Measures with locally finite support and spectrum

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The goal of this paper is the construction of measures μ on $\mathbb{R}^n$ enjoying three conflicting but fortunately compatible properties: (*i*) μ is a sum of weighted Dirac masses on a locally finite set, (*ii*) the Fourier transform $\hat{\mu}$ of μ is also a sum of weighted Dirac masses on a locally finite set, and (*iii*) μ is not a generalized Dirac comb. We give surprisingly simple examples of such measures. These unexpected patterns strongly differ from quasicrystals, they provide us with unusual Poisson's formulas, and they might give us an unconventional insight into aperiodic order.

Poisson formula | spectrum | almost periodic

The Dirac mass at $a\in\mathbb{R}^n$ is denoted by δ_a or $\delta_a(x)$. A purely atomic measure is a linear combination $\mu=\sum_{\lambda\in\Lambda}c(\lambda)\delta_\lambda$ of Dirac masses where the coefficients $c(\lambda)$ are real or complex numbers and $\sum_{|\lambda|\le R}|c(\lambda)|$ is finite for every $R>0$. Then Λ is a countable set of points of $\mathbb{R}^n$. If Λ is closed and if $c(\lambda)\neq 0, \forall\lambda\in\Lambda$, then Λ is the support of μ. A subset $\Lambda\subset\mathbb{R}^n$ is locally finite if $\Lambda\cap B$ is finite for every bounded set B. Equivalently Λ can be ordered as a sequence $\{\lambda_j, j=1,2,\ldots\}$ and $|\lambda_j|$ tends to infinity with j. A measure μ is a tempered distribution if it has a polynomial growth at infinity in the sense given by Laurent Schwartz in ref. 1. For instance, the measure $\sum_1^\infty 2^k\delta_k$ is not a tempered distribution whereas $\sum_1^\infty k^3\delta_k$ and $\sum_1^\infty 2^k[\delta_{(k+2^{-k})}-\delta_k]$ are tempered distributions. The Fourier transform $\mathcal{F}(f)=\hat{f}$ of a function f is defined by $\hat{f}(y)=\int_{\mathbb{R}^n}\exp(-2\pi\, ix\cdot y)f(x)dx$. The distributional Fourier transform $\hat{\mu}$ of μ is defined by the following condition: $\langle\hat{\mu},\phi\rangle=\langle\mu,\hat{\phi}\rangle$ shall hold for every test function ϕ belonging to the Schwartz class $\mathcal{S}(\mathbb{R}^n)$. The spectrum S of μ is the (closed) support of $\hat{\mu}$.

Definition 1: A purely atomic measure μ on $\mathbb{R}^n$ is a crystalline measure if

i) the support Λ of μ is a locally finite set,
ii) μ is a tempered distribution, and
iii) the distributional Fourier transform $\hat{\mu}$ of μ is also a purely atomic measure that is supported by a locally finite set S.

If μ is a crystalline measure, its Fourier transform is also a crystalline measure.

Definition 2: A measure μ on $\mathbb{R}$ is odd if for every compactly supported continuous function f we have $\int S(f)d\mu=-\int f\, d\mu$, where $S\,f(x)=f(-x)$.

For every set E of real numbers, let $E_+=E\cap\{x>0\}$ and $E_-=E\cap\{x<0\}$. Let us denote by $\mathbb{Q}$ the field of rational numbers. *Theorem 1* is proved in this article:

Theorem 1. *There exists an odd crystalline measure μ on $\mathbb{R}$ such that its support Λ and its spectrum S have the following properties*: (*i*) *Each finite subset of Λ_+ is linearly independent over $\mathbb{Q}$ and* (*ii*) *each finite subset of S_+ is linearly independent over $\mathbb{Q}$.*

The spectrum S of μ is an increasing sequence $s_k, k\in\mathbb{Z}$, of real numbers such that (*i*) $s_{-k}=-s_k, \forall k\in\mathbb{Z}$, and (*ii*) $s_1,s_2,\ldots,s_N$ are linearly independent over $\mathbb{Q}$ for each integer N. *Theorem 1* implies that $\sum_{-\infty}^{\infty}b(k)\exp(2\pi i s_k x)$ is a sum of Dirac masses on a locally finite set Λ. It is counterintuitive that these incoherent waves $b(k)\exp(2\pi i s_k x)$ can be piled up in harmony so that their sum yields Dirac masses.

Theorem 1 is valid in any dimension as the following proposition shows.

Theorem 2. *There exists a crystalline measure μ on $\mathbb{R}^n$ such that*

i) *μ is odd in the last variable x_n,*
ii) *the support Λ of μ is the union of $\Lambda_+=\Lambda\cap\{x_n>0\}$ and $\Lambda_-=\Lambda\cap\{x_n<0\}$,*
iii) *each finite subset of Λ_+ is linearly independent over $\mathbb{Q}$ and similarly for Λ_-,*
iv) *the spectrum S of μ is the union of $S_+=S\cap\{y_n>0\}$ and $S_-=S\cap\{y_n<0\}$*, and
v) *each finite subset of S_+ is linearly independent over $\mathbb{Q}$ and similarly for S_-.*

This line of investigation began with the Riemann–Weil explicit formula in number theory (2). The Riemann–Weil explicit formula can be written $\hat{\mu}=\omega+\sigma$, where μ is a series of Dirac masses on the nontrivial zeros of the zeta function; σ is a series of Dirac masses on $\log(p^m)$, p running over the set of prime numbers; $m=1,2,\ldots$; and $\omega(x)=-\log\pi+\Re\psi(1/4+ix/2)$, ψ being the logarithmic derivative of the Γ function. Moreover, an exponential decay is needed on the test function ϕ to give a meaning to $\langle\omega+\sigma,\phi\rangle$. The Selberg trace formula has a similar structure. Therefore, the measures μ studied by André Weil in 1952 (2) are not crystalline measures. André-Paul Guinand discovered other summation formulas in 1959 (3). Guinand's formulas do not contain an integral term ω but are spoiled by a derivative of the Dirac mass at 0. Using a completely distinct approach, Nir Lev and Alexander Olevskii (4) proved the existence of crystalline measures μ that are not generalized Dirac combs (*Definition 2* below). This theorem could have been deduced from Guinand's work, as is shown below. The Lev–Olevskii measures do not have closed-form expressions. To analyze Lev–Olevskii measures let us consider $\mathbb{R}^n$ as a vector space over the field $\mathbb{Q}$ and let us denote by $E_{\mathbb{Q}}$ the linear span of a set $E\subset\mathbb{R}^n$. If Λ is the support of a Lev–Olevskii measure μ, then $\Lambda_{\mathbb{Q}}$ is necessarily finite dimensional and the same is true for $S_{\mathbb{Q}}$, where S is the spectrum of μ (*Theorem 8*). Mihalis Kolountzakis improved on the Lev–Olevskii theorem in ref. 5. He built a crystalline measure for which both the dimension of $\Lambda_{\mathbb{Q}}$ and the dimension of $S_{\mathbb{Q}}$ are infinite. Kolountzakis' theorem is also implicit in Guinand's work and the support of Kolountzakis' measure cannot be linearly independent over $\mathbb{Q}$.

