

Rigorous passage to the limit in a system of reaction-diffusion equations towards a system including cross diffusions

F. CONFORTO

Dipartimento di Matematica
Università di Messina, Italy
Email: fiamma@unime.it

L. DESVILLETES

CMLA, ENS Cachan, IUF & CNRS, PRES UniverSud
61 Av. du Pdt. Wilson, 94235 Cachan Cedex, France
Email: desville@cmla.ens-cachan.fr

December 5, 2009

Abstract

We present in this work a rigorous passage to the limit (in dimension 1) in a system of three reaction-diffusion equations coming out of population dynamics, towards a system of two reaction-diffusion equations, one of which includes a cross diffusion term.

Subject Class: 35K57 35B25 35Q92 92D25

Keywords: reaction-diffusion, cross diffusion, entropy methods

1 Introduction

This paper is devoted to the mathematical study of the convergence of a reaction-diffusion system (first introduced by Iida, Mimura and Ninomiya, Cf. [12]) consisting of three equations towards a reaction-cross diffusion system consisting of two equations (introduced by Shigesada, Kawasaki and Teramoto, Cf. [19]), when a relaxation parameter tends to 0. We are concerned here with systems appearing in problems of population dynamics, with populations evolving through diffusion and a logistic-type reaction kernel (that is, the competition effects are assumed to be represented by quadratic terms).

One of the species appears in two different states, the corresponding populations are described by the number densities $u_A := u_A(t, x) \geq 0$ and $u_B := u_B(t, x) \geq 0$. An individual can switch from one state to the other with a

rate proportional to $1/\varepsilon$ and depending upon the density (at time t and point x) of a second species (described by the number density $v := v(t, x) \geq 0$).

When they are in the state B , the individuals of the first species have a larger diffusion rate than in the state A [for example, u_B represents the individuals which are scared or stressed, and they switch from the quieter state A when individuals of the second species compete with them for the resources].

Finally, for the sake of simplicity, we assume that all the individuals are confined in a domain Ω of \mathbb{R}^N ; that is, we impose homogeneous Neumann boundary conditions.

The following set of equations summarizes the assumptions discussed above:

$$\partial_t u_A^\varepsilon - d_u \Delta_x u_A^\varepsilon = (r_1 - a_1 (u_A^\varepsilon + u_B^\varepsilon) - b_1 v^\varepsilon) u_A^\varepsilon + \frac{1}{\varepsilon} (k(v^\varepsilon) u_B^\varepsilon - h(v^\varepsilon) u_A^\varepsilon), \quad (1)$$

$$\partial_t u_B^\varepsilon - (d_u + M\alpha) \Delta_x u_B^\varepsilon = (r_1 - a_1 (u_A^\varepsilon + u_B^\varepsilon) - b_1 v^\varepsilon) u_B^\varepsilon - \frac{1}{\varepsilon} (k(v^\varepsilon) u_B^\varepsilon - h(v^\varepsilon) u_A^\varepsilon), \quad (2)$$

$$\partial_t v^\varepsilon - d_v \Delta_x v^\varepsilon = (r_2 - b_2 (u_A^\varepsilon + u_B^\varepsilon) - a_2 v^\varepsilon) v^\varepsilon, \quad (3)$$

(for $t \geq 0$, $x \in \partial\Omega$)

$$\nabla_x u_A^\varepsilon(t, x) \cdot n(x) = 0, \quad \nabla_x u_B^\varepsilon(t, x) \cdot n(x) = 0, \quad \nabla_x v^\varepsilon(t, x) \cdot n(x) = 0. \quad (4)$$

We finally add the initial data (for $x \in \Omega$)

$$u_A^\varepsilon(0, x) = u_{A0}(x), \quad u_B^\varepsilon(0, x) = u_{B0}(x), \quad v^\varepsilon(0, x) = v_0(x). \quad (5)$$

This system was introduced by Iida, Mimura and Ninomiya in [12] (Cf. also [13]) as a prototype of reaction-diffusion systems which, in some limit, lead to a simple reaction-cross diffusion system (here, the cross diffusion appears only in one of the equations of the limiting system).

The strictly positive parameters r_1 , a_1 , b_1 , r_2 , a_2 , b_2 are related to the logistic competition between the individuals of various types, while the strictly positive parameters d_u , $d_u + M\alpha$, d_v (with $M\alpha > 0$) are the diffusion rates of the individuals [the reason for considering a constant $M\alpha$ which is a product of two terms will appear later]. Finally, h and k are nonnegative functions describing the effect of the presence of individuals of the second species on the possible switch between states A and B of individuals of the first species, while $\varepsilon > 0$ is the typical time scale of this switch. In [12] and [13], the assumptions proposed for h and k are the following: h is monotone increasing,

k is monotone decreasing, and

$$\frac{v}{M} = \frac{h(v)}{h(v) + k(v)}, \quad (6)$$

where $M \geq \max(\frac{r_2}{a_2}, \|v(0, \cdot)\|_\infty)$ is an upper bound for the number density of the second species.

We shall in the sequel consider the somewhat more general

Assumption 1: The coefficients $r_1, r_2, a_1, a_2, b_1, b_2, d_u, d_v, \alpha, M$ are strictly positive constants. Moreover, the functions h, k are of class $C^2(\mathbb{R}_+; \mathbb{R}_+)$, and satisfy

$$\{h = 0\} \subset \{0\}, \quad \{k = 0\} \subset \{0\}. \quad (7)$$

Note that the typical choices

$$h(v) = \frac{v}{M}, \quad k(v) = 1 - \frac{v}{M},$$

with $M > \max(\frac{r_2}{a_2}, \|v(0, \cdot)\|_\infty)$ satisfy both Assumption 1 and the assumptions of [13], thanks to the maximum principle applied to eq. (3), Cf. Proposition 2.1.

We also introduce the following hypothesis on the initial data:

Assumption 2: We assume that $v_0 \in H^1(\Omega)$,

$$0 < \inf_{\Omega} v_0 \leq \sup_{\Omega} v_0 < +\infty, \quad (8)$$

and $u_{A0} \geq 0, u_{B0} \geq 0, u_{A0} \log u_{A0}, u_{B0} \log u_{B0} \in L^1(\Omega)$.

The following Proposition states the existence and uniqueness of regular (for $t > 0$) solutions to system (1) – (5), for a given $\varepsilon > 0$:

Proposition 1.1. *We consider a smooth bounded subset $\Omega \subset \mathbb{R}^N$ for $N \in \mathbb{N}^*$, a relaxation time $\varepsilon > 0$, and coefficients/initial data satisfying Assumptions 1 and 2. Then, there exists a unique strong solution in $(C^\infty(\mathbb{R}_+^* \times \bar{\Omega}; \mathbb{R}_+))^3$ to system (1) – (5) [with initial data satisfied in the sense that $\lim_{t \rightarrow 0^+} u_A^\varepsilon(t, x) = u_{A0}(x)$ for a.e. $x \in \Omega$, etc.]. Moreover, this solution lies in $(C^\infty(\mathbb{R}_+ \times \bar{\Omega}; \mathbb{R}_+))^3$ if u_{A0}, u_{B0}, v_0 belong to $C^\infty(\bar{\Omega})$ and satisfy the compatibility conditions related to the Neumann boundary condition.*

This Proposition is a consequence of standard Theorems of existence/uniqueness for reaction-diffusion equations and of results of parabolic regularity. We refer for example to [8] for the existence/uniqueness, and to [15] for the parabolic regularity.

