation error which consists in replacing $u_j^n$ in the system (5) by the the value $u(x_j, t_n)$ of the exact solution at the centroid of the control volumes. Let us compute for $j = 1$ to $J$

$$\epsilon_j^n = \frac{u(x_j, t_{n+1}) - u(x_j, t_n)}{\Delta t_n} + \frac{1}{\Delta x_j}\left(\Phi_{j+\frac{1}{2}}(U^n) - \Phi_{j-\frac{1}{2}}(U^n)\right) \tag{7}$$

where $U^n = (u(x_j, t_n))_{j=1}^J$. Then one gets that the global error sequence denoted by $e^n = (e_j^n)_{j=1}^J$ with $e_j^n = u_j^n - u(x_j, t_n)$ satisfies for $j = 1$ to $J$

$$e_j^{n+1} = (\mathcal{L}^n e^n)_j - \Delta t_n \epsilon_j^n \,, \tag{8}$$

where the operator $\mathcal{L}^n$ acts on sequences $\xi = (\xi_j)_{j=1}^J$ and is defined by

$$(\mathcal{L}^n \xi)_j = \xi_j - \frac{\Delta t_n}{\Delta x_j}\left(-a^-_{j+\frac{1}{2}}\xi_{j+1} + (a^+_{j+\frac{1}{2}} + a^-_{j-\frac{1}{2}})\xi_j - a^+_{j-\frac{1}{2}}\xi_{j-1}\right) . \tag{9}$$

In classical finite differences theory, one transfers information on the smallness of the truncation error $\epsilon_j^n$ to the error $e_j^n$ via a stability property of the scheme. It amounts here to show that the norm of $\mathcal{L}^n$ is not greater than $1 + c\Delta t_n$ (a difference with constant $a$), once a C.F.L. number has been introduced as a limitation on the time step $\Delta t_n$. More precisely, we have the classical result

**Proposition 1.** *Under the C.F.L. condition*

$$(a^+_{j+\frac{1}{2}} + a^-_{j-\frac{1}{2}})\frac{\Delta t_n}{\Delta x_j} \le 1 \quad \textit{for} \quad j = 1, \ldots, J \tag{10}$$

*the operator $\mathcal{L}^n$ satisfies for every $p \in [1, +\infty]$ :*

$$||\mathcal{L}^n \xi||_p \le (1 + \Delta t_n ||a'||_\infty)||\xi||_p \,. \tag{11}$$

*Proof.* First let us prove the inequality for $p = \infty$. For $2 \le j \le J - 1$,

$$(\mathcal{L}^n \xi)_j = \left(1 - \frac{\Delta t_n}{\Delta x_j}(a^+_{j+\frac{1}{2}} + a^-_{j-\frac{1}{2}})\right)\xi_j + \frac{\Delta t_n}{\Delta x_j}a^-_{j+\frac{1}{2}}\xi_{j+1} + \frac{\Delta t_n}{\Delta x_j}a^+_{j-\frac{1}{2}}\xi_{j-1} \,,$$

where under the C.F.L. condition, all the terms of the r.h.s are positive. We can estimate

$$\begin{aligned} |(\mathcal{L}^n \xi)_j| &\le \left(1 + \frac{\Delta t_n}{\Delta x_j}(a_{j-\frac{1}{2}} - a_{j+\frac{1}{2}})\right)||\xi||_\infty \\ &\le (1 - \Delta t_n a'(\theta_j))\,||\xi||_\infty \quad \text{for some } \theta_j \in ]x_{j-\frac{1}{2}}, x_{j+\frac{1}{2}}[\,. \end{aligned}$$

For $j = 1$ (the case $j = J$ is identical), since in a similar way

$$\begin{aligned} |(\mathcal{L}^n \xi)_1| &\le \left(1 + \frac{\Delta t_n}{\Delta x_1}(a_{\frac{1}{2}} - a_{\frac{3}{2}})\right)||\xi||_\infty - \frac{\Delta t_n}{\Delta x_1}a^+_{\frac{1}{2}}||\xi||_\infty \\ &\le (1 - \Delta t_n a'(\theta_1))\,||\xi||_\infty \quad \text{for some } \theta_1 \in ]x_{\frac{1}{2}}, x_{\frac{3}{2}}[ \end{aligned}$$

we can infer the estimate (11) for $p = \infty$.

For $p = 1$, under the C.F.L. condition, the estimation comes from

$$\begin{aligned}
\sum_{j=1}^{J} \Delta x_j |(\mathcal{L}^n \xi)_j| \quad &\leq \quad \sum_{j=1}^{J} \Delta x_j |\xi_j| - \Delta t_n \left( \sum_{j=1}^{J} (a^+_{j+\frac{1}{2}} + a^-_{j-\frac{1}{2}}) |\xi_j| \right. \\
&\qquad \left. - \sum_{j=1}^{J-1} a^-_{j+\frac{1}{2}} |\xi_{j+1}| - \sum_{j=2}^{J} a^+_{j-\frac{1}{2}} |\xi_{j-1}| \right) \\
&\leq \quad \sum_{j=1}^{J} \Delta x_j |\xi_j| - \Delta t_n \left( a^-_{\frac{1}{2}} |\xi_1| + a^+_{J+\frac{1}{2}} |\xi_J| \right) \;\leq\; \sum_{j=1}^{J} \Delta x_j |\xi_j| \,.
\end{aligned}$$

□

This result, combined with (8), has the following straightforward corollary:

**Corollary 1.** *Under the C.F.L. condition* (10) *and for every* $p \in [1, +\infty]$ *we have the estimate:*

$$||e^n||_p \leq \exp(||a'||_\infty \, t_n) \left( ||e^0||_p + \sum_{i=0}^{n-1} \Delta t_i ||\epsilon^i||_p \right) . \tag{12}$$

*3.2. On the truncation error*

For the volume $j$, let us write the local error as $\epsilon^n_j = G^n_j + I^n_j$ where $G^n_j$ represents the centered part of the scheme

$$\begin{aligned}
G^n_j \quad &= \quad \frac{u(x_j, t_{n+1}) - u(x_j, t_n)}{\Delta t_n} + \frac{f(x_{j+\frac{1}{2}}, t_n) - f(x_{j-\frac{1}{2}}, t_n)}{\Delta x_j} \\
&= \quad \frac{\partial u(x_j, t_n)}{\partial t} + \mathcal{O}(\Delta t) + \frac{\partial f(x_j, t_n)}{\partial x} + \mathcal{O}(h) \quad = \quad \mathcal{O}(\Delta t) + \mathcal{O}(h)
\end{aligned}$$

and where $I^n_j$ represents the difference between the upwind part and the centered part of the scheme ($U^n$ is the sequence $u(x_j, t_n)$)

$$I^n_j \quad = \quad \frac{\Phi_{j+\frac{1}{2}}(U^n) - \Phi_{j-\frac{1}{2}}(U^n) + f(x_{j-\frac{1}{2}}, t_n) - f(x_{j+\frac{1}{2}}, t_n)}{\Delta x_j} \,. \tag{13}$$

An intensive use of Taylor's expansion leads to $I^n_j = \mathcal{O}(1)$ so that $\epsilon^n_j$ does not converge to zero as $h$ goes to zero: in the finite differences sense the scheme is not consistent.