Significance

An important problem in harmonic analysis is solved in this article: Is the Poisson summation formula unique or does it belong to a wider class? The latter is true. The method that is used to prove this statement is surprising. Our new Poisson's formulas were hidden inside an old and almost forgotten paper published in 1959 by A. P. Guinand. The role of number theory in this issue is fascinating.

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This article is organized as follows. Are crystalline measures almost periodic? This is answered in *Almost Periodic Measures*. In *Guinand's Distribution* Guinand's seminal work is described and used to prove Lev–Olevskii's theorem and Kolountzakis' theorem. In *Proof of Theorem 1*, *Theorem 1* is proved by an approach that extends and completes Guinand's work. A second proof of Kolountzakis' theorem is given in *Kolountzakis' Theorem*. In *The Crystalline Measures of Lev and Olevskii* the proof of the Lev–Olevskii theorem is sketched. *The Geometry of Crystalline Measures* is devoted to open problems.

Let $\Gamma \subset \mathbb{R}^n$ be a lattice. The distributional Fourier transform of the Dirac comb $\mu = \mathrm{vol}(\Gamma)\sum_{\gamma\in\Gamma}\delta_\gamma$ is the Dirac comb $\sum_{y\in\Gamma^*}\delta_y$ on the dual lattice Γ^*. We have

$$\mathrm{vol}(\Gamma)\sum_{\gamma\in\Gamma} f(\gamma) = \sum_{y\in\Gamma^*}\hat{f}(y). \qquad [1]$$

Poisson's formula [1] is valid for at least all test functions f belonging to the Schwartz class $\mathcal{S}(\mathbb{R}^n)$. In fact [1] is valid for a much larger class of functions (6, 7). A corollary of Poisson's formula that is used in the proof of *Theorem 5* is the following:

Lemma 1. *For every $\alpha,\beta\in\mathbb{R}^n$ and $f\in\mathcal{S}(\mathbb{R}^n)$ we have*

$$\mathrm{vol}(\Gamma)\sum_{\gamma\in\Gamma+\alpha} e^{2\pi i\beta\cdot\gamma}\hat{f}(\gamma) = e^{2\pi i\alpha\cdot\beta}\sum_{y\in\Gamma^*+\beta} e^{-2\pi i\alpha\cdot y} f(y).$$

Definition 3: Let σ_j be a Dirac comb supported by a coset $x_j+\Gamma_j$ of a lattice $\Gamma_j\subset\mathbb{R}^n, 1\le j\le N$. Let $F_j\subset\mathbb{R}^n$ be a finite set and $g_j(x)=\sum_{y\in F_j} c_j(y)\exp(2\pi i y\cdot x)$ be a trigonometric sum. Let $\mu_j = g_j\sigma_j$. Then $\mu=\mu_1+\cdots+\mu_N$ will be called a generalized Dirac comb.

The Fourier transform of a generalized Dirac comb is a generalized Dirac comb. Therefore, a generalized Dirac comb is a crystalline measure. When Λ is the support of a generalized Dirac comb μ, the dimension of $\Lambda_{\mathbb{Q}}$ is finite. The same property holds for the spectrum of μ. The crystalline measures of *Theorem 1* strongly differ from generalized Dirac combs.

Let μ be a crystalline measure. We then have $\mu=\sum_{\lambda\in\Lambda}a(\lambda)\delta_\lambda$ and $\hat{\mu}=\sum_{y\in S} b(y)\delta_y$, where $(a(\lambda))_{\lambda\in\Lambda}$ and $(b(y))_{y\in S}$ satisfy

$$a(\lambda)\neq 0, \lambda\in\Lambda, b(y)\neq 0, y\in S, \qquad [2]$$

and Λ, S are two locally finite sets. Then for every test function $f\in\mathcal{S}(\mathbb{R}^n)$ the following generalized Poisson's formula holds:

$$\sum_{\lambda\in\Lambda} a(\lambda)\hat{f}(\lambda) = \sum_{y\in S} b(y) f(y). \qquad [3]$$

It implies

$$\sum_{\lambda\in\Lambda} a(\lambda)\hat{f}(\lambda)\exp(2\pi i\lambda\cdot x) = \sum_{y\in S} b(y) f(x+y).$$

With the terminology of signal processing, sampling $\hat{f}$ on Λ yields an alias $\tilde{f}(x)=\sum_{y\in S} b(y) f(x+y)$ of the function f.

A locally finite set Λ is uniformly discrete if

$$\inf_{\{\lambda,\lambda'\in\Lambda;\ \lambda\neq\lambda'\}} |\lambda-\lambda'| = \beta > 0. \qquad [4]$$

Lev and Olevskii (8, 9) proved the following:

Theorem 3. *In one dimension if both the support Λ of a crystalline measure μ and the support S of its Fourier transform are uniformly discrete sets, then μ is a generalized Dirac comb.*

In ref. 9 Lev and Olevskii answered a problem raised by Jeffrey Lagarias in ref. 10. They proved *Theorem 2* in dimension $n\ge 2$ under the assumption that the measure μ is nonnegative. The problem is still open in the general case of a complex measure μ.

Almost Periodic Measures

Let μ be a crystalline measure. We have

$$\sum_{\lambda\in\Lambda} a(\lambda)\delta_\lambda(x) = \sum_{y\in S} b(y)\exp(2\pi i x\cdot y). \qquad [5]$$

This raises the following issue: Is the right-hand side of [5] an almost periodic measure?

A complex valued continuous function f defined on $\mathbb{R}^n$ is almost periodic in the sense of Bohr if for every positive ϵ one can find a finite subset $F\subset\mathbb{R}^n$ and a trigonometric sum $g(x)=\sum_{y\in F} a(y)\exp(2\pi i x\cdot y)$ such that $\|f-g\|_\infty = \sup_{x\in\mathbb{R}^n}|f(x)-g(x)|\le\epsilon$. Laurent Schwartz defined almost periodic distributions as follows (1):

Definition 4: A tempered distribution $S\in\mathcal{S}'(\mathbb{R}^n)$ is an almost periodic distribution if for every test function $\phi\in\mathcal{S}(\mathbb{R}^n)$ the convolution product $S\star\phi$ is an almost periodic function in the sense of Bohr.

This is less demanding than the definition of an almost periodic measure.

Definition 5: A Borel measure μ on $\mathbb{R}^n$ is an almost periodic measure if for every compactly supported continuous function ϕ the convolution product $\mu\star\phi$ is an almost periodic function in the sense of Bohr.

A generalized Dirac comb is an almost periodic measure.

Lemma 2. *Every crystalline measure μ is an almost periodic distribution.*

Indeed $g(x)=\mu\star\phi(x)=\sum_{y\in S} b(y)\hat{\phi}(y)\exp(2\pi i x\cdot y)$ is a finite trigonometric sum if $\phi\in\mathcal{S}(\mathbb{R}^n)$ has a compactly supported Fourier transform. Such test functions ϕ are dense in the Schwartz class that implies *Lemma 2*.