It is believed (Cf. [12] and [13]) that when $\varepsilon \rightarrow 0$, the solutions $(u_A^\varepsilon, u_B^\varepsilon, v^\varepsilon)$ of system (1) – (5) converge towards (u_A, u_B, v) , where

$$k(v) u_B = h(v) u_A, \quad (9)$$

and $(u_A + u_B, v)$ satisfies the cross diffusion-reaction system (of two equations)

$$\begin{aligned} \partial_t(u_A + u_B) - \Delta_x \left(d_u(u_A + u_B) + M \alpha \frac{h(v)}{k(v) + h(v)} (u_A + u_B) \right) \\ = (r_1 - a_1(u_A + u_B) - b_1 v)(u_A + u_B), \end{aligned} \quad (10)$$

$$\partial_t v - d_v \Delta_x v = (r_2 - b_2(u_A + u_B) - a_2 v) v. \quad (11)$$

This last system can be seen as a particular case of a class of systems introduced in [19] in order to explain the coexistence of individuals of species which, in absence of cross diffusion, would satisfy the competitive exclusion principle (Cf. [14]). More precisely, with respect to the general case of systems introduced in [19], we are here concerned by the case in which no self diffusion appears, and cross diffusion is present in only one of the equations (that is, from the point of view of the diffusion, the system is triangular).

Our main result is that the convergence of the solutions of the system of three reaction-diffusion equations towards the solutions of the system of two reaction-cross diffusion equations indeed holds when the dimension is $N = 1$. More precisely, we are able to prove the following:

Theorem 1.2. *We consider a bounded interval $\Omega \subset \mathbb{R}$, and coefficients/initial data satisfying Assumptions 1 and 2.*

Then, the unique solution $(u_A^\varepsilon, u_B^\varepsilon, v^\varepsilon)$ to system (1) – (5) [for $\varepsilon > 0$] converges when $\varepsilon \rightarrow 0$ towards (u_A, u_B, v) for a.e. $(t, x) \in \mathbb{R}_+ \times \Omega$, where (u_A, u_B, v) lie in $L^2([0, T] \times \Omega) \times L^2([0, T] \times \Omega) \times L^\infty([0, T] \times \Omega)$. Moreover, eq. (9) holds and $(u_A + u_B, v)$ is a weak solution of eq. (10), (11) with homogeneous Neumann boundary condition (for $t \geq 0, x \in \partial\Omega$)

$$\nabla_x(u_A + u_B)(t, x) \cdot n(x) = 0, \quad \nabla_x v(t, x) \cdot n(x) = 0,$$

and initial data (for $x \in \Omega$)

$$u_A(0, x) + u_B(0, x) = u_{A0}(x) + u_{B0}(x), \quad v(0, x) = v_0(x).$$

Finally, the following regularity result holds: the functions $u_A, u_B \in L^2(\log L)([0, T] \times \Omega)$ for all $T > 0$.

Note that this result is known (in any dimension of space) under the extra assumption that $(u_A^\varepsilon, u_B^\varepsilon, v^\varepsilon)$ is bounded (uniformly w.r.t ε) in $L^\infty([0, T] \times \Omega)$ (Cf. [12]). To obtain such a property is however still an open problem (at least for global weak solutions). Note also that the convergence of the steady solutions to (1) – (4) towards the steady solutions to (9) – (11) has been thoroughly investigated in [13], both from the numerical and analytical point of view.

Our result uses techniques involving weak solutions which are typically useful when maximum principles cannot be proven (as it is the case for $u_A^\varepsilon, u_B^\varepsilon$ here). Those techniques are based on Lyapounov functionals or duality lemmas for solutions of reaction-diffusion equations (such a lemma was first proven by M. Pierre and D. Schmitt, Cf. [17] and [18]). Those techniques are particularly helpful (when dealing with quadratic reaction terms) when they can be used for the equations satisfied by quantities like $u \log u$ instead of u . This is the case in [11] or, in the context of the quasi-steady state approximation, in [2] or [3]. They also have been successfully applied in the study of general cross diffusion-reaction equations with small initial data (Cf. [1]).

The difficulty in the present paper is that the reaction term is indeed quadratic, but no superlinear Lyapounov functional can easily be exhibited. We shall actually replace it by a functional which does not decrease along the flow of equations (1) – (4), but whose derivative can nevertheless be uniformly controlled w.r.t. ε . As a trade-off, this construction will only hold in dimension 1. We refer to [6] and [7] for the use of entropy-like functionals in the context of equations with cross diffusion and self diffusion.

Let us finally briefly explain how our result is related to works dealing with the mathematical theory of the limiting system. This system belongs to a general class of equations possibly including cross diffusions and self diffusions, first introduced in [19], and studied by many authors since that time: we refer for example to [20], [7] and [5] for recent papers in which many references are presented. In our case, the system has a special triangular structure (as far as cross diffusion is concerned) which leads to a possible partial use of the maximum principle (that is, for v). Moreover, no self diffusion appears. This case is treated in [16], where the method which is used (presented in dimension 2) also enables to get existence of (strong, global) solutions in dimension 1 (with initial data in a slightly smaller space than what is considered in the present work).

The rest of the paper is devoted to the Proof of Theorem 1.2. We begin in section 2 with preliminary estimates which hold in any dimension. We turn then in section 3 to estimates which are valid only in dimension 1 (Sobolev

estimates in dimension 1 are required there). In section 4 is presented the Lyapounov-like functional which enables to prove the main estimates of this paper. Then, section 5 is devoted to the proof of strong compactness results for the family $(u_A^\varepsilon, u_B^\varepsilon, v^\varepsilon)_{\varepsilon>0}$. Estimates are finally presented in section 6, which prevent this family to concentrate (in L^2). Thanks to the results of the two last sections, convergence in L^2 strong is obtained for $(u_A^\varepsilon, u_B^\varepsilon, v^\varepsilon)_{\varepsilon>0}$, and this is exactly what is needed to pass to the limit in system (1) – (5).