Our goal is now to construct a sequence $\gamma^n = (\gamma^n_j)_{j=1}^J$ in order to correct, in the mathematical analysis, the global error sequence $e^n$ and then to prove that under smoothness assumptions the scheme is however first order accurate. Indeed we have the following result

**Proposition 2.** *Assume that there exists a sequence* $\gamma^n = (\gamma^n_j)_{j=1}^J$ *such*

$$\|\gamma^n\|_p = \mathcal{O}(h) \tag{14}$$

*and such that the corrected error sequence defined by* $\underline{e}^n = (\underline{e}^n_j)_{j=1}^J$ *and*

$$\underline{e}^n_j = e^n_j + \gamma^n_j \tag{15}$$

*satisfies*

$$\underline{e}^{n+1} = (\mathcal{L}^n \underline{e}^n) + \Delta t_n \underline{\epsilon}^n \quad \textit{with} \quad \|\underline{\epsilon}^n\|_p = \mathcal{O}(h) \tag{16}$$

*then under the C.F.L. condition and if the initial error* $\|e^0\|_p = \mathcal{O}(h)$*, the explicit finite volume scheme is a first order convergent scheme i.e.*

$$\|e^n\|_p = \mathcal{O}(h) \quad \forall t_n \leq T\,. \tag{17}$$

*Proof.* This is a simple consequence of the stability. The inequality

$$||\underline{e}^n||_p \leq \exp(||a'||_\infty\, t_n) \left( ||\underline{e}^0||_p + \sum_{i=0}^{n-1} \Delta t_i ||\underline{\epsilon}^i||_p \right)$$

implies that the corrected error $\|\underline{e}^n\|_p$ is $\mathcal{O}(h)$. The triangular inequality and assumption on $\gamma^n$ finish the proof. □

*3.3. Introduction of a point where the error is evaluated*

We now build a such sequence $\gamma^n$. During the proof we shall assume that the zeros of the function $a$ are isolated. This simplification leaves aside some technicalities. We shall assume that $h$ is small enough, so that the function $a$ is equal to zero at most once every three consecutive volumes. To limit the number of cases, we also assume that zeros of the function $a$ are not located at the interface point $x_{j+\frac{1}{2}}$. From the definition (15) of $\underline{e}^n$ and from the link (8) between $e^n$ and $\epsilon^n$, the corrected errors readily satisfy

$$\underline{e}^n_{j+1} = (\mathcal{L}^n \underline{e}^n)_j + \gamma^{n+1}_j - (\mathcal{L}\gamma^n)_j - \Delta t_n \epsilon^n_j\,. \tag{18}$$

It can be put on the form (16) if we write

$$\underline{\epsilon}^n_j = \frac{\gamma^{n+1}_j - \gamma^n_j}{\Delta t_n} - G^n_j + \frac{Z^n_j}{\Delta x_j}\,. \tag{19}$$

Therefore if one looks for $\gamma^n$ in the form

$$\gamma^n_j = u(x_j, t_n) - w^n_j\,, \tag{20}$$

then one easily gets for $j = 1$ to $J$

$$\begin{aligned} Z^n_j \quad = \quad & a^+_{j+\frac{1}{2}} \big( u(x_{j+\frac{1}{2}}, t_n) - w^n_j \big) - a^-_{j+\frac{1}{2}} \big( u(x_{j+\frac{1}{2}}, t_n) - w^n_{j+1} \big) \\ + \quad & a^-_{j-\frac{1}{2}} \big( u(x_{j-\frac{1}{2}}, t_n) - w^n_j \big) - a^+_{j-\frac{1}{2}} \big( u(x_{j-\frac{1}{2}}, t_n) - w^n_{j-1} \big) \end{aligned} \tag{21}$$

with the convention $w^n_0 = u(x_{\frac{1}{2}}, t_n)$ and $w^n_{J+1} = u(x_{J+\frac{1}{2}}, t_n)$. We now discuss how to choose $w^n_j$ in order to get $Z^n_j = \mathcal{O}((\Delta x_j)^2)$. If the function $a$ does not change sign and remains for instance positive on $[\alpha, \beta]$, then with $w^n_j =$

$u(x_{j+\frac{1}{2}}, t_n)$ for all $j$, one easily gets $Z_j^n = 0$. Now, if $a$ remains negative, then with $w_j^n = u(x_{j-\frac{1}{2}}, t_n)$, one gets again $Z_j^n = 0$. From these observations, to take into account a change of sign of function $a$, let us introduce the point

$$z_j = \frac{1-\sigma_j}{2} x_{j-\frac{1}{2}} + \frac{1+\sigma_j}{2} x_{j+\frac{1}{2}} = x_j + \sigma_j \frac{\Delta x_j}{2} \quad \in [x_{j-\frac{1}{2}}, x_{j+\frac{1}{2}}] \tag{22}$$

and let us take $w_j^n = u(z_j, t_n)$ so that

$$\gamma_j^n = u(x_j, t_n) - u(z_j, t_n)\,. \tag{23}$$

Let us verify in order to conclude the proof, that for a such choice, $Z_j^n$ behaves like $\Delta x_j^2$. Applying several Taylor's expansions and using the local quasi-uniformity of the mesh (3) lead, for $j = 1$ to $J$ and with the convention $\Delta x_0 = \Delta x_{J+1} = 0$, to

$$\begin{aligned} Z_j^n = \Bigg( &\big(a^+_{j+\frac{1}{2}} \frac{1-\sigma_j}{2} - a^-_{j-\frac{1}{2}} \frac{1+\sigma_j}{2}\big)\Delta x_j + a^-_{j+\frac{1}{2}} \frac{1+\sigma_{j+1}}{2} \Delta x_{j+1} \\ &- a^+_{j-\frac{1}{2}} \frac{1-\sigma_{j-1}}{2} \Delta x_{j-1} \Bigg) \frac{\partial u(x_j, t_n)}{\partial x} + \mathcal{O}((\Delta x_j)^2)\,. \end{aligned} \tag{24}$$

We first study the coefficient $A \equiv a^+_{j+\frac{1}{2}} \frac{1-\sigma_j}{2} - a^-_{j-\frac{1}{2}} \frac{1+\sigma_j}{2}$ in factor of $\Delta x_j$. Several cases are to be considered:

- if $\sigma_j = 1$ and $a^-_{j-\frac{1}{2}} = 0$ then $A = 0$,
- if $\sigma_j = 1$ and $a^-_{j-\frac{1}{2}} = -a_{j-\frac{1}{2}} > 0$ then $a$ change of sign in $\zeta \in ]x_{j-\frac{1}{2}}, x_j[$ and from a Taylor expansion, it follows that $A \equiv a_{j-\frac{1}{2}} = \mathcal{O}(\Delta x_j)$,
- if $\sigma_j = -1$ and $a^+_{j+\frac{1}{2}} = 0$ then $A = 0$,
- if $\sigma_j = -1$ and $a^+_{j+\frac{1}{2}} = a_{j+\frac{1}{2}} > 0$ then $a$ changes of sign in $\zeta \in ]x_j, x_{j+\frac{1}{2}}[$ and from a Taylor expansion, it follows that $A \equiv -a_{j+\frac{1}{2}} = \mathcal{O}(\Delta x_j)$,
- if $\sigma_j = 0$ then $A \equiv a^+_{j+\frac{1}{2}} - a^-_{j-\frac{1}{2}} = \mathcal{O}(\Delta x_j)$ since $a(x_j) = 0$.

For the study of the coefficient $B \equiv a^-_{j+\frac{1}{2}} \frac{1+\sigma_{j+1}}{2}$ in factor of $\Delta x_{j+1}$ (a similar argument applies to the coefficient in factor of $\Delta x_{j-1}$), let us observe that

- if $\sigma_{j+1} = -1$ then $B = 0$
- if $\sigma_{j+1} = 1$ and $a^-_{j+\frac{1}{2}} = 0$ then $B = 0$
- if $\sigma_{j+1} = 1$ and $a^-_{j+\frac{1}{2}} = -a_{j+\frac{1}{2}} > 0$ then $B = \mathcal{O}(\Delta x_{j+1})$ since $a$ changes of sign
- if $\sigma_{j+1} = 0$ and $a^-_{j+\frac{1}{2}} = 0$ then $B = 0$

- if $\sigma_{j+1}=0$ and $a^-_{j+\frac12}=-a_{j+\frac12}>0$ then $B=\mathcal{O}(\Delta x_{j+1})$ since $a(x_{j+1})=0$.

Then for $\gamma_j^n = u(x_j,t_n)-u(z_j,t_n)$ with $z_j$ given by (22), we proved that $Z_j^n=\mathcal{O}((\Delta x_j)^2)$. This implies Theorem 1 since clearly $\frac{\gamma_j^{n+1}-\gamma_j^n}{\Delta t_n}=\mathcal{O}(\Delta x_j)$.