Surprisingly most crystalline measures are not almost periodic measures (*Theorems 4* and *5*). Are crystalline measures connected with quasicrystals? Today we know that model sets defined by the cut and projection scheme are modeling quasicrystals (11, 12). If Λ is a model set, the measure $\sigma_\Lambda=\sum_{\lambda\in\Lambda}\delta_\lambda$ is never a crystalline measure. Indeed σ_Λ is not even an almost periodic distribution. It is a generalized almost periodic measure (13). It means that for every $\epsilon>0$, one can find two almost periodic measures μ_ϵ and ν_ϵ such that $\mu_\epsilon\le\sigma\le\nu_\epsilon$ and $\mathcal{M}(\nu_\epsilon-\mu_\epsilon)\le\epsilon$, where $\mathcal{M}(\mu)=\lim_{T\to\infty}(|\mu|([-T,T])/2T)$.

Definition 6: Let us denote by $\mathcal{P}$ the Banach space of all Borel measures μ such that

i) μ is an almost periodic measure, and
ii) the distributional Fourier transform of μ is also an almost periodic measure.

The norm in the Banach space $\mathcal{P}$ is

$$\|\mu\|_{\mathcal{P}} = \sup_{x\in\mathbb{R}^n}|\mu|(x+U) + \sup_{\xi\in\mathbb{R}^n}|\hat{\mu}|(\xi+U), \qquad [6]$$

where U is the unit ball. Every $\mu\in\mathcal{P}$ is a purely atomic measure and its Fourier transform $\hat{\mu}$ is also a purely atomic measure.

We do not know whether there exists a crystalline measure $\sum_{\lambda\in\Lambda}c(\lambda)\delta_\lambda$ carried by a model set Λ. In ref. 12 we constructed some almost periodic measures μ that are supported by model sets (see proof of *Theorem 8*). They belong to $\mathcal{P}$ but they are not crystalline measures. In the construction of μ a test function ϕ supported by a compact set (the window of the model set) is being used. But the Fourier transform $\hat{\phi}$ of this function ϕ cannot be compactly supported, which implies that the support of $\hat{\mu}$ is dense in $\mathbb{R}^n$.

Guinand's Distribution

By Legendre's theorem, an integer $n \geq 0$ can be written as a sum of three squares (0^2 being admitted) if and only if n is not of the form $4^j(8k+7)$. For instance, 0, 1, 2, 3, 4, 5, 6 are sums of three squares but 7 is not. Let $r_3(n)$ be the number of decompositions of the integer $n \geq 1$ into a sum of three squares [with $r_3(n)=0$ if n is not a sum of three squares]. More precisely $r_3(n)$ is the number of points $k \in \mathbb{Z}^3$ such that $|k|^2 = n$. We have $r_3(4n) = r_3(n)$, $\forall n \in \mathbb{N}$, $r_3(0)=1$, $r_3(1)=6$, $r_3(2)=12$, …. Then $r_3(2^j)=6$ if j is even and 12 if j is odd. The behavior of $r_3(n)$ as $n \to \infty$ is erratic. The mean behavior is more regular because (14)

$$\sum_{0 \leq n \leq x} r_3(n) = \frac{4}{3}\pi x^{3/2} + O\left(x^{3/4+\epsilon}\right)$$

holds for every positive ϵ. Guinand began his seminal work (3) with *Lemma 3*:

Lemma 3. *For all $x>0$ we have*

$$1 + \sum_1^\infty r_3(n)\exp(-\pi n x) = x^{-3/2} + x^{-3/2}\sum_1^\infty r_3(n)\exp(-\pi n/x). \quad [7]$$

The functional equation satisfied by the Jacobi theta function

$$\sum_{-\infty}^{\infty} \exp\left(-\pi k^2 x\right) = x^{-1/2}\sum_{-\infty}^{\infty}\exp\left(-\pi k^2/x\right) \quad [8]$$

raised to the cubic power yields [7].

Let $f_x(t) = t\exp(-\pi x t^2)$, $t \in \mathbb{R}$, $x>0$. Then f_x is odd and its Fourier transform is $\hat{f}_x(y) = -i\,x^{-3/2}y\exp(-\pi y^2/x)$. Now [7] can be written

$$\frac{df_x}{dt}(0) + \sum_1^\infty r_3(n)n^{-1/2}f_x\left(\sqrt{n}\right) = i\,\frac{d\hat{f}_x}{dt}(0) + i\sum_1^\infty r_3(n)n^{-1/2}\hat{f}_x\left(\sqrt{n}\right). \quad [9]$$

Guinand introduced the odd distribution

$$\sigma = -2\frac{d}{dt}\delta_0 + \sum_1^\infty r_3(n)n^{-1/2}\left(\delta_{\sqrt{n}} - \delta_{-\sqrt{n}}\right), \quad [10]$$

which will be named Guinand's distribution. We have $\sum_0^N r_3(n)n^{-1/2} \sim 2\pi N$, $N \to \infty$, which implies that σ is a tempered distribution. Guinand proved the following:

Lemma 4. *The distributional Fourier transform of σ is $-i\sigma$.*

We need to prove $\langle \sigma, \hat{\phi}\rangle = -i\langle\sigma,\phi\rangle$ for every test function ϕ. But [9] can be written as $\langle\sigma, f_x\rangle = i\langle\sigma,\hat{f}_x\rangle$ or $\langle\sigma,f_x\rangle = i\langle\hat{\sigma},f_x\rangle$. The collection of odd functions f_x, $x>0$, is total in the subspace of odd functions of the Schwartz class. For even functions ϕ the identity $\langle\sigma,\hat{\phi}\rangle = -i\langle\sigma,\phi\rangle$ is trivial because σ is odd and $\langle\sigma,\hat{\phi}\rangle = -i\langle\sigma,\phi\rangle = 0$. *Lemma 4* is proved.