2 Proof of Theorem 1.2: preliminaries

We begin with a (uniform w.r.t. ε) estimate of $(u_A^\varepsilon, u_B^\varepsilon, v^\varepsilon)$, which is valid in all dimensions of space:

Proposition 2.1. *We consider a smooth bounded subset $\Omega \subset \mathbb{R}^N$ for $N \in \mathbb{N}^*$, and coefficients/initial data satisfying Assumptions 1 and 2. Then, the unique strong solution $u_A^\varepsilon, u_B^\varepsilon, v^\varepsilon$ in $(C^\infty(\mathbb{R}_+^* \times \bar{\Omega}; \mathbb{R}_+))^3$ to system (1) – (5) given by Proposition 1.1 satisfies for all $T > 0$ (and $i, j \in \{1, \dots, N\}$):*

$$\sup_{\varepsilon>0} \|u_A^\varepsilon\|_{L^2([0,T] \times \Omega)} < +\infty, \quad \sup_{\varepsilon>0} \|u_B^\varepsilon\|_{L^2([0,T] \times \Omega)} < +\infty, \quad (12)$$

$$\sup_{\varepsilon>0} \|v^\varepsilon\|_{L^\infty([0,T] \times \Omega)} < +\infty, \quad (13)$$

$$\sup_{\varepsilon>0} \left\| \frac{\partial v^\varepsilon}{\partial t} \right\|_{L^2([0,T] \times \Omega)} < +\infty, \quad \sup_{\varepsilon>0} \left\| \frac{\partial^2 v^\varepsilon}{\partial x_i \partial x_j} \right\|_{L^2([0,T] \times \Omega)} < +\infty, \quad (14)$$

$$\sup_{\varepsilon>0} \sup_{t \in [0,T]} \int_{\Omega} |\nabla_x v^\varepsilon(t, x)|^2 dx < +\infty. \quad (15)$$

Proof of Proposition 2.1: Adding (1) and (2), one gets the equation

$$\begin{aligned} & \partial_t(u_A^\varepsilon + u_B^\varepsilon) - \Delta_x \left(d_u u_A^\varepsilon + (d_u + M \alpha) u_B^\varepsilon \right) \\ &= (r_1 - a_1 (u_A^\varepsilon + u_B^\varepsilon) - b_1 v^\varepsilon) (u_A^\varepsilon + u_B^\varepsilon). \end{aligned} \quad (16)$$

At this point, it is possible to directly obtain (12) by integrating the above equation w.r.t. x , and by using the fact that $a_1 > 0$ (that is, the quadratic term has the right sign).

We also propose an alternative, more complicated proof, which still holds when the reaction term is not quadratic, but which requires an extra assumption on the initial datum. We can indeed write

$$\partial_t(u_A^\varepsilon + u_B^\varepsilon) - \Delta_x(K^\varepsilon (u_A^\varepsilon + u_B^\varepsilon)) \leq \frac{r_1^2}{4 a_1}, \quad (17)$$

with

$$0 < d_u \leq K^\varepsilon := \frac{d_u u_A^\varepsilon + (d_u + M \alpha) u_B^\varepsilon}{u_A^\varepsilon + u_B^\varepsilon} \leq d_u + M \alpha. \quad (18)$$

We can then use the following Lemma, due to M. Pierre and D. Schmitt (Cf. [17] et [18]):

Lemma 2.2. *We consider a smooth bounded subset $\Omega \subset \mathbb{R}^N$ for $N \in \mathbb{N}^*$, and assume that $\rho : [0, T] \times \Omega \rightarrow [0, +\infty[$ satisfies*

$$\begin{aligned} \partial_t \rho - \Delta_x (M \rho) &\leq 0 & \text{on } \Omega, \\ \nabla_x (\rho M) \cdot n &= 0 & \text{on } \partial\Omega, \end{aligned} \quad (19)$$

where $M : [0, T] \times \Omega \rightarrow \mathbb{R}$ is a function which satisfies $d_1 \geq M \geq d_0 > 0$ for some numbers d_1, d_0 . Then, for all $T > 0$,

$$\|\rho\|_{L^2([0, T] \times \Omega)} \leq \left(1 + \frac{d_1}{d_0}\right) T \|\rho(0, \cdot)\|_{L^2(\Omega)}.$$

Lemma 2.2 then yields (12), provided that the initial datum for u_A^ε and u_B^ε lies in L^2 .

We now turn to the estimates satisfied by v^ε . We observe that according to (3),

$$\begin{aligned} \partial_t v^\varepsilon - d_v \Delta_x v^\varepsilon &\leq (r_2 - a_2 v^\varepsilon) v^\varepsilon \\ &\leq \frac{r_2^2}{4 a_2}. \end{aligned} \quad (20)$$

Using (8) in Assumption 2, we deduce from (20) and the maximum principle that estimate (13) holds.

Denoting now

$$\theta^\varepsilon := (r_2 - b_2 (u_A^\varepsilon + u_B^\varepsilon) - a_2 v^\varepsilon) v^\varepsilon,$$

we know that thanks to (12) and (13), for all $T > 0$,

$$\sup_{\varepsilon > 0} \|\theta^\varepsilon\|_{L^2([0, T] \times \Omega)} < +\infty. \quad (21)$$

Taking (3) into account, we see that (for all $T > 0$)

$$\sup_{\varepsilon > 0} \left\| \frac{\partial v^\varepsilon}{\partial t} - d_v \Delta_x v^\varepsilon \right\|_{L^2([0, T] \times \Omega)} < +\infty. \quad (22)$$

Using standard theorems of parabolic regularization (Cf. [15] for example), estimate (22) implies estimate (14).

Multiplying (3) by $\Delta_x v^\varepsilon$, we see that

$$\Delta_x v^\varepsilon \partial_t v^\varepsilon - d_v (\Delta_x v^\varepsilon)^2 = \theta^\varepsilon \Delta_x v^\varepsilon,$$

so that (for all $t > 0$)

$$\begin{aligned} & \frac{1}{2} \int_{\Omega} |\nabla_x v^\varepsilon(t, x)|^2 dx + d_v \int_0^t \int_{\Omega} |\Delta_x v^\varepsilon(s, x)|^2 dx ds \\ &= - \int_0^t \int_{\Omega} \Delta_x v^\varepsilon(s, x) \theta^\varepsilon(s, x) dx ds + \frac{1}{2} \int_{\Omega} |\nabla_x v^\varepsilon(0, x)|^2 dx \\ &\leq \frac{1}{2} d_v \int_0^t \int_{\Omega} |\Delta_x v^\varepsilon(s, x)|^2 dx ds + \frac{1}{2 d_v} \int_0^t \int_{\Omega} |\theta^\varepsilon(s, x)|^2 dx ds + \frac{1}{2} \int_{\Omega} |\nabla_x v^\varepsilon(0, x)|^2 dx. \end{aligned}$$

Using (21) and Assumption 2, we get estimate (15).

This ends the Proof of Proposition 2.1 \square

3 Proof of Theorem 1.2: dimension-dependent estimates

We now turn to estimates which do not necessarily hold in higher dimension. We begin with the proof of the following Proposition:

Proposition 3.1. *We consider a bounded interval $\Omega \subset \mathbb{R}$ and coefficients/initial data satisfying Assumptions 1 and 2. Then, the unique strong solution $(u_A^\varepsilon, u_B^\varepsilon, v^\varepsilon)$ in $(C^\infty(\mathbb{R}_+^* \times \bar{\Omega}; \mathbb{R}_+))^3$ to system (1) – (5) given by Proposition 1.1 satisfies for all $T > 0$:*

$$\sup_{\varepsilon > 0} \|\partial_x v^\varepsilon\|_{L^4([0, T] \times \Omega)} < +\infty. \quad (23)$$

Proof of Proposition 3.1: This result is a direct consequence of embedding Theorems in [15], and can in fact be improved (the space L^4 can be replaced by L^6). We nevertheless propose an elementary proof below.