*3.4. Interpretation in term of geometric corrector*

The above result can be understood by using the notion of geometric corrector. Indeed, a Taylor expansion applied to (23) leads to

$$\gamma_j^n = -\sigma_j \frac{\Delta x_j}{2}\frac{\partial u(x_j,t_n)}{\partial x} + \mathcal{O}((\Delta x_j)^2)\,, \tag{25}$$

then one might consider a geometric corrector in the sense of Bouche *et al* [2] given by

$$\Gamma_j = \sigma_j \frac{\Delta x_j}{2}\,, \tag{26}$$

and which satisfies to $\mathcal{O}((\Delta x_j)^2)$ the following system of equations:

$$a^+_{j+\frac12}(\frac{\Delta x_j}{2}-\Gamma_j)-a^-_{j-\frac12}(\frac{\Delta x_j}{2}+\Gamma_j)+a^-_{j+\frac12}(\frac{\Delta x_{j+1}}{2}+\Gamma_{j+1})-a^+_{j-\frac12}(\frac{\Delta x_{j-1}}{2}-\Gamma_{j-1})=0 \tag{27}$$

where by convention

$$\Delta x_0=\Gamma_0=\Delta x_{J+1}=\Gamma_{J+1}=0\,.$$

Indeed one gets with $C_j$ equal to the left hand side of (27)

$$Z_j^n = C_j\frac{\partial u(x_j,t_n)}{\partial x}+\mathcal{O}(\Delta x_j)\mathcal{O}(||\Gamma||)+\mathcal{O}((\Delta x_j)^2)\,. \tag{28}$$

## 4. A flux finite volume method

We now consider a flux scheme (see [13]) where the sequence $u^n=(u_j^n)_{j=1}^J$ is solution to the following system:

$$\frac{u_j^{n+1}-u_j^n}{\Delta t_n}+\frac{\Psi_{j+\frac12}(u^n)-\Psi_{j-\frac12}(u^n)}{\Delta x_j}=0 \quad \text{for} \quad j=1,\ldots,J\,. \tag{29}$$

If, by convention, we write $u_0^n=\psi_\alpha(t_n)$ and $u_{J+1}^n=\psi_\beta(t_n)$, the numerical flux is given, in an equivalent way, by

$$\begin{aligned}\Psi_{j+\frac12}(u^n) &= \frac{a_ju_j^n+a_{j+1}u_{j+1}^n}{2}-\sigma_{j+\frac12}\frac{a_{j+1}u_{j+1}^n-a_ju_j^n}{2}\\ &= \frac{\sigma_{j+\frac12}+1}{2}a_ju_j^n-\frac{\sigma_{j+\frac12}-1}{2}a_{j+1}u_{j+1}^n\end{aligned} \tag{30}$$

A simple rearrangement of the terms leads to the following expression

$$\frac{u_j^{n+1}-u_j^n}{\Delta t_n}+\frac{1}{\Delta x_j}\left(\frac{1-\sigma_{j+\frac{1}{2}}}{2}(\varphi_{j+1}^n-\varphi_j^n)+\frac{1+\sigma_{j-\frac{1}{2}}}{2}(\varphi_j^n-\varphi_{j-1}^n)\right)=0 \quad (31)$$

where $\varphi_j^n = a_j u_j^n$ for $j=1$ to $J$ and $\varphi_0^n = a_{\frac{1}{2}}\psi_\alpha(t_n)$ and $\varphi_{J+1}^n = a_{J+\frac{1}{2}}\psi_\beta(t_n)$. Let us observe that by multiplying (31) by $a_j$, the scheme can be written in term of the fluxes and can be interpreted as a finite volume discretization of the convection equation (2) on the flux i.e. the sequence $\varphi^n = (\varphi_j^n)_{j=1}^J$ satisfies

$$\frac{\varphi_j^{n+1}-\varphi_j^n}{\Delta t_n}+\frac{a_j}{\Delta x_j}\left(\Psi_{j+\frac{1}{2}}(\varphi^n)-\Psi_{j-\frac{1}{2}}(\varphi^n)\right)=0 \quad \text{for} \quad j=1,\dots,J \quad (32)$$

where now the numerical flux for $j=0$ to $J$ is given by

$$\begin{aligned}\Psi_{j+\frac{1}{2}}(\varphi^n) &= \frac{\varphi_j^n+\varphi_{j+1}^n}{2}-\sigma_{j+\frac{1}{2}}\frac{\varphi_{j+1}^n-\varphi_j^n}{2}\\ &= \frac{\sigma_{j+\frac{1}{2}}+1}{2}\varphi_j^n-\frac{\sigma_{j+\frac{1}{2}}-1}{2}\varphi_{j+1}^n\,.\end{aligned} \quad (33)$$

Our goal in the forthcoming subsections is to establish the following result.

**Theorem 2.** *The explicit flux finite volume scheme (29)-(30) applied to the system* (1) *is first order accurate in the sense defined in Proposition 4 below.*

*4.1. Stability*

Let us evaluate the truncation error by substituting the exact solution in (32), namely by replacing $\varphi_j^n$ by the exact flux function $f(x_j,t_n)=a_j u(x_j,t_n)$ for $j=1$ to $J$. We denote $F^n=(f(x_j,t_n))_{j=1}^J$ and we compute

$$\epsilon_j^n=\frac{f(x_j,t_{n+1})-f(x_j,t_n)}{\Delta t_n}+\frac{a_j}{\Delta x_j}\left(\Psi_{j+\frac{1}{2}}(F^n)-\Psi_{j-\frac{1}{2}}(F^n)\right) \quad (34)$$

where by convention $f(x_0,t_n)=a_{\frac{1}{2}}\psi_\alpha(t_n)$ and $f(x_{J+1},t_n)=a_{J+\frac{1}{2}}\psi_\beta(t_n)$. Then from (32) and (34), the global error on the flux denoted by $e_j^n=\varphi_j^n-f(x_j,t_n)$ satisfies the following formula

$$e_j^{n+1}=(\mathcal{L}_F^n e^n)_j-\Delta t_n\epsilon_j^n\,, \quad \text{for} \quad j=1,\dots,J \quad (35)$$

where $e^n=(e_j^n)_{j=1}^J$ and where $\mathcal{L}_F^n$ acts on sequences $\xi=(\xi_j)_{j=1}^J$ such

$$(\mathcal{L}_F^n\xi)_j=\xi_j-\frac{\Delta t_n}{\Delta x_j}a_j\left(\frac{\sigma_{j+\frac{1}{2}}+\sigma_{j-\frac{1}{2}}}{2}\xi_j^n-\frac{\sigma_{j+\frac{1}{2}}-1}{2}\xi_{j+1}^n-\frac{\sigma_{j-\frac{1}{2}}+1}{2}\xi_{j-1}^n\right)\,.$$

Here again, one transfers information on the smallness of the truncation error $\epsilon_j^n$ to the error $e_j^n$ via a stability property of the scheme. It amounts here to show the following result:

**Proposition 3.** *Under the C.F.L. condition*

$$a_j \frac{\sigma_{j+\frac{1}{2}} + \sigma_{j-\frac{1}{2}}}{2} \frac{\Delta t_n}{\Delta x_j} \leq 1 \quad \textit{for} \quad j = 1, \dots, J \tag{36}$$

*the operator* $\mathcal{L}_F^n$ *satisfies for every* $p \in [1, +\infty]$ *:*

$$||\mathcal{L}_F^n \xi||_p \leq (1 + (\kappa + 1)\Delta t_n ||a'||_\infty)||\xi||_p \,. \tag{37}$$

*where* $\kappa$ *is the constant in* (3).