A variant on Guinand's distribution σ is the measure $\tilde{\sigma} = -4\pi t + \sum_1^\infty r_3(n)n^{-1/2}(\delta_{\sqrt{n}} - \delta_{-\sqrt{n}})$. Because $\mathcal{F}((d/dt)\delta_0 - 2\pi t) = -i((d/dt)\delta_0 - 2\pi t)$, we also have

$$\mathcal{F}(\tilde{\sigma}) = -i\tilde{\sigma}.$$

We now move one small step beyond Guinand's work and prove both Kolountzakis' theorem and Lev–Olevskii's theorem. Let $\alpha \in (0,1)$ and set

$$\tau_\alpha(t) = \left(\alpha^2 + \frac{1}{\alpha}\right)\sigma(t) - \alpha\sigma(\alpha t) - \sigma(t/\alpha). \quad [11]$$

Then the derivative of the Dirac mass at 0 disappears from this linear combination. On the Fourier transform side

$$\hat{\tau}_\alpha(y) = \left(\alpha^2 + \frac{1}{\alpha}\right)\hat{\sigma}(y) - \hat{\sigma}(y/\alpha) - \alpha\hat{\sigma}(\alpha y) = -i\tau_\alpha.$$

Fix $\alpha = 1/2$ in the preceding construction, let $\tau = \tau_{1/2}$, and define $\chi(n) = -1/2$ if $n \in \mathbb{N}\backslash 4\,\mathbb{N}$, $\chi(n)=4$ if $n \in 4\,\mathbb{N}\backslash 16\,\mathbb{N}$, and $\chi(n)=0$ if $n \in 16\,\mathbb{N}$. Then we have the following:

Theorem 4. *The Fourier transform of the measure*

$$\tau = \sum_1^\infty \chi(n)r_3(n)n^{-1/2}\left(\delta_{\sqrt{n}/2} - \delta_{-\sqrt{n}/2}\right) \quad [12]$$

is $-i\tau$.

Another proof of *Theorem 4* is given in the next section. Let us observe that $\sum_1^N r_3(n)n^{-1/2} = 2\pi N + O(N^{1/2})$, $N \to \infty$, whereas we have $\left|\sum_1^N \chi(n)r_3(n)n^{-1/2}\right| = O(N^{1/2})$. If χ was erased from [12], τ would no longer be a crystalline measure. The cancellations provided by χ are playing a key role.

The support Λ of τ is the set of $\pm\sqrt{n}/2$, $n \in E$, where $E \subset \mathbb{N}$ is defined by the two conditions $r_3(n) \neq 0$, $\chi(n) \neq 0$. It amounts to $n \neq 4^j(8k+7)$, $j=0,1$, and $n \notin 16\mathbb{N}$. There are infinitely many primes p that are not congruent to 7 modulo 8 and the square roots of these primes are linearly independent over $\mathbb{Q}$. Therefore, the dimension of the span over $\mathbb{Q}$ of the support of τ is infinite, which yields Kolountzakis' theorem. The positive half of the support of τ is not linearly independent over $\mathbb{Q}$. *Theorem 4* does not imply *Theorem 1*. The measure τ is not an almost periodic measure. Indeed $|\tau|([x,x+1]) \to \infty$, $x \to \infty$. It is, however, an almost periodic distribution.

Here is our second example. Let us observe that for every function f the Fourier transform of $\cos(\pi x)[f(x-1/2) - f(x+1/2)]$ is $i\cos(\pi y)[\hat{f}(y-1/2) - \hat{f}(y+1/2)]$. This simple observation leads to a variant on the measure τ of *Theorem 4*. Let σ be the Guinand distribution and consider the measure $\rho = \cos(\pi x)[\sigma(x-1/2) - \sigma(x+1/2)]$. The derivative of the Dirac mass at 0 is moved to 1/2 or $-1/2$ and then transformed into Dirac masses after being multiplied by $\cos(\pi x)$. On the Fourier transform side the derivative of the Dirac mass at 0 is transformed into a Dirac mass after multiplication by $\sin(\pi y)$ and then the resulting measure is translated by $\pm 1/2$. Finally the Fourier transform of ρ is ρ. We have $\rho =$

$$2\pi\delta_{1/2} + 2\pi\delta_{-1/2} + \sum_1^\infty \sin\left(\pi\sqrt{n}\right)r_3(n)n^{-1/2} \times \left(\delta_{(\sqrt{n}+1/2)} + \delta_{(\sqrt{n}-1/2)} + \delta_{(-\sqrt{n}+1/2)} + \delta_{(-\sqrt{n}-1/2)}\right).$$

One is tempted to replace 1/2 by 0 in the definition of ρ. But $\tilde{\rho} = 2\pi\delta_0 + \sum_1^\infty \sin(\pi\sqrt{n})r_3(n)n^{-1/2}(\delta_{\sqrt{n}} + \delta_{-\sqrt{n}})$ is not a crystalline measure: A derivative of the Dirac mass at $\pm 1/2$ peeps out from the Fourier transform of $\tilde{\rho}$.

Here is another variant on Guinand's construction. Let $\chi(k)=0$ if $k \in 4\mathbb{Z}$, $\chi(k)=2$ if $k \in 4\mathbb{Z}+2$, and $\chi(k)=-1$ if $k \in 4\mathbb{Z}\pm 1$. Then the Fourier transform of $\sigma = \sum_{-\infty}^\infty \chi(k)\delta_{k/2}$ is σ. Following Guinand's approach, it implies $\sum_{\infty}^\infty \chi(k)\exp(-\pi(k^2/4)x) = (1/\sqrt{x})\sum_{-\infty}^\infty \chi(k)\exp(-\pi(k^2/4x))$ for $x>0$. This identity is now raised to the third power. We obtain $\sum_1^\infty \rho_3(n)\exp(-\pi(n/4)x) = (1/x^{3/2})\sum_1^\infty \rho_3(n)\exp(-\pi(n/4x))$, where

$$\rho_3(n) = \sum_{k_1^2+k_2^2+k_3^2=n} \chi(k_1)\chi(k_2)\chi(k_3).$$

Finally we consider the odd measure

$$\theta = \sum_1^\infty \frac{\rho_3(n)}{\sqrt{n}} \left(\delta_{\sqrt{n}/2} - \delta_{-\sqrt{n}/2}\right)$$

and conclude that the Fourier transform of θ is $-i\theta$. In this construction we did not face the issue of a derivative of the Dirac mass at the origin. This issue was solved by the cancellation provided by χ. This measure θ is distinct from the measure τ of *Theorem 4*.

Kolountzakis' theorem holds in any dimension. The tensor product $\mu = \mu_1 \otimes \mu_2 \otimes \cdots \otimes \mu_n$ between n copies of the measure of *Theorem 3* is a crystalline measure with the required properties.

The obvious identity

$$\sum_1^\infty r_3(n) n^{-1/2} \left(\delta_{\sqrt{n}} - \delta_{-\sqrt{n}}\right) = \sum_{k\in\mathbb{Z}^3, k\neq 0} \frac{1}{|k|} \left(\delta_{|k|} - \delta_{-|k|}\right)$$

paves the road to the examples of the next section.

