Géométrie et intégrabilité

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ENS Cachan, 16 juin 2016.

Outline

The two fundamental quadratic forms

Gauss 1827

(Surfaces in $\mathbb{R}^3$) $\Leftrightarrow$ (two “fundamental” quadratic forms):
$\mathrm{I} =< \mathrm{d}\mathbf{F}, \mathrm{d}\mathbf{F} >$, $\mathrm{II} = - < \mathrm{d}\mathbf{F}, \mathrm{d}\mathbf{N} >$,
$\mathbf{F}(x_1, x_2)$:=point on the surface, d$\mathbf{F}$=vector in the tangent plane,
$\mathbf{N}$:=any unit vector normal to the tangent plane.
In “conformal” coordinates, these quadratic forms

$$\mathrm{I} =< \mathrm{d}\mathbf{F}, \mathrm{d}\mathbf{F} >= e^u \mathrm{d}z \, \mathrm{d}\bar{z},$$
$$\mathrm{II} = - < \mathrm{d}\mathbf{F}, \mathrm{d}\mathbf{N} >= Q \, \mathrm{d}z^2 + e^u H \mathrm{d}z \, \mathrm{d}\bar{z} + \overline{Q} \, \mathrm{d}\bar{z}^2,$$

define three fields: u real, Q complex, H real.
Link with the two principal curvatures $1/R_1$ and $1/R_2$:

$$\frac{1}{2}\left(\frac{1}{R_1} + \frac{1}{R_2}\right) = \text{mean curvature} = H,$$
$$\frac{1}{R_1 R_2} = \text{total(=Gaussian) curvature} = H^2 - 4e^{-2u}|Q|^2 = -2e^{-u} u_{z\bar{z}}.$$

The moving frame equations, a linear system

Weingarten, Acta mathematatica 1897; Bobenko 1994; Bobenko and Eitner 2000

The moving frame
$\sigma = {}^{\mathrm{t}}(\mathbf{F}_z, \mathbf{F}_{\bar{z}}, \mathbf{N})$
evolves linearly as $\sigma_z = \mathbb{U}\sigma,\ \sigma_{\bar{z}} = \mathbb{V}\sigma$, and these third order matrices can be represented by second order traceless matrices (Bobenko and Eitner 2000)

$$\mathbb{U} = \begin{pmatrix} u_z/4 & -Qe^{-u/2} \\ (H+c)e^{u/2}/2 & -u_z/4 \end{pmatrix}, \mathbb{V} = \begin{pmatrix} -u_{\bar{z}}/4 & -(H-c)e^{u/2}/2 \\ \overline{Q}e^{-u/2} & u_{\bar{z}}/4 \end{pmatrix}.$$

c:=additional parameter when replacing $\mathbb{R}^3$ by ($\mathbb{S}^3$, $\mathbb{R}^3$ or $\mathbb{H}^3$) $\subset \mathbb{R}^4$.

The three nonlinear equations

Gauss 1827; Karl Peterson 1853 = Mainardi 1856 = Codazzi 1868. Also Bour, Bonnet

The condition $(\sigma_z)_{\bar z} = (\sigma_{\bar z})_z$ generates three nonlinear PDEs,

$$\begin{cases} u_{z\bar z} + \frac{1}{2}(H^2 - c^2)e^u - 2|Q|^2 e^{-u} = 0, \ \text{(Gauss)} \\ Q_{\bar z} - \frac{1}{2} H_z e^u = 0, \ \overline{Q}_z - \frac{1}{2} H_{\bar z} e^u = 0. \ \text{(Codazzi)} \end{cases}$$

This system (**Gauss-Codazzi** equations) is **underdetermined**, i.e. one additional condition can be enforced.
(Any solution $(u, H, Q, \overline{Q})$) $\Longrightarrow$ (unique surface up to rigid motion (déplacement)).

Outline

Global (explicit) integration *vs.* local

See for instance Cargèse lecture notes, http://arXiv.org/abs/solv-int/9710020

Problem: given some ODE, to integrate it by a closed form expression valid in the whole domain of definition.

Example: the ODE $u' + u^2 = 0$ is integrated by $u = 1/(x - c)$, c=arbitrary constant.

Counterexample: the ODE $u' + u^2 = 0$ is NOT integrated by $u = u_0 \sum_{j=0}^{+\infty} [-(x - x_0)u_0]^j$ (solution of Cauchy).

Fact: whenever one succeeds to integrate, the closed form expression is built from solutions of either any order linear ODEs or same or lower order nonlinear ODEs.

(Bad) vocabulory: these elementary functions are often called "special functions", while they are just functions.

$\mathrm{P_{VI}}$, the unique special function beyond the elliptic fn.