*Proof.* Without loss of generality, $h$ is taken small enough in order to get the same assumptions as in Section 3.3. Hence $\sigma_{j+\frac{1}{2}} + \sigma_{j-\frac{1}{2}}$ can only be equal to 2, $-2$ or 0. Now if $\sigma_{j+\frac{1}{2}} + \sigma_{j-\frac{1}{2}} = 2$ then $a_j \geq 0$ because $a$ cancels at most once in a cell and if $\sigma_{j+\frac{1}{2}} + \sigma_{j-\frac{1}{2}} = -2$ then $a_j \leq 0$. We then infer that

$$0 \leq a_j \frac{\sigma_{j+\frac{1}{2}} + \sigma_{j-\frac{1}{2}}}{2} \,.$$

We shall first prove the inequality for $p = \infty$. For $2 \leq j \leq J - 1$, one gets

$$|(\mathcal{L}_F^n \xi)_j| \leq \left(1 - \frac{\Delta t_n}{\Delta x_j} a_j \frac{\sigma_{j+\frac{1}{2}} + \sigma_{j-\frac{1}{2}}}{2}\right) ||\xi||_\infty + \frac{\Delta t_n}{\Delta x_j} |a_j| \frac{2 - \sigma_{j+\frac{1}{2}} + \sigma_{j-\frac{1}{2}}}{2} ||\xi||_\infty \,.$$

Several cases are to be considered:

- If $\sigma_{j+\frac{1}{2}} + \sigma_{j-\frac{1}{2}} = 2$ then $-\sigma_{j+\frac{1}{2}} + \sigma_{j-\frac{1}{2}} = 0$, $a_j \geq 0$ and $|(\mathcal{L}_F^n \xi)_j| \leq ||\xi||_\infty$.
- If $\sigma_{j+\frac{1}{2}} + \sigma_{j-\frac{1}{2}} = -2$ then $-\sigma_{j+\frac{1}{2}} + \sigma_{j-\frac{1}{2}} = 0$, $a_j \leq 0$ and $|(\mathcal{L}_F^n \xi)_j| \leq ||\xi||_\infty$.
- If $\sigma_{j+\frac{1}{2}} = -\sigma_{j-\frac{1}{2}} = -1$ then $|(\mathcal{L}_F^n \xi)_j| \leq ||\xi||_\infty$.
- If $\sigma_{j+\frac{1}{2}} = -\sigma_{j-\frac{1}{2}} = 1$ then $|(\mathcal{L}_F^n \xi)_j| \leq \left(1 + \frac{\Delta t_n}{\Delta x_j} 2|a_j|\right) ||\xi||_\infty$ and from $\zeta \in ]x_{j-\frac{1}{2}}, x_{j+\frac{1}{2}}[$ zero of $a$, one gets $|(\mathcal{L}_F^n \xi)_j| \leq (1 + \Delta t_n ||a'||_\infty)||\xi||_\infty$.

For $j = 1$ (the treatment of $j = J$ is identical), one gets

$$|(\mathcal{L}_F^n \xi)_1| \quad \leq \quad \left(1 - \frac{\Delta t_n}{\Delta x_1} a_1 \frac{\sigma_{\frac{3}{2}} + \sigma_{\frac{1}{2}}}{2}\right) ||\xi||_\infty + \frac{\Delta t_n}{\Delta x_1} |a_1| \frac{1 - \sigma_{\frac{3}{2}}}{2} ||\xi||_\infty \tag{38}$$

and a similar discussion leads to $|(\mathcal{L}_F^n \xi)_1| \leq \left(1 + \Delta t_n \frac{||a'||_\infty}{2}\right) ||\xi||_\infty$.

Now, we consider $p = 1$. Under the C.F.L. condition, a simple rearrangement gives with $a_0 = a_{J+1} = 0$ by convention

$$\begin{aligned} \sum_{j=1}^{J} \Delta x_j (\mathcal{L}_F^n \xi)_j \quad & \leq \sum_{j=1}^{J} \Delta x_j |\xi_j| + \Delta t_n \left( \sum_{j=1}^{J} (|a_{j-1}| + a_j) \frac{1 - \sigma_{j-\frac{1}{2}}}{2} |\xi_j| \right. \\ & \qquad \left. + \sum_{j=1}^{J} (|a_{j+1}| - a_j) \frac{1 + \sigma_{j+\frac{1}{2}}}{2} |\xi_j| \right) \end{aligned}$$

In the extra term with $\Delta t_n$ in factor, let us consider the first sum, the second one is treated in a similar way. For $j = 2$ to $J$ and for $\sigma_{j-\frac{1}{2}} = -1$

- if $a_{j-1} \leq 0$ then the contribution of the volume $j$ is equal to $(a_j - a_{j-1})|\xi_j|$ and is bounded by $||a'||_\infty \frac{\kappa+1}{2}\Delta x_j |\xi_j|$ thanks to a Taylor formula,

- if $a_{j-1} \geq 0$ then the contribution is $(a_j + a_{j-1})|\xi_j|$ and is by $||a'||_\infty \frac{\kappa+1}{2}\Delta x_j |\xi_j|$ thanks to a Taylor formula applied in $\zeta \in ]x_{j-1}, x_{j-\frac{1}{2}}[$ zero of $a$.

For $j = 1$ and $\sigma_{\frac{1}{2}} = -1$, if $a_1 \leq 0$ then the contribution is negative and if $a_1 \geq 0$ then it is bounded by $||a'||_\infty \frac{\Delta_1}{2}|\xi_1|$ since there is $\zeta \in ]x_{\frac{1}{2}}, x_1[$ where $a(\zeta) = 0$. This infers the estimation (37) for $p = 1$ and finishes the proof. $\square$

This result, combined with (35), has the following straightforward corollary.

**Corollary 2.** *Under the C.F.L. condition* (36) *and for every* $p \in [1, +\infty]$ *we have the estimate:*

$$||e^n||_p \leq \exp((\kappa+1)||a'||_\infty\, t_n) \left( ||e^0||_p + \sum_{i=0}^{n-1} \Delta t_i ||\epsilon^i||_p \right). \qquad (39)$$

*4.2. On the truncation error*

For the volume $j$, let us write the local error as $\epsilon_j^n = G_j^n + a_j I_j^n$ where $G_j^n$ represents the centered part of the scheme

$$\begin{aligned} G_j^n &= \frac{f(x_j, t_{n+1}) - f(x_j, t_n)}{\Delta t_n} + a_j \frac{f(x_{j+\frac{1}{2}}, t_n) - f(x_{j-\frac{1}{2}}, t_n)}{\Delta x_j} \\ &= \frac{\partial f(x_j, t_n)}{\partial t} + \mathcal{O}(\Delta t) + a(x_j)\frac{\partial f(x_j, t_n)}{\partial x} + \mathcal{O}(h) \quad = \quad \mathcal{O}(\Delta t) + \mathcal{O}(h) \end{aligned}$$

and where $I_j^n$ represents the difference between the upwind and the centered part

$$I_j^n \quad = \quad \frac{\tilde{\Psi}_{j+\frac{1}{2}}(F^n) - \tilde{\Psi}_{j-\frac{1}{2}}(F^n) + f(x_{j-\frac{1}{2}}, t_n) - f(x_{j+\frac{1}{2}}, t_n)}{\Delta x_j}.$$

Here again $I_j^n = \mathcal{O}(1)$. So like in the previous scheme, our goal is to construct a sequence $\gamma^n = (\gamma_j^n)_{j=1}^J$ in order to correct, in the mathematical analysis, the global errors and then to prove that this finite volume variant is first order accurate. Indeed we have the following Proposition. The proof is similar to that of Proposition 2 except that the norm concerns the error on the fluxes.