Using the Sobolev inequality $H^1(\Omega) \subset L^\infty(\Omega)$, we get the estimate

$$\sup_{x \in \Omega} |\partial_x v^\varepsilon(t, x)|^2 \leq C(\Omega) \left\{ \int_{\Omega} |\partial_x v^\varepsilon(t, x)|^2 dx + \int_{\Omega} |\partial_{xx}^2 v^\varepsilon(t, x)|^2 dx \right\}, \quad (24)$$

where $C(\Omega)$ is the constant in the Sobolev inequality.

Using estimates (14) and (15), and integrating with respect to $t \in [0, T]$, we end up with

$$\sup_{\varepsilon > 0} \int_0^T \sup_{x \in \Omega} |\partial_x v^\varepsilon(t, x)|^2 dt < +\infty. \quad (25)$$

Interpolating between estimates (15) and (25), we get

$$\begin{aligned} \int_0^T \int_\Omega |\partial_x v^\varepsilon(t, x)|^4 dx dt &\leq \left(\sup_{s \in [0, T]} \int_\Omega |\partial_x v^\varepsilon(s, x)|^2 dx \right) \\ &\quad \times \left(\int_0^T \sup_{y \in \Omega} |\partial_x v^\varepsilon(t, y)|^2 dt \right), \end{aligned}$$

which yields estimate (23).

This ends the Proof of Proposition 3.1.

We now prove bounds from below for v^ε .

Proposition 3.2. *We consider a bounded interval $\Omega \subset \mathbb{R}$ and coefficients/initial data satisfying Assumptions 1 and 2. Then, the unique strong solution $(u_A^\varepsilon, u_B^\varepsilon, v^\varepsilon)$ in $(C^\infty(\mathbb{R}_+^* \times \bar{\Omega}; \mathbb{R}_+))^3$ to system (1) – (5) given by Proposition 1.1 satisfies for all $T > 0$:*

$$\inf_{\varepsilon > 0} \inf_{(t, x) \in [0, T] \times \Omega} h(v^\varepsilon(t, x)) > 0, \quad \inf_{\varepsilon > 0} \inf_{(t, x) \in [0, T] \times \Omega} k(v^\varepsilon(t, x)) > 0. \quad (26)$$

Proof of Proposition 3.2: We first observe that (3) implies that

$$\begin{aligned} (\partial_t - d_v \partial_{xx})(\log v^\varepsilon) &= d_v \frac{|\nabla_x v^\varepsilon|^2}{(v^\varepsilon)^2} + r_2 - b_2(u_A^\varepsilon + u_B^\varepsilon) - a_2 v^\varepsilon \\ &\geq -b_2(u_A^\varepsilon + u_B^\varepsilon) - a_2 v^\varepsilon. \end{aligned} \quad (27)$$

As a consequence, for any $t \in [0, T]$, $x \in \Omega$,

$$\log v^\varepsilon(t, x) \geq w^\varepsilon(t, x),$$

where

$$\sup_{\varepsilon > 0} \|(\partial_t - d_v \partial_{xx})w^\varepsilon\|_{L^2([0, T] \times \Omega)} < +\infty$$

(thanks to estimates (12), (13)) and $\sup_{x \in \Omega} [-w^\varepsilon(0, x)] < +\infty$ (thanks to hypothesis (8) in Assumption 2), so that (see for example the appendix in [10])

$$\inf_{\varepsilon > 0} \inf_{(t, x) \in [0, T] \times \Omega} w^\varepsilon(t, x) > 0,$$

and finally

$$\inf_{\varepsilon > 0} \inf_{(t,x) \in [0,T] \times \Omega} v^\varepsilon(t,x) > 0. \quad (28)$$

This estimate, together with (7) in Assumption 1, yields estimate (26).

This ends the Proof of Proposition 3.2. \square

4 Proof of Theorem 1.2: entropy estimates

In this section, we introduce a functional which is close to the entropies used in reaction-diffusion systems coming out of reversible chemistry (Cf. for example [9], [10]), and which is close to being a Lyapounov functional. Note that it is also reminiscent of functionals used in [7]. As a consequence, we end up with new estimates for the solution to system (1) – (5). More precisely, we prove the following proposition:

Proposition 4.1. *We consider a bounded interval $\Omega \subset \mathbb{R}$ and coefficients/initial data satisfying Assumptions 1 and 2. Then, the unique strong solution $(u_A^\varepsilon, u_B^\varepsilon, v^\varepsilon)$ in $(C^\infty(\mathbb{R}_+^* \times \bar{\Omega}; \mathbb{R}_+))^3$ to system (1) – (5) given by Proposition 1.1 satisfies for all $T > 0$:*

$$\sup_{\varepsilon > 0} \sup_{t \in [0,T]} \int_{\Omega} u_A^\varepsilon(t,x) |\log u_A^\varepsilon(t,x)| dx < +\infty, \quad (29)$$

$$\sup_{\varepsilon > 0} \sup_{t \in [0,T]} \int_{\Omega} u_B^\varepsilon(t,x) |\log u_B^\varepsilon(t,x)| dx < +\infty, \quad (30)$$

$$\sup_{\varepsilon > 0} \int_0^T \int_{\Omega} |\partial_x \sqrt{u_A^\varepsilon}|^2 dx dt < +\infty, \quad (31)$$

$$\sup_{\varepsilon > 0} \int_0^T \int_{\Omega} |\partial_x \sqrt{u_B^\varepsilon}|^2 dx dt < +\infty, \quad (32)$$

$$\sup_{\varepsilon > 0} \frac{1}{\varepsilon} \int_0^T \int_{\Omega} |\sqrt{k(v^\varepsilon) u_B^\varepsilon} - \sqrt{h(v^\varepsilon) u_A^\varepsilon}|^2 dx dt < +\infty. \quad (33)$$

Proof of Proposition 4.1: We introduce the quantity

$$H^\varepsilon = u_A^\varepsilon (\log[h(v^\varepsilon)]) + u_B^\varepsilon (\log[k(v^\varepsilon)]) + u_A^\varepsilon \log u_A^\varepsilon + u_B^\varepsilon \log u_B^\varepsilon - u_A^\varepsilon - u_B^\varepsilon.$$