L. Fuchs, H. Poincaré, É. Picard, R. Fuchs, P. Painlevé, B. Gambier

The so-called "special functions" belong to 3 disjoint subsets:
($\exists$ LODE) e^x, x^n, Gauss ${}_1F_2(a,b,c;x)$, Bessel $J_\nu(x)$, Airy $\mathrm{Ai}(x)$,
($\exists$ NLODE) elliptic function $\wp(x,g_2,g_3)$ (Weierstrass, Jacobi),
($\nexists$ ODE) $\Gamma(x)$ (Hölder 1886), Riemann $\zeta(x)$, ...
Can this set be extended in some systematic way?

Problem (L. Fuchs, H. Poincaré). To define (new) functions by algebraic ordinary differential equations.

Solution (Painlevé 1900, Gambier 1910):
First order ODEs. Except linearizable ODEs, only **one new**: the elliptic function.
Second order ODEs. Except linearizable and elliptic ODEs, only **six new**: the master function $\mathrm{P_{VI}}$ and its 5 degeneracies $\mathrm{P_V}$–$\mathrm{P_I}$.

P_{VI} function/equation. Ex nihilo definition

Picard, Journal de Liouville 1889; R. Fuchs, CRAS 1905; Painlevé, CRAS 1906

Une équation différentielle curieuse (Picard, 1889).

The exponential function is defined by the equation $u' = u$.
Similarly, the P_{VI} function is **defined** by the P_{VI} equation

$$- u'' + \frac{1}{2}\left[\frac{1}{u} + \frac{1}{u-1} + \frac{1}{u-x}\right] u'^2 - \left[\frac{1}{x} + \frac{1}{x-1} + \frac{1}{u-x}\right] u'$$
$$+ \frac{u(u-1)(u-x)}{2x^2(x-1)^2}\left[\theta_\infty^2 - \theta_0^2 \frac{x}{u^2} + \theta_1^2 \frac{x-1}{(u-1)^2} + (1-\theta_x^2)\frac{x(x-1)}{(u-x)^2}\right] = 0.$$

(Painlevé) Property: its general solution is singlevalued except at the fixed singularities $x = \infty, 0, 1$.

$\mathrm{P_{VI}}$. Beautiful physical definition

R. Fuchs, CRAS 141 (1905) 555–558; Painlevé, CRAS 143 (1906) 1111–1117; rediscovered by Manin 1998, Babich and Bordag 1999

Under the point transformation $(u, x) \to (U, X)$

$$\begin{cases} \text{crossratio } (u, 0, 1, x) = (\wp(U), e_1, e_2, e_3), \\ X = \text{ratio of the two periods of } \wp, \end{cases}$$

$\mathrm{P_{VI}}$ can be written as the "physical" Hamiltonian ($Q \equiv U$)

$$H = \frac{P^2}{2} + V(Q, X), \; V = \sum_{j=\infty,0,1,x} \theta_j^2 \wp(Q + \omega_j(X)),$$

$$\frac{\mathrm{d}^2 Q}{\mathrm{d}X^2} + \frac{\partial V}{\partial Q} = 0,$$

in which $\omega_j(X)$ are the four (X-dependent!) half-periods $0, \omega_1, \omega_2, \omega_1 + \omega_2$ of the Weierstrass elliptic function $\wp$.

Outline

Motivation for finding a $\mathrm{P_{VI}}$ solution to geometry

- In $\mathbb{R}^3$, looking for applicable surfaces conserving R_1 and R_2, Bonnet (1867) isolated, among others, the *Bonnet surfaces*, in which H is the Hamiltonian of a codim-3 $\mathrm{P_{VI}}$,

$$(\theta_\infty^2, \theta_0^2, \theta_1^2, \theta_X^2) = (0, \theta_0^2, \theta_0^2, 1),\ \theta_0 \text{ arbitrary.}$$

- In $\mathbb{S}^3$, assuming $1/H$ harmonic, BEK (1997) found the general solution; $\Re(Q)$ is the Hamiltonian of a codim-2 $\mathrm{P_{VI}}$

$$(\theta_\infty^2, \theta_0^2, \theta_1^2, \theta_X^2) = (0, \theta_0^2, \theta_1^2, 1),\ \theta_0, \theta_1 \text{ arbitrary.}$$

- In $\mathbb{R}^3$, the isothermic case $Q = \overline{Q}$ passes the Painlevé test (Cieśliński, Goldstein, Sym 1995).
- $\mathrm{P_{VI}}$ is **complete** (impossible to add terms and preserve the PP).
- Gauss-Codazzi **completely** describes the geometry.

Invariances of Gauss-Codazzi equations

Bobenko 1994; Springborn 2002

First: conformal invariance

$$\forall G(z): \ (z, e^u, H, Q) \to \left(G(z), |G'(z)|^2 e^u, H, G'(z)^2 Q\right).$$

Second: under the condition $Q - \overline{Q} = c$, the involution,

$$(u, H, Q, \overline{Q}) \to \left(-u, 2Q - c = 2\overline{Q} + c, \frac{H+c}{2}, \frac{H-c}{2}\right).$$

Conclusion: the solutions of Bonnet 1867 and of BEK 1997 are exchanged by this involution.