**Proposition 4.** *Assume that there exists a sequence* $\gamma^n = (\gamma_j^n)_{j=1}^J$ *such*

$$\|\gamma^n\|_p = \mathcal{O}(h) \qquad (40)$$

*and such that the corrected error sequence defined by* $\underline{e}^n = (\underline{e}_j^n)_{j=1}^J$ *and*

$$\underline{e}_j^n = e_j^n + \gamma_j^n \qquad (41)$$

*satisfies*

$$\underline{e}^{n+1} = (\mathcal{L}_F^n \underline{e}^n) + \Delta t_n \underline{\epsilon}^n \quad \textit{with} \quad \|\underline{\epsilon}^n\|_p = \mathcal{O}(h) \qquad (42)$$

*then under the C.F.L. condition and if the initial error* $\|e^0\|_p = \mathcal{O}(h)$*, the explicit finite volume scheme is a first order convergent scheme in the following sense*

$$\|(a_j u_j^n - a_j u(x_j,t_n))_{j=1}^J\|_p = \mathcal{O}(h) \quad \forall t_n \le T\,. \tag{43}$$

*4.3. Introduction of a point where the error is estimated*

The corrected errors readily satisfy

$$\underline{e}_{j+1}^n \quad = \quad (\mathcal{L}_F \underline{e}^n)_j + \Delta t_n \left( \frac{\gamma_j^{n+1} - \gamma_j^n}{\Delta t_n} - G_j^n + a_j \frac{Z_j^n}{\Delta x_j} \right) \tag{44}$$

We again look for $\gamma^n$ in the following form

$$\gamma_j^n = f(x_j,t_n) - w_j^n\,. \tag{45}$$

Then one easily gets for $j = 1$ to $J$

$$\begin{aligned} Z_j^n \quad &= \quad \frac{\sigma_{j+\frac12}+1}{2}\big(f(x_{j+\frac12},t_n) - w_j^n\big) - \frac{\sigma_{j+\frac12}-1}{2}\big(f(x_{j+\frac12},t_n) - w_{j+1}^n\big) \\ &+ \quad \frac{\sigma_{j-\frac12}-1}{2}\big(f(x_{j-\frac12},t_n) - w_j^n\big) - \frac{\sigma_{j-\frac12}+1}{2}\big(f(x_{j-\frac12},t_n) - w_{j-1}^n\big) \end{aligned}$$

with the convention $w_0^n = f(x_{\frac12},t_n)$ and $w_{J+1}^n = f(x_{J+\frac12},t_n)$. We now discuss the choice of $w_j^n$ in order to have $a_j Z_j^n = \mathcal{O}((\Delta x_j)^2)$. If the function $a$ remains positive, then with $w_j^n = f(x_{j+\frac12},t_n)$, one easily gets $Z_j^n = 0$. Now if $a$ is negative, then with $w_j^n = f(x_{j-\frac12},t_n)$, one gets again $Z_j^n = 0$. From these observations, let us write with $z_j$ defined by (22)

$$w_j^n = f(z_j,t_n)\,. \tag{46}$$

Several Taylor's expansions and the local quasi-uniformity of the mesh yield to

$$a_j Z_j^n = a_j\left(A\Delta x_j + B\Delta x_{j+1} + C\Delta x_{j-1}\right)\frac{\partial f(x_j,t_n)}{\partial x} + \mathcal{O}((\Delta x_j)^2) \tag{47}$$

with

$$\begin{aligned} A &= \frac{\sigma_{j+\frac12}+1}{2}\frac{1-\sigma_j}{2} - \frac{\sigma_{j-\frac12}-1}{2}\frac{1+\sigma_j}{2} \\ B &= \frac{\sigma_{j+\frac12}-1}{2}\frac{1+\sigma_{j+1}}{2} \qquad\qquad C = -\frac{\sigma_{j-\frac12}+1}{2}\frac{1-\sigma_{j-1}}{2} \end{aligned}$$

For $a_j A$, several cases are to be considered

- if $\sigma_j = 0$ then $a_j A = 0$.
- if $\sigma_j = 1$ and $\sigma_{j-\frac12} = 1$ then $a_j A = 0$
- if $\sigma_j = 1$ and $\sigma_{j-\frac12} = -1$ then $A = 1$ but $a$ changes of sign in $]x_{j-\frac12}, x_j[$ and then $a_j A = \mathcal{O}(\Delta x_j)$

- if $\sigma_j = -1$ and $\sigma_{j+\frac{1}{2}} = -1$ then $a_j A = 0$
- if $\sigma_j = -1$ and $\sigma_{j+\frac{1}{2}} = 1$ then $a$ changes of sign in $]x_j, x_{j+\frac{1}{2}}[$ and then $a_j A = \mathcal{O}(\Delta x_j)$ .

Concerning $B$, (a similar argument applies to C)

- if $\sigma_{j+1} = -1$ then $a_j B = 0$
- if $\sigma_{j+1} = 1$ and $\sigma_{j+\frac{1}{2}} = 1$ then $a_j B = 0$
- if $\sigma_{j+1} = 1$ and $\sigma_{j+\frac{1}{2}} = -1$ then $a_j B = \mathcal{O}(h)$ since $a$ changes of sign
- if $\sigma_{j+1} = 0$ and $\sigma_{j+\frac{1}{2}} = -1$ then $a_j B = 0$
- if $\sigma_{j+1} = 0$ and $\sigma_{j+\frac{1}{2}} = 1$ then $a_j B = \mathcal{O}(h)$ since $a(x_{j+1}) = 0$ .

This proves that $Z_j^n = \mathcal{O}((\Delta x_j)^2)$ and achieves the proof of Theorem 2.

*4.4. Interpretation in term of geometric corrector*

Let us observe that a Taylor expansion applied to the corrector defined in (45) leads to

$$\gamma_j^n = -\sigma_j \frac{\Delta x_j}{2} \frac{\partial f(x_j, t_n)}{\partial x} + \mathcal{O}(h^2)\,. \tag{48}$$

Therefore, one may consider a geometric corrector given by

$$\Gamma_j = \sigma_j \frac{\Delta x_j}{2}\,. \tag{49}$$

and which satisfies to $\mathcal{O}((\Delta x_j)^2)$ the system of equations

$$\begin{aligned} a_j \Bigg( &\frac{\sigma_{j+\frac{1}{2}} + 1}{2} \Big(\frac{\Delta x_j}{2} - \Gamma_j\Big) - \frac{\sigma_{j-\frac{1}{2}} - 1}{2} \Big(\frac{\Delta x_j}{2} + \Gamma_j\Big) \\ &+ \frac{\sigma_{j+\frac{1}{2}} - 1}{2} \Big(\frac{\Delta x_{j+1}}{2} + \Gamma_{j+1}\Big) - \frac{\sigma_{j-\frac{1}{2}} + 1}{2} \Big(\frac{\Delta x_{j-1}}{2} - \Gamma_{j-1}\Big) \Bigg) = 0 \end{aligned} \tag{50}$$

where by convention

$$\Delta x_0 = \Gamma_0 = \Delta x_{J+1} = \Gamma_{J+1} = 0\,.$$

Indeed one gets with $a_j D_j$ equal to the left hand side of (50)

$$a_j Z_j^n = a_j D_j \frac{\partial f(x_j, t_n)}{\partial x} + \mathcal{O}(\Delta x_j)\mathcal{O}(||\Gamma||) + \mathcal{O}((\Delta x_j)^2) \tag{51}$$

## 5. A discussion about the linear case

As a first step from linear to nonlinear one dimensional equation, we have addressed the case of a linear advection with variable coefficients in conservative form (1). In contrast with the constant coefficient case, there are (at least) two different ways to consider such a question. The first one (studied in Section 3), the usual one in the classical bibliography, consists in finding the numerical flux at the interface, $x = x_{j+\frac{1}{2}}$, by using the Riemann solver for the linear equation

$$\frac{\partial u(x,t)}{\partial t} + a_{j+\frac{1}{2}} \frac{\partial u(x,t)}{\partial x} = 0$$

and this leads to the standard linear upwind finite volume method. The second one (studied in Section 4), which is more in the spirit of the nonlinear case, consists in solving the same linear equation but this time for the flux $f(x,t) = a(x)u(x,t)$:

$$\frac{\partial f(x,t)}{\partial t} + a_{j+\frac{1}{2}} \frac{\partial f(x,t)}{\partial x} = 0\,.$$

These two schemes give the same kind of results except in the case of sonic points, namely points $x$ where $a(x)$ vanishes. In particular the error estimate on the conservative variable $u_j^n$ holds true for the second scheme in an adapted norm, given by formula (43).