The integral of this quantity will be close to a Lyapounov functional. The computation is performed in any dimension. We first observe that

$$\frac{\partial H^\varepsilon}{\partial t} = \frac{h'(v^\varepsilon)}{h(v^\varepsilon)} \frac{\partial v^\varepsilon}{\partial t} u_A^\varepsilon + \log[h(v^\varepsilon)] \frac{\partial u_A^\varepsilon}{\partial t} + \frac{k'(v^\varepsilon)}{k(v^\varepsilon)} \frac{\partial v^\varepsilon}{\partial t} u_B^\varepsilon + \log[k(v^\varepsilon)] \frac{\partial u_B^\varepsilon}{\partial t}$$

$$+ \log u_A^\varepsilon \frac{\partial u_A^\varepsilon}{\partial t} + \log u_B^\varepsilon \frac{\partial u_B^\varepsilon}{\partial t}. \quad (34)$$

Then, we compute

$$\begin{aligned} \Delta_x \left\{ \log[h(v^\varepsilon)] u_A^\varepsilon + u_A^\varepsilon \log u_A^\varepsilon - u_A^\varepsilon \right\} &= 2 \frac{h'(v^\varepsilon)}{h(v^\varepsilon)} \nabla_x u_A^\varepsilon \cdot \nabla_x v^\varepsilon \\ &+ \log[h(v^\varepsilon)] \Delta_x u_A^\varepsilon + \left(\frac{h'}{h} \right)' (v^\varepsilon) u_A^\varepsilon \Delta_x v^\varepsilon + \log u_A^\varepsilon \Delta_x u_A^\varepsilon + \frac{|\nabla_x u_A^\varepsilon|^2}{u_A^\varepsilon}, \end{aligned} \quad (35)$$

and

$$\begin{aligned} \Delta_x \left\{ \log[k(v^\varepsilon)] u_B^\varepsilon + u_B^\varepsilon \log u_B^\varepsilon - u_B^\varepsilon \right\} &= 2 \frac{k'(v^\varepsilon)}{k(v^\varepsilon)} \nabla_x u_B^\varepsilon \cdot \nabla_x v^\varepsilon \\ &+ \log[k(v^\varepsilon)] \Delta_x u_B^\varepsilon + \left(\frac{k'}{k} \right)' (v^\varepsilon) u_B^\varepsilon \Delta_x v^\varepsilon + \log u_B^\varepsilon \Delta_x u_B^\varepsilon + \frac{|\nabla_x u_B^\varepsilon|^2}{u_B^\varepsilon}. \end{aligned} \quad (36)$$

As a consequence, if we denote

$$\begin{aligned} J^\varepsilon &= d_u \left((\log[h(v^\varepsilon)]) u_A^\varepsilon + u_A^\varepsilon \log u_A^\varepsilon - u_A^\varepsilon \right) \\ &+ (d_u + M \alpha) \left((\log[k(v^\varepsilon)]) u_B^\varepsilon + u_B^\varepsilon \log u_B^\varepsilon - u_B^\varepsilon \right), \end{aligned}$$

we see that

$$\frac{\partial H^\varepsilon}{\partial t} - \Delta_x J^\varepsilon = \quad (37)$$

$$\log[h(v^\varepsilon)] u_A^\varepsilon (r_1 - a_1 (u_A^\varepsilon + u_B^\varepsilon) - b_1 v^\varepsilon) u_A^\varepsilon \quad (38)$$

$$+ \log[k(v^\varepsilon)] u_B^\varepsilon (r_1 - a_1 (u_A^\varepsilon + u_B^\varepsilon) - b_1 v^\varepsilon) u_B^\varepsilon \quad (39)$$

$$- \frac{1}{\varepsilon} \left[k(v^\varepsilon) u_B^\varepsilon - h(v^\varepsilon) u_A^\varepsilon \right] \left(\log[k(v^\varepsilon) u_B^\varepsilon] - \log[h(v^\varepsilon) u_A^\varepsilon] \right) \quad (40)$$

$$+ \left\{ \frac{h'(v^\varepsilon)}{h(v^\varepsilon)} u_A^\varepsilon + \frac{k'(v^\varepsilon)}{k(v^\varepsilon)} u_B^\varepsilon \right\} \frac{\partial v^\varepsilon}{\partial t} \quad (41)$$

$$- \left\{ d_u \frac{h'(v^\varepsilon)}{h(v^\varepsilon)} u_A^\varepsilon + (d_u + M \alpha) \frac{k'(v^\varepsilon)}{k(v^\varepsilon)} u_B^\varepsilon \right\} \Delta_x v^\varepsilon \quad (42)$$

$$- 2 \left\{ d_u \frac{h'(v^\varepsilon)}{h(v^\varepsilon)} \nabla_x u_A^\varepsilon + (d_u + M \alpha) \frac{k'(v^\varepsilon)}{k(v^\varepsilon)} \nabla_x u_B^\varepsilon \right\} \cdot \nabla_x v^\varepsilon \quad (43)$$

$$- \left\{ d_u \left(\frac{h'}{h} \right)' (v^\varepsilon) u_A^\varepsilon + (d_u + M \alpha) \left(\frac{k'}{k} \right)' (v^\varepsilon) u_B^\varepsilon \right\} |\nabla_x v^\varepsilon|^2 \quad (44)$$

$$- d_u \frac{|\nabla_x u_A^\varepsilon|^2}{u_A^\varepsilon} - (d_u + M \alpha) \frac{|\nabla_x u_B^\varepsilon|^2}{u_B^\varepsilon}. \quad (45)$$

The estimates are now done in dimension 1 only.

We observe that thanks to estimates (13), (26) and Assumption 1,

$$|(41)| \leq Cst (u_A^\varepsilon + u_B^\varepsilon) \left| \frac{\partial v^\varepsilon}{\partial t} \right|, \quad (46)$$

$$|(42)| \leq Cst (u_A^\varepsilon + u_B^\varepsilon) |\partial_{xx} v^\varepsilon|, \quad (47)$$

$$\begin{aligned} |(43)| &\leq Cst \left(|\partial_x u_A^\varepsilon| + |\partial_x u_B^\varepsilon| \right) |\partial_x v^\varepsilon| \\ &\leq \left[\frac{d_u}{2} \frac{|\partial_x u_A^\varepsilon|^2}{u_A^\varepsilon} + \frac{d_u + M\alpha}{2} \frac{|\partial_x u_B^\varepsilon|^2}{u_B^\varepsilon} \right] + Cst |\partial_x v^\varepsilon|^2 (u_A^\varepsilon + u_B^\varepsilon), \end{aligned} \quad (48)$$

$$|(44)| \leq Cst (u_A^\varepsilon + u_B^\varepsilon) |\partial_x v^\varepsilon|^2. \quad (49)$$