Proof of Theorem 1

The proof of *Theorem 1* is based on *Theorem 5*:

Theorem 5. *Let $\alpha = (\alpha_1, \alpha_2, \alpha_3) \notin \mathbb{Z}^3$ and $\beta = (\beta_1, \beta_2, \beta_3) \notin \mathbb{Z}^3$. Then the distributional Fourier transform of the measure*

$$\sigma_{(\alpha,\beta)} = \sum_{k\in\mathbb{Z}^3} \frac{\exp(2\pi i k\cdot\beta)}{|k+\alpha|} \left(\delta_{|k+\alpha|} - \delta_{-|k+\alpha|}\right)$$

is

$$\mathcal{F}\left(\sigma_{(\alpha,\beta)}\right) = -i \exp(-2\pi i \alpha\cdot\beta)\sigma_{(\beta,-\alpha)}.$$

The proof of *Theorem 5* is postponed to make room for a few comments and for the proof of *Theorem 1*. Let us observe that $\sigma_{(\alpha,\beta)}$ is an odd measure. We have $\sigma_{(\alpha,\beta)} = \sigma_{(-\alpha,-\beta)}$ and $\sigma_{(\alpha,-\beta)} = \overline{\sigma_{(\alpha,\beta)}}$. Moreover $\sigma_{(\alpha,\beta)}$ is $\mathbb{Z}^3$ periodic in β and $\exp(2\pi i\alpha\cdot\beta)\sigma_{(\alpha,\beta)}$ is $\mathbb{Z}^3$ periodic in α. What happens if $\beta = 0$? The Fourier transform of the measure

$$\sigma_{(\alpha,0)} = \sum_{k\in\mathbb{Z}^3} \frac{1}{|k+\alpha|} \left(\delta_{|k+\alpha|} - \delta_{-|k+\alpha|}\right)$$

is not a measure and the cancellations that are introduced in $\sigma_{(\alpha,\beta)}$ by the phase factor $\exp(2\pi ik\cdot\beta)$ are playing a seminal role. If $1, \beta_1, \beta_2, \beta_3$ are linearly independent over $\mathbb{Q}$, then for $k\neq l$, $k, l\in\mathbb{Z}^3$, $|k+\beta| \neq |l+\beta|$. It implies $\int_x^{x+1} d|\sigma_{(\alpha,\beta)}|(t) \simeq x$, $x\to\infty$, and this estimate is optimal. Therefore, $\sigma_{(\alpha,\beta)}$ is a tempered measure that is not an almost periodic measure. The fourth construction will yield an almost periodic crystalline measure.

The support of $\sigma_{(\alpha,\beta)}$ is the set $\Lambda = \{\pm|k+\alpha|; k\in\mathbb{Z}^3\}$. Then every finite subset of $\Lambda\cap(0,\infty)$ is linearly independent over $\mathbb{Q}$ for every α in a residual set in the sense of Baire's category. Similarly let S denote the spectrum of $\sigma_{(\alpha,\beta)}$. Then every finite subset of $S\cap(0,\infty)$ is linearly independent over $\mathbb{Q}$ for every β in a residual set. *Theorem 5* and these two observations imply *Theorem 1*.

Lemma 5 shows that an odd crystalline measure $\mu = \sum_{\lambda\in\Lambda} a(\lambda)\delta_\lambda$ whose spectrum S is linearly independent over $\mathbb{Q}$ cannot be an almost periodic measure.

Lemma 5. *With the preceding notations let us assume that μ is an odd crystalline measure and that $S\cap(0,\infty)$ is $\mathbb{Q}$-linearly independent. Then μ is not an almost periodic measure.*

We argue by contradiction and suppose that for every compactly supported continuous function ϕ the convolution product $f(x) = \mu\star\phi = \sum_{\lambda\in\Lambda} a(\lambda)\phi(x-\lambda)$ is an almost periodic function in the sense of Bohr. We are assuming now that ϕ is even, is real valued, is supported by a small interval $[-\eta,\eta]$ (specified below), and does not belong to the Wiener algebra $A(\mathbb{R})$. The Wiener algebra (15) is the algebra consisting of Fourier transforms of functions in $L^1(\mathbb{R})$. Using [**5**] we have $f(x) = \sum_{s\in S} b(s)\hat{\phi}(s)\exp(2\pi isx)$. Let $z_s\in\mathbb{C}$ be any sequence of $\pm i$ such that $z_{-s} = \bar{z}_s, s\in S$. Then there exists a sequence x_k tending to infinity such that $\exp(2\pi ix_k s)\to z_s, s\in S$, as k tends to infinity. This implies $f(x_k)\to\sum_{s\in S} z_s b(s)\hat{\phi}(s)$ and $\left|\sum_{s\in S, s>0} b(s)\hat{\phi}(s)(z_s-\bar{z}_s)\right| \leq \|f\|_\infty$. By an appropriate choice of $z_s = \pm i, s\in S$, we obtain $\sum_{s\in S, s>0}|b(s)\hat{\phi}(s)| \leq 2\|f\|_\infty$. Therefore, $f(x)$ locally belongs to the Wiener algebra. It means that $f g\in A(\mathbb{R})$ for every compactly supported test function g. If $a(\lambda_0)\neq 0$ and if the support of the continuous function ϕ is contained in $[-\eta,\eta]$ where η is small enough, then $f(x)$ coincides with $a(\lambda_0)\phi(x-\lambda_0)$ on $[\lambda_0-\eta, \lambda_0+\eta]$. This implies that ϕ belongs to the Wiener algebra. We reach a contradiction.

Returning to *Theorem 5* it is interesting to let α and β tend to 0. Then the limit $\sigma_{(0,0)}$ of $\sigma_{(\alpha,\beta)}$ is the Guinand distribution

$$-2\frac{d}{dt}\delta_0 + \sum_{k\in\mathbb{Z}^3; k\neq 0} \frac{1}{|k|} \left(\delta_{|k|} - \delta_{-|k|}\right).$$

Therefore, *Theorem 5* gives another proof of *Lemma 4*: The Fourier transform of the Guinand distribution $\sigma_{(0,0)}$ is $-i\sigma_{(0,0)}$.

Are there n-dimensional crystalline measures that are not constructed as a tensor product between one-dimensional crystalline measures? Here is the answer. Let $\Gamma\subset\mathbb{R}^{n-1}\times\mathbb{R}^3$ be an oblique lattice. It means that the two coordinate maps $p_1: \mathbb{R}^{n-1}\times\mathbb{R}^3 \mapsto \mathbb{R}^{n-1}$, $p_2: \mathbb{R}^{n-1}\times\mathbb{R}^3\mapsto\mathbb{R}^3$, once restricted to Γ, are injective with a dense range. Let Γ^* be the dual lattice and let us assume that $\mathrm{vol}(\Gamma) = 1$. Then we have the following:

Theorem 6. *Let $\alpha\notin\Gamma$ and $\beta\notin\Gamma^*$. Then the atomic measure $\sigma_\Gamma^{[\alpha,\beta]}$ defined on $\mathbb{R}^n = \mathbb{R}^{n-1}\times\mathbb{R}^1$ by*

$$\sum_{\gamma\in\Gamma+\alpha} \frac{\exp(2\pi i\beta\cdot\gamma)}{|p_2(\gamma)|} \left(\delta_{(p_1(\gamma), |p_2(\gamma)|)} - \delta_{(p_1(\gamma), -|p_2(\gamma)|)}\right)$$

is crystalline and its Fourier transform is

$$\mathcal{F}\left(\sigma_\Gamma^{[\alpha,\beta]}\right) = -i \exp(2\pi i\alpha\cdot\beta)\, \sigma_{\Gamma^*}^{[\beta,-\alpha]}.$$

We return to the proof of *Theorem 5*. *Theorem 5* is a corollary of a more general statement:

Theorem 7. *Let μ be a crystalline measure on $\mathbb{R}^3$. We then have $\mu = \sum_{\lambda\in\Lambda} a(\lambda)\delta_\lambda$ and $\hat{\mu} = \sum_{y\in S} b(y)\delta_y$. Let us assume that $0\notin\Lambda$, $0\notin S$, and consider the one-dimensional measure*

$$\sigma_\Lambda = \sum_{\lambda\in\Lambda} \frac{a(\lambda)}{|\lambda|} \left(\delta_{|\lambda|} - \delta_{-|\lambda|}\right). \qquad [13]$$

Then σ_Λ is a crystalline measure and the distributional Fourier transform of σ_Λ is $-i\sigma_S$, where $\sigma_S = \sum_{y\in S}(b(y)/|y|)(\delta_{|y|} - \delta_{-|y|})$.