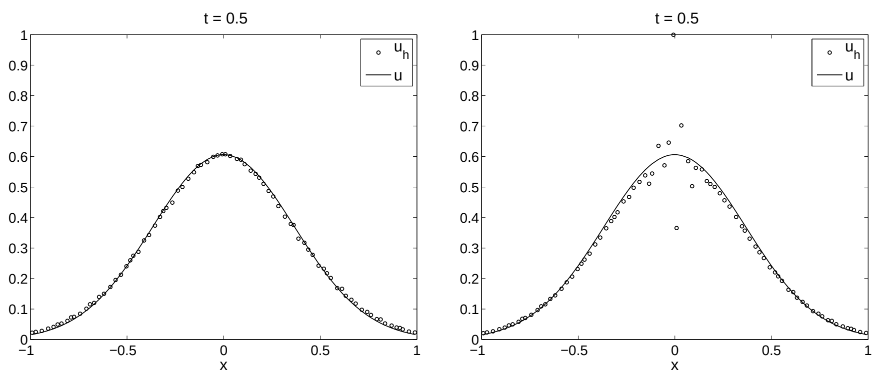


Figure 2: Approximated solution with $J = 80$ at $t = 0.5$ obtained with the standard finite volume scheme (left) and the flux scheme (right).

From a numerical point of view, we consider a problem with a sonic point by taking the non constant vector $a(x) = x$. And we examine the following solution of (1) on $[\alpha, \beta] = [-1, 1]$

$$u(x,t) = e^{-10x^2 e^{-2t} - t}\,.$$

Let us remark that since $a(-1) < 0$ and $a(1) > 0$, we don't need any boundary functions like $\psi_\alpha$ or $\psi_\beta$. The solution presents a sonic point at $x = 0$ and on

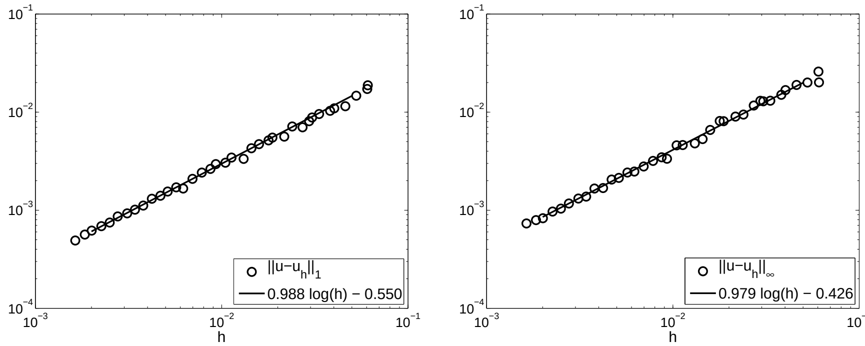


Figure 3: The $l^1$ norm error (left) and the $l^\infty$ norm error (right) versus $h$ for the approximated solution computed with the standard finite volume scheme

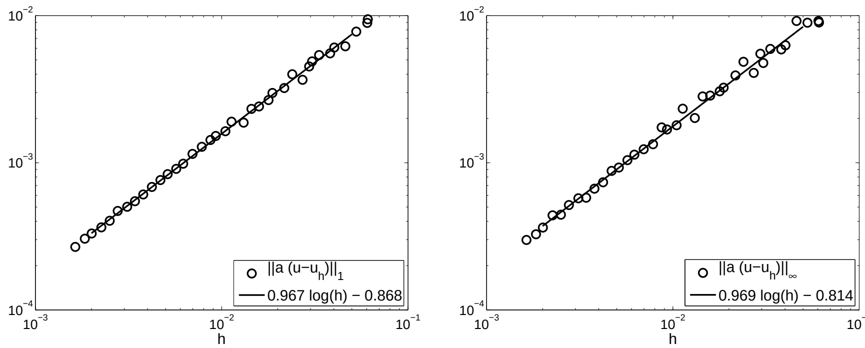


Figure 4: The adapted $l^1$ norm error (left) and the adapted $l^\infty$ norm error (right) versus $h$ for the approximated solution computed with the flux finite volume scheme

Figure 2 we observe that for a given $J$ and time $t$ the standard finite volume solution on a non uniform grid is similar to the exact solution. On the other hand, the flux finite volume scheme solution is far from the exact solution close to the sonic point and the usual error estimate given by (17) does not hold good. Nevertheless, as one can see in Figure 4, the error estimate in the adapted norm defined in (43) for the flux scheme does likewise the usual error estimate for the standard scheme presented in Figure 3: it behaves like a first order scheme.

## 6. The nonlinear case

In this part, we prove error estimates for the Murman-Roe finite volume scheme applied to the nonlinear scalar conservation law of the form

$$\begin{aligned}
\frac{\partial u(x,t)}{\partial t} + \frac{\partial f(u(x,t))}{\partial x} &= 0\,, \quad (x,t) \in ]\alpha,\beta[\times]0,T[\,, && (52)\\
u(x,0) &= u_0(x)\,, \quad x \in ]\alpha,\beta[\,, && (53)\\
u(\alpha,t) &= \psi_\alpha(t)\,, \quad \text{if } f'(u(\alpha,t)) > 0\,, && (54)\\
u(\beta,t) &= \psi_\beta(t)\,, \quad \text{if } f'(u(\beta,t)) < 0\,. && (55)
\end{aligned}$$

The data $u_0$, $\psi_\alpha$ and $\psi_\beta$ are smooth functions satisfying compatibility conditions. It is classical ([21]), using the method of characteristics, that this system of equations admits a unique smooth solution on a time interval $[0,T[$. The maximal time, $T$, is finite when two characteristics cross themselves, otherwise, $T = +\infty$. We shall use the following property of solutions to (52):

**Proposition 5.** *For smooth solutions to (52), the sign of $f'(u(x,t))$ is independent of $t$.*

*Proof.* The characteristic curve starting from $x_0$ at time $t_0$ is the solution to the differential equation $\frac{dx}{dt}(t) = f'(x(t),t)$ satisfying $x(t_0) = x_0$. If $f'(u(x_0,t_0))$ vanishes for a given $(x_0,t_0)$ then this curve which goes through $(x_0,t_0)$ is the vertical straight line $x(t) = x_0$. Hence the result. □

### 6.1. On the finite volume method

We consider the finite volume method of the form

$$\frac{u_j^{n+1} - u_j^n}{\Delta t_n} + \frac{\Phi^n_{j+\frac{1}{2}}(u^n) - \Phi^n_{j-\frac{1}{2}}(u^n)}{\Delta x_j} = 0 \quad \text{for} \quad j = 1,\ldots,J\,. \qquad (56)$$

The numerical flux $\Phi^n_{j+\frac{1}{2}}(u^n)$ is an approximation to the average flux along the interface $x = x_{j+\frac{1}{2}}$ and takes into account the direction where the information comes from. We will use the following formula for $j = 0$ to $J$

$$\Phi^n_{j+\frac{1}{2}}(u^n) = \frac{f(u^n_{j+1}) + f(u^n_j)}{2} - \sigma^n_{j+\frac{1}{2}} \frac{f(u^n_{j+1}) - f(u^n_j)}{2} \qquad (57)$$

where by convention $u^n_0 = \psi_\alpha(t_n)$ and $u^n_{J+1} = \psi_\beta(t_n)$. In the diffusion term,

$$\sigma^n_{j+\frac{1}{2}} = \operatorname{sign}(s^n_{j+\frac{1}{2}})$$

is the sign of the difference quotient defined by

$$s^n_{j+\frac{1}{2}} = \begin{cases} \dfrac{f(u^n_{j+1}) - f(u^n_j)}{u^n_{j+1} - u^n_j} & \text{if } u^n_{j+1} \neq u^n_j \\ f'(u^n_j) & \text{if } u^n_{j+1} = u^n_j \end{cases} \qquad (58)$$

Let us remark that solutions computed with this scheme fail to satisfy the entropy condition in case of transonic rarefaction wave. Since we are interested in the convergence analysis, we do not try to add any entropy fix (see for instance De Vuyst *et al* [9]).

*6.2. On the nature of this flux scheme*

Murman-Roe scheme (56)-(58) is a natural extension to the nonlinear case of the flux scheme (29)-(30) in the linear case. Of course such a scheme can not be straightforwardly extended to systems of equations since (58) involves the division by $u^n_{j+1} - u^n_j$. This deep subject has led in the early 1980's to the famous Roe's scheme [29] in the context of Euler's equation for perfect gas. This scheme has been extended to arbitrary hyperbolic system of conservation laws by Ghidaglia *et al* [14], [15]. One of the main features of these schemes is that the numerical flux (57) at an interface appears to be a *linear* combination of the left and right flux in the two neighboring cells. This property, named as "flux scheme" in [13], implies in particular that the scheme behaves very well with respect to genuine shocks (Rankine-Hugoniot type shocks).