So that

$$\begin{aligned} &\frac{\partial H^\varepsilon}{\partial t} - \Delta_x J^\varepsilon + \frac{d_u}{2} \frac{|\partial_x u_A^\varepsilon|^2}{u_A^\varepsilon} + \frac{d_u + M\alpha}{2} \frac{|\partial_x u_B^\varepsilon|^2}{u_B^\varepsilon} \\ &+ \frac{1}{\varepsilon} \left[k(v^\varepsilon) u_B^\varepsilon - h(v^\varepsilon) u_A^\varepsilon \right] \left(\log[k(v^\varepsilon) u_B^\varepsilon] - \log[h(v^\varepsilon) u_A^\varepsilon] \right) \\ &\leq \log[h(v^\varepsilon) u_A^\varepsilon] (r_1 - a_1 (u_A^\varepsilon + u_B^\varepsilon) - b_1 v^\varepsilon) u_A^\varepsilon \\ &\quad + \log[k(v^\varepsilon) u_B^\varepsilon] (r_1 - a_1 (u_A^\varepsilon + u_B^\varepsilon) - b_1 v^\varepsilon) u_B^\varepsilon \\ &\quad + Cst (u_A^\varepsilon + u_B^\varepsilon) \left[\left| \frac{\partial v^\varepsilon}{\partial t} \right| + |\partial_{xx} v^\varepsilon| + |\partial_x v^\varepsilon|^2 \right]. \end{aligned} \quad (50)$$

Then, we use the elementary inequalities (for $x > 0$)

$$x \log x - x + 1 \geq 0, \quad x \log x - x + 1 \leq Cst (1 + x |\log x|),$$

in order to estimate (using (13) and (26))

$$\begin{aligned} &\log[h(v^\varepsilon) u_A^\varepsilon] (r_1 - a_1 (u_A^\varepsilon + u_B^\varepsilon) - b_1 v^\varepsilon) u_A^\varepsilon \\ &= \left\{ \log[h(v^\varepsilon) u_A^\varepsilon] (h(v^\varepsilon) u_A^\varepsilon) - h(v^\varepsilon) u_A^\varepsilon + 1 \right\} \frac{1}{h(v^\varepsilon)} (r_1 - a_1 (u_A^\varepsilon + u_B^\varepsilon) - b_1 v^\varepsilon) \\ &\quad + (h(v^\varepsilon) u_A^\varepsilon - 1) \frac{1}{h(v^\varepsilon)} (r_1 - a_1 (u_A^\varepsilon + u_B^\varepsilon) - b_1 v^\varepsilon) \\ &\leq Cst \left(1 + h(v^\varepsilon) u_A^\varepsilon |\log[h(v^\varepsilon) u_A^\varepsilon]| \right) \frac{r_1}{h(v^\varepsilon)} + r_1 u_A^\varepsilon \\ &\quad + \frac{a_1}{h(v^\varepsilon)} (u_A^\varepsilon + u_B^\varepsilon) + \frac{b_1}{h(v^\varepsilon)} v^\varepsilon \end{aligned}$$

$$\leq Cst (1 + (u_A^\varepsilon)^2 + (u_B^\varepsilon)^2). \quad (51)$$

A similar computation leads to

$$\begin{aligned} & \log[k(v^\varepsilon) u_B^\varepsilon] (r_1 - a_1 (u_A^\varepsilon + u_B^\varepsilon) - b_1 v^\varepsilon) u_B^\varepsilon \\ & \leq Cst (1 + (u_A^\varepsilon)^2 + (u_B^\varepsilon)^2). \end{aligned} \quad (52)$$

Integrating (50) on $[0, T] \times \Omega$ and using (51), (52), we end up with

$$\begin{aligned} & \int_{\Omega} H^\varepsilon(T, x) dx + \frac{d_u}{2} \int_0^T \int_{\Omega} \frac{|\partial_x u_A^\varepsilon|^2}{u_A^\varepsilon} dxdt + \frac{d_u + M\alpha}{2} \int_0^T \int_{\Omega} \frac{|\partial_x u_B^\varepsilon|^2}{u_B^\varepsilon} dxdt \\ & + \frac{1}{\varepsilon} \int_0^T \int_{\Omega} \left[k(v^\varepsilon) u_B^\varepsilon - h(v^\varepsilon) u_A^\varepsilon \right] \left(\log[k(v^\varepsilon) u_B^\varepsilon] - \log[h(v^\varepsilon) u_A^\varepsilon] \right) dxdt \\ & \leq \int_{\Omega} H^\varepsilon(0, x) dx + Cst \int_0^T \int_{\Omega} (1 + (u_A^\varepsilon)^2 + (u_B^\varepsilon)^2) dxdt \\ & + Cst \int_0^T \int_{\Omega} \left(\left| \frac{\partial v^\varepsilon}{\partial t} \right|^2 + |\partial_{xx} v^\varepsilon|^2 + |\partial_x v^\varepsilon|^4 \right) dxdt. \end{aligned} \quad (53)$$

We now use estimates (12), (14), (23) and (26) (and Assumption 2) together with the elementary inequality (which holds for all $K > 1$)

$$\frac{1}{2} y \log y - \frac{1}{2} y + (\log K) y \geq -\frac{2}{K}, \quad (54)$$

in order to get the estimate

$$\begin{aligned} & \int_{\Omega} H^\varepsilon(T, x) dx + \frac{d_u}{2} \int_0^T \int_{\Omega} \frac{|\partial_x u_A^\varepsilon|^2}{u_A^\varepsilon} dxdt + \frac{d_u + M\alpha}{2} \int_0^T \int_{\Omega} \frac{|\partial_x u_B^\varepsilon|^2}{u_B^\varepsilon} dxdt \\ & + \frac{1}{\varepsilon} \int_0^T \int_{\Omega} \left[k(v^\varepsilon) u_B^\varepsilon - h(v^\varepsilon) u_A^\varepsilon \right] \left(\log[k(v^\varepsilon) u_B^\varepsilon] - \log[h(v^\varepsilon) u_A^\varepsilon] \right) dxdt \leq Cst. \end{aligned} \quad (55)$$

This immediately ensures that estimates (29) – (32) hold. Finally, estimate (33) is a consequence of (55) and the elementary inequality (for any $x, y > 0$)

$$(x - y) (\log x - \log y) \geq Cst (\sqrt{x} - \sqrt{y})^2.$$

This ends the Proof of Proposition 4.1.

5 Proof of Theorem 1.2: Strong compactness

We show in this section that the solution to system (1) – (5) converges a.e. More precisely, we prove the following proposition:

Proposition 5.1. *We consider a bounded interval $\Omega \subset \mathbb{R}$, and coefficients/initial data satisfying Assumptions 1 and 2. Then, the unique strong solution $(u_A^\varepsilon, u_B^\varepsilon, v^\varepsilon)$ in $(C^\infty(\mathbb{R}_+^* \times \bar{\Omega}; \mathbb{R}_+))^3$ to system (1) – (5) given by Proposition 1.1 is such that for a.e. $(t, x) \in \mathbb{R}_+ \times \Omega$,*

$$u_A^\varepsilon(t, x) \rightarrow u_A(t, x), \quad u_B^\varepsilon(t, x) \rightarrow u_B(t, x), \quad v^\varepsilon(t, x) \rightarrow v(t, x), \quad (56)$$

where $v \in L_{loc}^\infty(\mathbb{R}_+ \times \Omega)$, $u_A, u_B \in L_{loc}^2(\mathbb{R}_+ \times \Omega)$, and (equivalently to (9))

$$u_A = \frac{k(v)}{k(v) + h(v)} (u_A + u_B), \quad (57)$$

$$u_B = \frac{h(v)}{k(v) + h(v)} (u_A + u_B). \quad (58)$$

Moreover, eq. (11) is satisfied in the sense of distributions (by $u_A + u_B, v$).