We prove *Theorem 7*. The measures σ_Λ and σ_S are odd. To check the identity

$$\langle\sigma_\Lambda, \hat{\phi}\rangle = -i\langle\sigma_S, \phi\rangle \qquad [14]$$

for every test function ϕ it suffices to do it for every odd ϕ. Let $\omega = \hat{\phi}$ be the 1D Fourier transform of ϕ. Then ω is also an odd function in the Schwartz class $\mathcal{S}(\mathbb{R})$ and the left-hand side of [**14**] is

$$s(\omega) = 2\sum_{\lambda\in\Lambda} a(\lambda)\frac{\omega(|\lambda|)}{|\lambda|}. \quad [15]$$

We introduce the radial function $\Phi(x) = \omega(|x|)/|x|$, which belongs to $\mathcal{S}(\mathbb{R}^3)$. Then

$$s(\omega) = 2\sum_{\lambda\in\Lambda} a(\lambda)\Phi(\lambda). \quad [16]$$

We have for every test function F

$$\sum_{\lambda\in\Lambda} a(\lambda)\hat{F}(\lambda) = \langle \mu, \hat{F}\rangle = \langle \hat{\mu}, F\rangle = \sum_{y\in S} b(y)F(y). \quad [17]$$

Lemma 6. *The* 3D *Fourier transform of the radial function* $F(y) = -i\phi(|y|)/|y|$ *is* $\hat{F}(x) = \Phi(x)$.

Indeed the 3D Fourier transform $\hat{F}$ of a radial function $F \in L^1(\mathbb{R}^3)$ is given by

$$\hat{F}(x) = 4\pi\int_0^\infty F(r)\frac{\sin(2\pi|x|r)}{2\pi|x|r}r^2\,dr. \quad [18]$$

We apply this to $F(y) = -i(\phi(|y|)/|y|)$. Then we are left with $\hat{F}(x) = (-2i/|x|)\int_0^\infty \phi(r)\sin(2\pi|x|r)dr$. Because ϕ is an odd function in $\mathcal{S}(\mathbb{R})$, this integral is the 1D Fourier transform of ϕ, which is precisely ω. We have proved the identity

$$-i\mathcal{F}\left(\frac{\phi(|\cdot|)}{|\cdot|}\right) = \frac{\omega(|\cdot|)}{|\cdot|}, \quad [19]$$

which is *Lemma 6*.

Then **[16]**, **[17]**, and *Lemma 6* yield

$$s(\omega) = -2i\sum_{y\in S} b(y)\frac{\phi(|y|)}{|y|}.$$

This is $-i\langle\sigma_S, \phi\rangle$, which ends the proof.

Theorem 5 is a corollary of *Theorem 7*. Indeed by *Lemma 1* the Fourier transform of $\sum_{k\in\mathbb{Z}^3}\exp(2\pi ik\cdot\beta)\delta_{k+\alpha}$ is $\exp(-2\pi i\beta\cdot\alpha)\sum_{k\in\mathbb{Z}^3}\exp(-2\pi ik\cdot\alpha)\delta_{k+\beta}$.

This gives a new proof of *Theorem 4*. One starts with $\tilde{\chi}:\mathbb{Z}^3\mapsto\{-1/2, 0, 4\}$ defined by $\tilde{\chi}(k) = 0$ if $k\in 4\mathbb{Z}^3$, $\tilde{\chi}(k) = 4$ if $k\in 2\mathbb{Z}^3\backslash 4\mathbb{Z}^3$, and $\tilde{\chi}(k) = -1/2$ if $k\in\mathbb{Z}^3\backslash 2\mathbb{Z}^3$. Then the 3D Fourier transform of $\mu = \sum_{k\in\mathbb{Z}^3}\tilde{\chi}(k)\delta_{k/2}$ is identical to μ. It suffices to use *Theorem 7* to conclude.

Theorem 5 will be proved if we can show that

$$s(\phi) = \left\langle \sigma_\Gamma^{[\alpha,\beta]}, \hat{\phi}\right\rangle = -i\,\exp(2\pi i\alpha\cdot\beta)\left\langle\sigma_{\Gamma^*}^{[\beta,-\alpha]}, \phi\right\rangle \quad [20]$$

holds for every test function $\phi = \phi(u, v)$, $u\in\mathbb{R}^{n-1}, v\in\mathbb{R}$. It suffices to do it when $\phi(u,v) = \phi_1(u)\phi_2(v)$ because this collection is total in the space of test functions. The measures $\sigma_\Gamma^{[\alpha,\beta]}$ and $\sigma_{\Gamma^*}^{[\beta,-\alpha]}$ are odd in the second variable v and we can restrict the proof of **[20]** to odd test functions ϕ_2. We apply the ordinary Poisson formula to the coset $\Gamma^*+\beta$ of the lattice Γ^* and to the function $F(x) = \exp(2\pi ix\cdot\beta)\phi_1(x_1)(\phi_2(|x_2|)/|x_2|)$, where $x = (x_1, x_2)$, $x_1\in\mathbb{R}^{n-1}$, $x_2\in\mathbb{R}^3$. Then the proof of **[20]** is identical to the proof of *Theorem 4*.

We now return to *Theorem 2*. Let Λ be the support of $\sigma_\Gamma^{[\alpha,\beta]}$. For almost every α the set $\Lambda\cap\{x_n > 0\} = \{(p_1(\gamma), |p_2(\gamma+\alpha)|);\ \gamma\in\Gamma\}$ is linearly independent over $\mathbb{Q}$ and the same holds for the spectrum of $\sigma_\Gamma^{[\alpha,\beta]}$. This ends the proof.