*6.3. Stability*

Under the C.F.L. condition that reads

$$\left(\frac{\sigma^n_{j+\frac{1}{2}} - 1}{2} s^n_{j+\frac{1}{2}} + \frac{\sigma^n_{j-\frac{1}{2}} + 1}{2} s^n_{j-\frac{1}{2}}\right) \frac{\Delta t_n}{\Delta x_j} \leq 1 \tag{59}$$

and that ensures that the shocks with slope $s_{j-\frac{1}{2}}$ and $s_{j+\frac{1}{2}}$ does not intersect, the scheme defined by (56)-(58) is a T.V.D. and monotone scheme. It is a well known result (see Kröner [18], Barth and Ohlberger [1]) that the approximating solution is uniformly bounded by

$$\begin{aligned}\min\left\{\inf_{[\alpha,\beta]} u_0(x), \inf_{[0,T]} \psi_\alpha(t), \inf_{[0,T]} \psi_\beta(t)\right\} &\leq u^{n+1}_j \\ &\leq \max\left\{\sup_{[\alpha,\beta]} u_0(x), \sup_{[0,T]} \psi_\alpha(t), \sup_{[0,T]} \psi_\beta(t)\right\}\end{aligned} \tag{60}$$

and if $\Delta t_n$ and $h$ converge to zero such that the ratio $\frac{\Delta t_n}{h}$ remains bounded, then the approximating solution $u_h(x,t) = u^n_i$ for $t \in ]t_n, t_{n+1}]$ and $x \in ]x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}}]$ converges almost every where on $]\alpha, \beta[\times]0, T[$ to the solution $u$ of (52)-(53).

*6.4. On the truncation error*

A simple rearrangement using (58) leads to

$$\begin{aligned}u^{n+1}_j = u^n_j - \frac{\Delta t_n}{\Delta x_j}\Bigg(&\frac{1-\sigma^n_{j+\frac{1}{2}}}{2} s_{j+\frac{1}{2}}(u^n_{j+1} - u^n_j) \\ &+ \frac{1+\sigma^n_{j-\frac{1}{2}}}{2} s_{j-\frac{1}{2}}(u^n_j - u^n_{j-1})\Bigg).\end{aligned} \tag{61}$$

In order to evaluate the global error, we introduce a point that takes into account the direction where the information comes from

$$z_j = \frac{1-\delta_j}{2} x_{j-\frac{1}{2}} + \frac{1+\delta_j}{2} x_{j+\frac{1}{2}} = x_j + \delta_j \frac{\Delta x_j}{2} \quad \in [x_{j-\frac{1}{2}}, x_{j+\frac{1}{2}}]. \tag{62}$$

Here $\delta_j$ does not depend on time according to Proposition 5 and is equal to

$$\delta_j = \text{sign}(f'(u(x_j,\cdot)) = \text{sign}(f'(u_0(x_j)) \,.$$

As usual, we define the error in the finite differences sense by the difference between the cell-centered finite volume solution $u_j^n$ and the exact solution at $x_j$

$$e_j^n = u_j^n - u(x_j,t_n) \qquad j = 0,\cdots,J, \quad \forall t_n \leq T \tag{63}$$

and in order to establish an estimate, we introduce the following corrected error

$$\underline{e}_j^n = u_j^n - u(z_j,t_n) = e_j^n + \gamma_j^n \quad \text{ with } \quad \gamma_j^n = u(x_j,t_n) - u(z_j,t_n) \,. \tag{64}$$

A Taylor expansion leads to

$$\gamma_j^n = -\delta_j \frac{\Delta x_j}{2} \frac{\partial u(x_j,t_n)}{\partial x} + \mathcal{O}((\Delta x_j)^2) \,. \tag{65}$$

Then it is sufficient to estimate the corrected error and it allows to extend the notion of geometric corrector in the sense defined in [2] and now given by

$$\Gamma_j = \delta_j \frac{\Delta x_j}{2} \,.$$

In order to simplify the proof and since we are interested in the convergence property of the scheme, we assume that $h$ is sufficiently small such that $f'(u(.,t))$ changes of sign at most once every three consecutive cells and such that if $f'(u(.,t))$ does not change of sign on cell $j-1$, $j$ and $j+1$ then

$$\sigma_{j+\frac{1}{2}} = \sigma_{j-\frac{1}{2}} = \delta_j = \delta_{j-1} = \delta_{j+1} \,. \tag{66}$$

We now define the local truncation error $\underline{\epsilon}_j^n$ for $j = 1$ to $J$ by

$$\begin{aligned}\underline{e}_j^{n+1} = \underline{e}_j^n - \frac{\Delta t_n}{\Delta x_j} \Bigg( & \frac{1-\sigma_{j+\frac{1}{2}}^n}{2} s_{j+\frac{1}{2}}^n (\underline{e}_{j+1}^n - \underline{e}_j^n) \\ & + \frac{1+\sigma_{j-\frac{1}{2}}^n}{2} s_{j-\frac{1}{2}}^n (\underline{e}_j^n - \underline{e}_{j-1}^n) \Bigg) - \Delta t_n \underline{\epsilon}_j^n\end{aligned} \tag{67}$$

where by convention $z_{J+1} = \beta$, $e_{J+1} = 0$, $z_0 = \alpha$ and $e_0 = 0$. First we shall state the following estimate on the local error:

**Lemma 3.**

$$|\underline{\epsilon}_j^n| \leq c(\Delta t_n + \Delta x_j + |\underline{e}_{j-1}^n| + |\underline{e}_j^n| + |\underline{e}_{j+1}^n|) \,. \tag{68}$$

*Proof.* Indeed thanks to the definition of the scheme, we obtain from (67)

$$\begin{aligned}\underline{\epsilon}_j^n = \frac{u(z_j,t_{n+1}) - u(z_j,t_n)}{\Delta t_n} & + \frac{1-\sigma_{j+\frac{1}{2}}^n}{2} s_{j+\frac{1}{2}}^n \frac{u(z_{j+1},t_n) - u(z_j,t_n)}{\Delta x_j} \\ & + \frac{1+\sigma_{j-\frac{1}{2}}^n}{2} s_{j-\frac{1}{2}}^n \frac{u(z_j,t_n) - u(z_{j-1},t_n)}{\Delta x_j}\end{aligned} \tag{69}$$

Some rearrangements and several Taylor expansion lead to

$$
\begin{aligned}
\underline{\epsilon}_j^n = & \frac{1-\sigma^n_{j+\frac12}}{2}(s^n_{j+\frac12} - f'(u(z_j,t_n)))\frac{u(z_{j+1},t_n)-u(z_j,t_n)}{\Delta x_j} \\
& + \frac{1+\sigma^n_{j-\frac12}}{2}(s^n_{j-\frac12} - f'(u(z_j,t_n)))\frac{u(z_j,t_n)-u(z_{j-1},t_n)}{\Delta x_j} \\
& + \frac{\partial f(u(z_j,t_n))}{\partial x}\left(\frac{(1-\sigma^n_{j+\frac12})(z_{j+1}-z_j)+(1+\sigma^n_{j-\frac12})(z_j-z_{j-1})}{2\Delta x_j} - 1\right) \\
& + \frac{\partial u(z_j,t_n)}{\partial t} + \frac{\partial f(u(z_j,t_n))}{\partial x} + \mathcal{O}(\Delta t_n) + \mathcal{O}(\Delta x_j)\,.
\end{aligned} \tag{70}
$$