Proof of Proposition 5.1: According to (13) and (14), it is clear that $(v^\varepsilon)_{\varepsilon>0}$ belongs to a strongly compact set of $L^1([0, T] \times \Omega)$ (for all $T > 0$). As a consequence, we can extract from $(v^\varepsilon)_{\varepsilon>0}$ a subsequence (still denoted by $(v^\varepsilon)_{\varepsilon>0}$) such that

$$v^\varepsilon(t, x) \rightarrow v(t, x) \quad \text{for a.e. } (t, x) \in \mathbb{R}_+ \times \Omega.$$

Using now (31) and (32), we see that

$$\sup_{\varepsilon>0} \int_0^T \int_\Omega |\partial_x u_A^\varepsilon| dx dt < +\infty; \quad \sup_{\varepsilon>0} \int_0^T \int_\Omega |\partial_x u_B^\varepsilon| dx dt < +\infty, \quad (59)$$

thanks to the estimate:

$$\int_0^T \int_\Omega |\partial_x u| dx dt \leq \left(\int_0^T \int_\Omega \frac{|\partial_x u|^2}{u} dx dt \right)^{1/2} \left(\int_0^T \int_\Omega |u| dx dt \right)^{1/2}. \quad (60)$$

According to (12), we can extract from the sequences $(u_A^\varepsilon)_{\varepsilon>0}$ and $(u_B^\varepsilon)_{\varepsilon>0}$ subsequences (still denoted by $(u_A^\varepsilon)_{\varepsilon>0}$ and $(u_B^\varepsilon)_{\varepsilon>0}$) such that

$$u_A^\varepsilon \rightharpoonup u_A, \quad u_B^\varepsilon \rightharpoonup u_B \quad \text{in } L_{loc}^2(\mathbb{R}_+ \times \Omega), \quad (61)$$

where u_A and u_B both belong to $L_{loc}^2(\mathbb{R}_+ \times \Omega)$.

Thanks to (16) (and (12), (13)), we know that $\frac{\partial}{\partial t}(u_A^\varepsilon + u_B^\varepsilon)$ is bounded in $L^1([0, T]; W^{-2,1}(\Omega))$. Interpolating this bound with (59) [that is, applying Aubin's lemma] we obtain the strong compactness of $(u_A^\varepsilon + u_B^\varepsilon)_{\varepsilon>0}$ in $L^1([0, T] \times \Omega)$ (for all $T > 0$). Therefore, we can extract from $(u_A^\varepsilon + u_B^\varepsilon)_{\varepsilon>0}$ a subsequence (still denoted by $(u_A^\varepsilon + u_B^\varepsilon)_{\varepsilon>0}$) such that

$$(u_A^\varepsilon + u_B^\varepsilon)(t, x) \rightarrow (u_A + u_B)(t, x) \quad \text{for a.e.} \quad (t, x) \in \mathbb{R}_+ \times \Omega. \quad (62)$$

At this level, we see that thanks to (12), (13), (56) and (62),

$$(u_A^\varepsilon + u_B^\varepsilon) v^\varepsilon \rightharpoonup (u_A + u_B) v, \quad (v^\varepsilon)^2 \rightharpoonup v^2 \quad \text{in} \quad \mathcal{D}'(\mathbb{R}_+^* \times \Omega). \quad (63)$$

We can therefore pass to the limit in (3) in order to get eq. (11) (in the sense of distributions).

But

$$\begin{aligned} \int_0^T \int_\Omega |k(v^\varepsilon) u_B^\varepsilon - h(v^\varepsilon) u_A^\varepsilon|^2 dx dt &\leq \left(\int_0^T \int_\Omega |\sqrt{k(v^\varepsilon) u_B^\varepsilon} - \sqrt{h(v^\varepsilon) u_A^\varepsilon}|^2 dx dt \right)^{1/2} \\ &\quad \times \left(\int_0^T \int_\Omega \left[\sqrt{k(v^\varepsilon) u_B^\varepsilon} + \sqrt{h(v^\varepsilon) u_A^\varepsilon} \right]^2 dx dt \right)^{1/2} \\ &\leq C st \varepsilon^{1/2}, \end{aligned}$$

thanks to (12), (13), (33) and Assumption 1.

Therefore, up to extraction of a subsequence,

$$k(v^\varepsilon(t, x)) u_B^\varepsilon(t, x) - h(v^\varepsilon(t, x)) u_A^\varepsilon(t, x) \rightarrow 0 \quad \text{for a.e.} \quad (t, x) \in \mathbb{R}_+ \times \Omega. \quad (64)$$

But

$$\begin{aligned} u_A^\varepsilon &= \frac{k(v^\varepsilon) (u_A^\varepsilon + u_B^\varepsilon) - [k(v^\varepsilon) u_B^\varepsilon - h(v^\varepsilon) u_A^\varepsilon]}{k(v^\varepsilon) + h(v^\varepsilon)}, \\ u_B^\varepsilon &= \frac{h(v^\varepsilon) (u_A^\varepsilon + u_B^\varepsilon) + [k(v^\varepsilon) u_B^\varepsilon - h(v^\varepsilon) u_A^\varepsilon]}{k(v^\varepsilon) + h(v^\varepsilon)}, \end{aligned}$$

so that thanks to the convergences (56), (62), (64) [and estimate (26)], formulas (57) and (58) hold, together with the convergences in (56).

This concludes the Proof of Proposition 5.1. \square

In order to pass to the limit in (16), it remains to verify that $(u_A^\varepsilon + u_B^\varepsilon)^2$ converges in L^1 towards $(u_A + u_B)^2$ (that is, no L^1 concentration occurs in the sequence $(u_A^\varepsilon + u_B^\varepsilon)^2$). In order to do so, we shall provide an estimate of u_A^ε and u_B^ε in a slightly better space than L^2 .