Kolountzakis' Theorem

The construction that is detailed below was discovered independently by the author and by Kolountzakis (5). It is nothing but the 2-adic analog of the approach by Lev and Olevskii in (4). The crystalline measure σ is given by a series

$$\sigma = \sum_0^\infty \epsilon_j\sigma_j, \quad [21]$$

where

i) The atomic measure σ_j is 2^j periodic and its support is contained in $M_j = 2^{-j-1}+\Lambda_j$, where $\Lambda_j = 2^{-j}\mathbb{Z}\backslash[-2^{j-3}, 2^{j-3}]$. More precisely, this support is the union $\cup_{k\in\mathbb{Z}}(2^{-j}\mathbb{Z}\cap I_k)$, where I_k is the interval centered at $2^{-j-1}+(k+1/2)2^j$ with length 2^{j-2}.

ii) The support of the Fourier transform $\hat{\sigma}_j$ of σ_j is contained in Λ_j.

iii) The choice of $\epsilon_j > 0$ ensures the convergence of the series $\sigma = \sum_0^\infty \epsilon_j\|\sigma_j\|_{\mathcal{P}}$.

Let us observe that the sets M_j, $j\geq 0$, are pairwise disjoint in such a way that σ is not a generalized Dirac comb. Moreover, $\cup_0^\infty M_j$ is a locally finite set. The support Λ of σ is contained in $\mathbb{Q}$. In ref. 5 Kolountzakis used another construction of σ for which the dimension of the linear span of Λ over the field $\mathbb{Q}$ is infinite.

How does one construct these σ_j? It suffices to use *Lemma 7*:

Lemma 7. *Let* $\alpha\in(0, 1/6)$. *For every integer* $N\geq N_\alpha$ *there exists an* N*-periodic atomic measure* $\sigma = \sigma_N$ *that is a sum of Dirac masses on* $\Lambda_N = N^{-1}\mathbb{Z}\backslash[-\alpha N, \alpha N]$ *and whose Fourier transform is also supported by* Λ_N. *Moreover, the support of* σ *is* $\tilde{\Lambda}_N = \cup_{k\in\mathbb{Z}}I_k$, *where* I_k *is the interval centered at* $(k+1/2)N$ *with length* $(1-2\alpha)N$.

Let us prove *Lemma 7*. If σ is N periodic, we have

$$\sigma = \tau\star\nu, \quad [22]$$

where ν is the Dirac comb $\sum_{k\in\mathbb{Z}}\delta_{kN}$ and

$$\tau = \sum_0^{N^2-1} c_k\delta_{k/N}.$$

It implies

$$\hat{\sigma} = N^{-1}\sum_{m\in\mathbb{Z}} P(m)\delta_{m/N} \quad [23]$$

with

$$P(y) = \sum_0^{N^2-1} c_k\,\exp\left(-2\pi ikyN^{-2}\right). \quad [24]$$

Finally N-periodic measures on $N^{-1}\mathbb{Z}$ are in a 1–1 correspondence with trigonometric polynomials given by **[24]**. We now use *Lemma 8*:

Lemma 8. *Let* $M\in\mathbb{N}$, $M\geq 2$, *and let* $E, F\subset\mathbb{Z}/M\mathbb{Z}$ *be two sets of cardinality* $|E|, |F|$. *If* $|E|+|F| < M$, *there exists a nontrivial trigonometric polynomial*

$$P(y) = \sum_0^{M-1} c_k\,\exp(2\pi iky/M) \quad [25]$$

such that

$$c_k = 0,\quad k\in E,\quad P(y) = 0,\ y\in F. \quad [26]$$

Moreover, if E *and* F *are two intervals and* $|E|+2|F| < M$, *we can impose* $c_k\neq 0$ *for every* $k\notin E$.

A simple dimension-counting argument implies the first statement. The proof of the second statement runs by contradiction. Assuming that the linear space G defined by **[26]** does not intersect the open set defined by $c_k \neq 0,\ \forall k \notin E$, it implies that for some $k_0 \notin E$ the linear space G is contained in the hyperplane defined by $c_{k_0} = 0$. By linear algebra it implies that coefficients $a_k,\ k \in E, b_l, l \in F$, exist such that for every trigonometric polynomial $P(y) = \sum_0^{M-1} c_k \exp(2\pi iky/M)$ we have

$$c_{k_0} = \sum_{k \in E} a_k c_k + \sum_{l \in F} b_l P(l). \qquad [27]$$

It implies

$$\sum_{l \in F} b_l \exp(2\pi i k_0 l/M) = 1 \qquad [28]$$

$$\sum_{l \in F} b_l \exp(2\pi ikl/M) = 0, k \in \left(E \cup \{k_0\}\right)^c. \qquad [29]$$

Because $M > |E| + 2|F|$, the set $(E \cup \{k_0\})^c$ contains an interval J of length equal to $|F|$ and the matrix

$$((\exp(2\pi ikl/M)))_{k \in J, l \in F}$$

is an invertible Vandermonde matrix. Therefore, **[29]** implies $b_l = 0,\ \forall l \in F$, which contradicts **[28]**.

We now conclude the proof of *Lemma 7*. Our first demand is that σ be an N-periodic measure carried by $N^{-1}\mathbb{Z} \backslash (N\mathbb{Z} + [-\alpha N, \alpha N])$. The restriction of this measure σ to $[0, N)$ is $\tau_N = \sum_{k \in T_N} c(k, N)\delta_{kN^{-1}}$, where $T_N = \mathbb{Z} \cap (\alpha N^2, (1-\alpha)N^2)$. Then **[23]** yields

$$\hat{\sigma} = N^{-1} \sum_{-\infty}^{\infty} P(l)\delta_{lN^{-1}}, \qquad [30]$$

where

$$P(y) = \hat{\tau}_N(y) = \sum_{k \in T_N} c(k, N)\exp\left(-2\pi ikN^{-2}y\right). \qquad [31]$$

We have $(1-2\alpha)N^2 - 1 \leq |T_N| \leq (1-2\alpha)N^2 + 1$. *Lemma 8* with $E_N = [0, N^2 - 1] \backslash T_N$ and $F_N = ([0, \alpha N^2] \cup [(1-\alpha)N^2, N^2]) \cap \mathbb{Z}$ yields nontrivial coefficients $c(k,j)$ such that $P_j(l) = 0$ when $|l| \leq \alpha N^2$, which ends the proof. It works if $|E_N| + 2|F_N| < N^2$, which reads $\alpha < 1/6$. We set $\alpha = 1/8$ in the construction of σ_j.

The Crystalline Measures of Lev and Olevskii

Let $\Gamma \subset \mathbb{R}^n \times \mathbb{R}$ be an oblique lattice. As introduced earlier it means that the two projections $p_1 : \mathbb{R}^n \times \mathbb{R} \mapsto \mathbb{R}^n$, $p_2 : \mathbb{R}^n \times \mathbb{R} \mapsto \mathbb{R}$, once restricted to Γ, are injective with a dense range. Let $I = [-a, a]$. The model set Λ_I is defined by the standard cut and projection scheme (11, 13, 16). Then

$$\Lambda_I = \{\lambda = p_1(\gamma); \gamma \in \Gamma, p_2(\gamma) \in I\}. \qquad [32]$$

Let $0 < h_1 < h_2 < \ldots$ be an increasing sequence of positive numbers tending to infinity. We set $I_1 = [-h_1, h_1] \subset I_2 = [-h_2, h_2] \subset \ldots$ and we have $\cup_1^\infty I_k = \mathbb{R}$. The corresponding sequence of model sets is $\Lambda_k,\ k \in \mathbb{N}$. We have

$$\Lambda_1 \subset \Lambda_2 \subset \ldots \subset \Lambda_k \subset \ldots \qquad [33]$$

and $\cup_1^\infty \Lambda_k = p_1(\Gamma)$ is dense in $\mathbb{R}^n$.