By using the local quasi-uniformity of meshes and noticing that

$$
z_{j+1} - z_j = \frac{1+\delta_{j+1}}{2}\Delta x_{j+1} + \frac{1-\delta_j}{2}\Delta x_j \tag{71}
$$

one gets $z_{j+1} - z_j = \mathcal{O}(\Delta x_j)$. So in the first and second term, we have

$$
\left|\frac{u(z_{j+1},t_n)-u(z_j,t_n)}{\Delta x_j}\right| \le c\left\|\frac{\partial u}{\partial x}\right\|_\infty \quad , \quad \left|\frac{u(z_j,t_n)-u(z_{j-1},t_n)}{\Delta x_j}\right| \le c\left\|\frac{\partial u}{\partial x}\right\|_\infty
$$

and for some $\bar{u}^n_j \in (u^n_{j+1}, u^n_j)$

$$
s^n_{j+\frac12} - f'(u(z_j,t_n)) = \begin{cases} f'(u^n_j) - f'(u(z_j,t_n)) + f''(\bar{u}^n_j)(u^n_{j+1}-u^n_j) & \text{if } u^n_{j+1} \neq u^n_j \\ f'(u^n_j) - f'(u(z_j,t_n)) & \text{if } u^n_{j+1} = u^n_j \end{cases}
$$

Therefore, we obtain the following estimates

$$
\begin{aligned}
\left|s^n_{j+\frac12} - f'(u(z_j,t_n))\right| &\le c\left(|\underline{e}^n_j| + |\underline{e}^n_{j+1}| + \Delta x_j\right) \\
\left|s^n_{j-\frac12} - f'(u(z_j,t_n))\right| &\le c\left(|\underline{e}^n_j| + |\underline{e}^n_{j-1}| + \Delta x_j\right)
\end{aligned} \tag{72}
$$

where the generic constant $c$ depends on $\|f''\|_\infty$ and $\left\|\frac{\partial u}{\partial x}\right\|_\infty$.

Concerning the third term, two cases are to be considered. First, if $f'(u(x,t_n))$ does not change of sign in the cells $j-1$, $j$ and $j+1$, we get from (66) and since $z_j$ has been selected for this reasons

$$
\frac{(1-\sigma^n_{j+\frac12})(z_{j+1}-z_j)+(1+\sigma^n_{j-\frac12})(z_j-z_{j-1}) - 2\Delta x_j}{2\Delta x_j} = 0\,.
$$

Second, if $f'(u(x,t_n))$ changes of sign in one of the cells $j-1$, $j$ and $j+1$, we denote $\zeta$ the point where $f'(u(\zeta,t)) = 0$, a Taylor expansion between $z_j$ and $\zeta$ implies that

$$
\frac{\partial f(u(z_j,t_n))}{\partial x} = \mathcal{O}(\Delta x_j)
$$

while the ratio in factor remains bounded independently on $\Delta x_j$ from the local quasi-uniformity assumption.

Gathering these estimates and using (52), we obtain the desired result (68). $\square$

*6.5. Order of convergence*

We are now able to prove the following theorem.

**Theorem 4.** *We assume that the discretization of the initial data is such that*

$$|u_j^0 - u_0(x_j)| \leq ch\,, \quad \forall j = 1, \ldots, J\,. \tag{73}$$

*i) Under the local quasi-uniformity* (3) *of meshes and the C.F.L. condition* (59)*, the error for the finite volume scheme satisfies the first order estimate:*

$$||(e_j^n)_{j=1}^J||_\infty \leq C'_\infty h, \quad t_n \leq T\,. \tag{74}$$

*ii) Under the global quasi-uniformity of meshes i.e. if there is a constant $\kappa$ such that for all $h < h_0$*

$$\frac{1}{\kappa}\, h \leq \Delta x_j \leq h\,, \quad \forall j = 1, \ldots, J\,. \tag{75}$$

*and the C.F.L. condition* (59)*, the error for the finite volume scheme satisfies the first order estimate: for all $p \in [1, \infty]$,*

$$||(e_j^n)_{j=1}^J||_p \leq C'_p h, \quad t_n \leq T\,. \tag{76}$$

*Proof.* Let us go back to (67) and use the C.F.L. condition to estimate

$$|\underline{e}_j^{n+1}| \leq \left(1 - \frac{\Delta t_n}{\Delta x_j}\Big(\frac{\sigma^n_{j+\frac12} - 1}{2} s^n_{j+\frac12} + \frac{\sigma^n_{j-\frac12} + 1}{2} s^n_{j-\frac12}\Big)\right)|\underline{e}_j^n|$$

$$+\frac{\Delta t_n}{\Delta x_j}\frac{\sigma^n_{j+\frac12} - 1}{2} s^n_{j+\frac12}|\underline{e}^n_{j+1}| + \frac{\Delta t_n}{\Delta x_j}\frac{\sigma^n_{j-\frac12} + 1}{2} s^n_{j-\frac12}|\underline{e}^n_{j-1}| + \Delta t_n|\underline{\epsilon}_j^n|\,.$$

Applying the estimate (68) on $\underline{\epsilon}_j^n$ and the fact that the ratio $\Delta t_n/h$ remains bounded, we get

$$|\underline{e}_j^{n+1}| \leq (1 + c\Delta t_n)\|\underline{e}^n\|_\infty + c\Delta t_n\, h \tag{77}$$

and we easily conclude by induction since $\|\underline{e}^0\|_\infty \leq ch$.

In view of (75), in order to show (76) it is sufficient to consider the case $p = 1$. Multiplying (67) by $\Delta x_j$ and summing, one gets after some simplifications

$$\begin{aligned}\sum_{j=1}^J \Delta x_j|\underline{e}_j^{n+1}| \leq \sum_{j=1}^J \Delta x_j|\underline{e}_j^n| + \Delta t_n \sum_{j=1}^J \left(s^n_{j+\frac12} - s^n_{j-\frac12}\right)|\underline{e}_j^n| \\ +\Delta t_n \frac{1 - \sigma^n_{\frac12}}{2} s^n_{\frac12}|\underline{e}_1^n| - \Delta t_n \frac{\sigma^n_{J+\frac12} + 1}{2} s^n_{J+\frac12}|\underline{e}_J^n| + \Delta t_n \sum_{j=1}^J \Delta x_j|\underline{\epsilon}_j^n|\,.\end{aligned} \tag{78}$$

Lemma 3, the estimate (74) and the fact that $\Delta t_n/h$ remains bounded imply

$$\Delta t_n \sum_{j=1}^{J} \Delta x_j |\underline{\epsilon}_j^n| \leq c\Delta t_n h \,. \tag{79}$$

In order to estimate the second term of the r.h.s. in (78), we introduce the term $f'(u(z_j, t_n))$ and use (72) to get

$$\Delta t_n \sum_{j=1}^{J} \left( s_{j+\frac{1}{2}}^n - s_{j-\frac{1}{2}}^n \right) |\underline{e}_j^n| \leq c\Delta t_n \sum_{j=1}^{J} (|\underline{e}_j^n| + |\underline{e}_{j-1}^n| + |\underline{e}_{j+1}^n| + \Delta x_j)|\underline{e}_j^n| \,. \tag{80}$$

We conclude with the estimate (74). At this step, the global quasi-uniformity is necessary to bound the difference $|s_{j+\frac{1}{2}}^n - s_{j-\frac{1}{2}}^n|$ by $c\Delta x_j$ and to write

$$\Delta t_n \sum_{j=1}^{J} \left( s_{j+\frac{1}{2}}^n - s_{j-\frac{1}{2}}^n \right) |\underline{e}_j^n| \leq c\Delta t_n \sum_{j=1}^{J} \Delta x_j |\underline{e}_j^n| \,. \tag{81}$$

The last two remaining terms in formula (78) are negative. Altogether, we obtain

$$\sum_{j=1}^{J} \Delta x_j |\underline{e}_j^{n+1}| \leq (1 + c\Delta t_n) \sum_{j=1}^{J} \Delta x_j |\underline{e}_j^n| + c\Delta t_n h \,. \tag{82}$$

This completes the proof.

$\square$

## 7. Conclusion

This work is a first account in the generalization of our previous work dealing with the search for an optimal error estimate for upwind finite volume method. In the previous paper, we addressed the linear constant advection equation in arbitrary space dimension. In the present study, we extend the notion of geometric corrector to non constant (at least in space) coefficient and nonlinear scalar problems. We provide a mathematical analysis of so-called one order schemes and show that despite the lack of consistency, they are indeed of order one. In a work in progress, we address the case of non linear hyperbolic systems of conservation laws.

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