6 Proof of Theorem 1.2: non concentration estimate for $(u_A^\varepsilon + u_B^\varepsilon)^2$ and conclusion

We begin with the following Proposition:

Proposition 6.1. *We consider a bounded interval $\Omega \subset \mathbb{R}$ and coefficients/initial data satisfying Assumptions 1 and 2. Then, the unique strong solution $(u_A^\varepsilon, u_B^\varepsilon, v^\varepsilon)$ in $(C^\infty(\mathbb{R}_+^* \times \bar{\Omega}; \mathbb{R}_+))^3$ to system (1) – (5) given by Proposition 1.1 satisfies for all $T > 0$:*

$$\sup_{\varepsilon > 0} \int_0^T \int_{\Omega} |u_A^\varepsilon|^2 |\log u_A^\varepsilon| dx dt < +\infty, \quad (65)$$

$$\sup_{\varepsilon > 0} \int_0^T \int_{\Omega} |u_B^\varepsilon|^2 |\log u_B^\varepsilon| dx dt < +\infty. \quad (66)$$

Proof of Proposition 6.1: Using a Sobolev inequality (with constant $C(\Omega)$) in dimension 1, we get (for any $t \geq 0$)

$$\begin{aligned} \sup_{x \in \Omega} |u_A^\varepsilon(t, x)| &= \left(\sup_{x \in \Omega} |\sqrt{u_A^\varepsilon(t, x)}| \right)^2 \\ &\leq Cst \int_{\Omega} |\partial_x \sqrt{u_A^\varepsilon}|^2 dx, \end{aligned} \quad (67)$$

so that according to estimate (31),

$$\sup_{\varepsilon > 0} \int_0^T \left(\sup_{x \in \Omega} |u_A^\varepsilon(t, x)| dx \right) dt < +\infty. \quad (68)$$

Therefore,

$$\begin{aligned} &\sup_{\varepsilon > 0} \int_0^T \int_{\Omega} |u_A^\varepsilon|^2 |\log u_A^\varepsilon| dx dt \\ &\leq \sup_{\varepsilon > 0} \int_0^T \sup_{y \in \Omega} |u_A^\varepsilon(t, y)| \left(\int_{\Omega} u_A^\varepsilon(t, x) |\log u_A^\varepsilon(t, x)| dx \right) dt \\ &\leq \sup_{\varepsilon > 0} \left(\int_0^T \sup_{y \in \Omega} |u_A^\varepsilon(t, y)| dt \right) \left(\sup_{s \in [0, T]} \int_{\Omega} |u_A^\varepsilon(s, x)| |\log u_A^\varepsilon(s, x)| dx \right). \end{aligned}$$

Using (29) and (68) [and doing the same proof for u_B^ε] leads to the conclusion of the Proof of Proposition 6.1. \square

We can now provide the

Conclusion of the Proof of Theorem 1.2: Thanks to the convergence (56) and estimates (65), (66) we see that

$$(u_A^\varepsilon + u_B^\varepsilon)^2 \rightarrow (u_A + u_B)^2 \quad \text{in} \quad L_{loc}^1(\mathbb{R}_+ \times \Omega),$$

so that it is possible to pass to the limit in equation (16) and get (using identities (57) and (58)) the limit equation (10). The fact that $u_A, u_B \in L^2(\log L)([0, T] \times \Omega)$ is a direct consequence of Proposition 6.1 This concludes the Proof of Theorem 1.2. \square

References

- [1] Bendahmane, Mostafa; Lepoutre, Thomas; Marrocco, Americo; Perthame, Benoit; *Conservative cross diffusions and pattern formation through relaxation*. Preprint of the laboratory LJLL, (2009), <http://www.ann.jussieu.fr/publications/2009/R09056.html>
- [2] Bisi, Marzia; Conforto, Fiammetta; Desvillettes, Laurent; *Quasi-steady-state approximation for reaction-diffusion equations*. Bull. Inst. Math. Acad. Sin. (N.S.) **2** (2007), no. 4, 823–850.
- [3] Bothe, Dieter; Pierre, Michel; *Quasi-steady-state approximation for a reaction-diffusion system with fast intermediate*. Preprint.
- [4] Carrillo, Jose-Antonio; Desvillettes, Laurent; Fellner, Klemens; *Rigorous Derivation of a Nonlinear Diffusion Equation as Fast-Reaction Limit of a Continuous Coagulation–Fragmentation Model with Diffusion*. Preprint n. 2008-38 of the CMLA, ENS Cachan. Accepted for publication in Commun. Partial Differential Equations.
- [5] Choi, Y.S.; Lui, Roger; Yamada, Yoshio; *Existence of global solutions for the Shigesada-Kawasaki-Teramoto model with strongly coupled cross-diffusion*. Discrete Contin. Dynam. Systems **10** (2004), no. 3, 719–730.
- [6] Chen, Li; Jüngel, Ansgar; *Analysis of a multi-dimensional parabolic population model with strong cross diffusion*. SIAM J. Math. Anal. **36** (2004), 301–322.
- [7] Chen, Li; Jüngel, Ansgar; *Analysis of a parabolic cross-diffusion population model without self-diffusion*. J. Differential Equations **224** (2006), no. 1, 39–59.
- [8] Desvillettes, Laurent; *About entropy methods for reaction-diffusion equations*. Riv. Mat. Univ. Parma **7** (2007), no. 7, 81–123.

- [9] Desvillettes, Laurent; Fellner, Klemens; *Exponential Decay toward Equilibrium via Entropy Methods for Reaction-Diffusion Equations*. J. Math. Anal. Appl. **319** (2006), 157-176.
- [10] Desvillettes, Laurent; Fellner, Klemens; *Entropy methods for Reaction-Diffusion Equations: Degenerate Diffusion and Slowly Growing A-priori bounds*. Rev. Matem. Iber. **24**, (2008), no. 2, 407-432.
- [11] Desvillettes, Laurent; Fellner, Klemens; Pierre, Michel; Vovelle, Julien; *Global existence for quadratic systems of reaction-diffusion*. Adv. Non-linear Stud. **7** (2007), no. 3, 491-511.
- [12] Iida, Masato; Mimura, Masayasu; Ninomiya, Hirokazu; *Diffusion, cross-diffusion and competitive interaction*. J. Math. Biol. **53** (2006), no. 4, 617-641.
- [13] Izuhara, Hirofumi; Mimura, Masayasu; *Reaction-diffusion system approximation to the cross-diffusion competition system*. Hiroshima Math. J. **38** (2008), no. 2, 315-347.
- [14] Kishimoto, Kazuo; Weinberger, Hans F.; *The spatial homogeneity of stable equilibria of some reaction-diffusion systems on convex domains*. J. Differential Equations **58** (1985), no. 1, 15-21.
- [15] Ladyzhenskaya, Olga A.; Ural'tseva, Nina N.; *Linear and quasilinear elliptic equations*. Translated from the Russian by Scripta Technica, Inc. Translation editor: Leon Ehrenpreis Academic Press, New York-London 1968 xviii+495 pp. 35.47
- [16] Lou, Yuan; Ni, Wei-Ming; Wu, Yaping; *On the global existence of a cross-diffusion system*. Discrete Contin. Dynam. Systems **4** (1998), no. 2, 193-203.
- [17] Martin, R.H.; Pierre, Michel; *Nonlinear reaction-diffusion systems*. Mathematics in science and engineering, **185**, (1992), 363-398.
- [18] Pierre, Michel; Schmitt, Didier; *Blowup in reaction-diffusion systems with dissipation of mass*. SIAM Rev. **42** (2000), 93-106 (electronic).
- [19] Shigesada, Nanako; Kawasaki, Kohkichi; Teramoto, Ei; *Spatial segregation of interacting species*. J. Theoret. Biol. **79** (1979), no. 1, 83-99.
- [20] Wang, Yi; *The global existence of solutions for a cross-diffusion system*. Acta Math. Appl. Sin. Engl. Ser. **21** (2005), no. 3, 519-528.