Enriched model sets are defined by Lev and Olevskii as follows:

Definition 7: Let $a_0 = 0 < a_1 < a_2 < \ldots$ be an increasing sequence of positive numbers tending to infinity. An enriched model set is defined by

$$\tilde{\Lambda} = \bigcup_1^\infty \tilde{\Lambda}_k, \qquad [34]$$

where

$$\tilde{\Lambda}_k = \{\lambda \in \Lambda_k; |\lambda| \geq a_{k-1}\}. \qquad [35]$$

Lemma 9. *Let $E = \{k + m\sqrt{2}; (k, m) \in \mathbb{N}^2\}$. Then $\Lambda = E \cup (-E)$ is an enriched model set.*

For proving *Lemma 9* it suffices to use *Definition 7* with

$$\Gamma = \left\{\left(k + m\sqrt{2}, k - m\sqrt{2}\right); k, m \in \mathbb{Z}\right\}, \quad h_k = a_k = k.$$

An enriched model set is locally finite. A model set is never an enriched model set. The density of a model set is finite whereas the density of an enriched model set is infinite. Lev and Olevskii proved the following:

Theorem 8. *Every enriched model set $\tilde{\Lambda}$ contains the support of a measure μ such that*

i) μ is not a generalized Dirac comb, and
ii) the Fourier transform $\hat{\mu}$ of μ is also supported by an enriched model set S.

Let us sketch the construction of this measure μ.

We have $\mathbb{R}^2 = A \cup B,\ A \cap B = \emptyset$, where

$$A = \bigcup_1^\infty \{(x, y); |x| \geq a_{n-1}, |y| \leq h_n\} \qquad [36]$$

$$B = \bigcup_1^\infty \{(x, y); |x| < a_n, |y| > h_n\}. \qquad [37]$$

A similar partition into A^* and B^* is provided by two other sequences a_n^*, h_n^*, $n \in \mathbb{N}$. The three sequences a_n, h_n, a_n^*, $n \in \mathbb{N}$, are arbitrary but the last one h_n^* is the result of a subtle induction. The enriched model set defined by A is

$$\Lambda = \{p_1(\gamma); \gamma \in \Gamma \cap A\}. \qquad [38]$$

Similarly

$$Q = \{p_2(\gamma); \gamma \in \Gamma \cap B\} \qquad [39]$$

$$S = \left\{p_1^*(\gamma^*); \gamma^* \in \Gamma^* \cap A^*\right\} \qquad [40]$$

$$Z = \left\{p_1^*(\gamma^*); \gamma^* \in \Gamma^* \cap B^*\right\}. \qquad [41]$$

Lemma 10 is seminal in the proof.

Lemma 10. *There exist a sequence h_n^* and a nontrivial function $\phi \in \mathcal{S}(\mathbb{R})$ such that*

$$\phi = 0 \text{ on } Z, \quad \hat{\phi} = 0 \text{ on } Q. \qquad [42]$$

Then μ is defined by

$$\mu = \sum_{(x,y) \in \Gamma} \hat{\phi}(y)\delta_x = \sum_{(x,y) \in \Gamma \cap A} \hat{\phi}(y)\delta_x = \sum_{\lambda \in \Lambda} \hat{\phi}\left(\overline{\lambda}\right)\delta_\lambda \qquad [43]$$

with $(\lambda, \overline{\lambda}) \in \Gamma$. It implies

$$\hat{\mu} = \sum_{(u,v)\in\Gamma^*} \phi(v)\delta_u = \sum_{(u,v)\in\Gamma^*\cap A^*} \phi(v)\delta_u = \sum_{u\in S} \phi(\overline{u})\delta_u. \qquad [44]$$

Theorem 9. *If ϕ is defined by Lemma 10, then the measure μ defined by* [**43**] *is carried by the locally finite set Λ defined in* [**38**] *and its Fourier transform $\hat{\mu}$ is carried by the locally finite set S defined by* [**40**].

These properties follow from the construction of μ and the reader is referred to ref. 4 for the proof of *Lemma 10*. Our fourth approach was inspired by this construction.

The Geometry of Crystalline Measures

The structure of crystalline measures is still unclear. Some crystalline measures are almost periodic measures and others are not. What can be said about the geometrical properties of the support of a crystalline measure? Let us denote by $\mathcal{L}$ the collection of these supports. Is it possible to characterize this collection $\mathcal{L}$ by additive properties? On the one hand there exists a locally finite set $T\subset(0,\infty)$ that is linearly independent over $\mathbb{Q}$ and such that $\Lambda = T\cup(-T)\in\mathcal{L}$. This is *Theorem 1*. But there are infinitely many locally finite sets $T\subset(0,\infty)$ that are linearly independent over $\mathbb{Q}$ and such that $\Lambda = T\cup(-T)\notin\mathcal{L}$. On the other hand $\Lambda=\mathbb{Z}$ belongs to $\mathcal{L}$. Sitting in between, the support of the crystalline measure discovered by Kolountzakis is contained in $\mathbb{Q}$.

Do there exist uniformly discrete sets $\Lambda\in\mathcal{L}$? A trivial answer is given by Dirac combs. Are there other examples? *Theorem 3* implies that in one dimension the spectrum of such a measure μ cannot be a uniformly discrete set.

Moving a single point in a set $\Lambda\in\mathcal{L}$ can be destructive as the following example shows. We start from $\mathbb{Z}$ and move 0 to $1/2$. Then the resulting set does not belong to $\mathcal{L}$.

Lemma 11. *The set $\Lambda = \{1/2\}\cup\mathbb{Z}\backslash\{0\}$ does not belong to $\mathcal{L}$.*

We argue by contradiction. If Λ was the support of a crystalline measure μ, we would have $\hat{\mu}(y) = c\,\exp(\pi i y) + F(y)$, where $F(y)$ is 1 periodic and c is a constant. Therefore, $\hat{\mu}(y+1)-\hat{\mu}(y) = -2c\exp(\pi i y)$ would coincide with a purely atomic measure. It implies $c=0$, $\mu(\{1/2\})=0$, and Λ is not the support of μ. Let us observe that $\mathbb{Z}\backslash\{0\}$ belongs to $\mathcal{L}$. Indeed if σ is the Dirac comb on $\mathbb{Z}$, we set $\mu = P\sigma, P(x) = \sin(2\pi\sqrt{2}x)$. Then μ is a generalized Dirac comb whose support is $\mathbb{Z}\backslash\{0\}$.

Excepting Dirac combs, do there exist nonnegative crystalline measures?

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