$\mathbb{R}^3$. The solution *Bonnet surface*

Bonnet, J.Éc. polytechnique (1867) p. 84; Haddidakis, Crelle's journal 117 (1897) p. 43; Bobenko and Eitner 2000

$(1/Q = f(z) + \bar{f}(\bar{z})) \Rightarrow e^u$ and H only depend on $\xi = \Re(w)$, with $\mathrm{d}w/\mathrm{d}z = \mathrm{d}z/\mathrm{d}f$,

$$Q = \frac{4c_{\mathrm{z}} \sinh(2c_{\mathrm{z}} \bar{w})}{\sinh(2c_{\mathrm{z}} w) \sinh(4c_{\mathrm{z}} \Re(w))}, e^u = v(\xi),\ H = h(\xi),\ vh' = -\frac{16c_{\mathrm{z}}^2}{\sinh^2(4c_{\mathrm{z}}\xi)}.$$

The moving frame equations provide the first integral (Hazzidakis 1897)

$$K = \left(\frac{v'}{v}\right)^2 + 16\left[(h^2 - c^2)v + \frac{(4c_{\mathrm{z}})^2}{\sinh^2(4c_{\mathrm{z}}\xi)} v^{-1} - 2(4c_{\mathrm{z}} \coth(4c_{\mathrm{z}}\xi))h\right].$$

h is the Hamiltonian (Chazy 1911) of a codimension-2 $\mathrm{P_{VI}}$, first recognized as such by Bobenko and Eitner, with

$$(\theta_\infty^2, \theta_0^2, \theta_1^2, \theta_X^2) = (0, \theta_0^2, \theta_0^2, 1),\ \theta_0 \text{ arbitrary}.$$

$\mathbb{R}^3$. The solution $1/H$ harmonic (dual of Hazzidakis)

Bobenko, Eitner, Kitaev, Geometriae dedicata 1997

Assuming $(1/H)_{z\bar{z}} = 0$, one obtains $c = 0$ and

$$\frac{1}{H} = \frac{2c_{\mathrm{z}}}{c_{\mathrm{h}}} \left(\coth(2c_{\mathrm{z}} z) + \coth(2c_{\mathrm{z}} \bar{z})\right).$$

Then $e^u H^2$ and Q only depend on x,

$$e^u H^2 = 4c_{\mathrm{u}} v(x),\ \ Q = \frac{4c_{\mathrm{q}}}{c_{\mathrm{h}}} \left(\frac{2c_{\mathrm{z}}}{\sinh(2c_{\mathrm{z}} x)}\right)^2 (q(x) + i\theta),\ \ v = \frac{c_{\mathrm{q}}}{c_{\mathrm{u}}} q',$$

in which θ is an arbitrary constant. First integral from GW eqs,

$$K = \left(\frac{v'}{v} + 8c_{\mathrm{z}} \coth(4c_{\mathrm{z}} x)\right)^2$$

$$+ 16 \left(\frac{4c_{\mathrm{z}}}{\sinh(4c_{\mathrm{z}} x)}\right)^2 \frac{c_{\mathrm{q}}^2}{c_{\mathrm{u}}} (q + i\theta)(r - i\theta) + 16 c_{\mathrm{u}} v + 16 c_{\mathrm{q}} \frac{4c_{\mathrm{z}}}{\sinh(4c_{\mathrm{z}} x)} (q + r).$$

For $q = r$ (Q real), q is the Hamiltonian of a codim-2 $\mathrm{P_{VI}}$

$$(\theta_\infty^2, \theta_0^2, \theta_1^2, \theta_x^2) = (0, \theta_0^2, \theta_1^2, 1),\ \ \theta_0, \theta_1 \text{ arbitrary}.$$

Outline

Lie point symmetries

Infinite-dimensional Lie algebra (notation $U = e^u$)

$$\left\{ \begin{array}{l} X(F) = F(z)\partial_z + F'(z)(-2Q\partial_Q - U\partial_U), \\ Y(G) = G(\bar{z})\partial_{\bar{z}} + G'(\bar{z})(-2\overline{Q}\partial_{\overline{Q}} - U\partial_U), \\ (c = 0 \text{ only})\ a = -H\partial_H + Q\partial_Q + \overline{Q}\partial_{\overline{Q}} + 2U\partial_U, \end{array} \right. \quad (1)$$

F, G arbitrary functions of one variable.

Table of commutation

$$\left\{ \begin{array}{l} [X(F_1), X(F_2)] = X(F_1 F_2' - F_1' F_2), \\ [Y(G_1), Y(G_2)] = Y(G_1 G_2' - G_1' G_2), \\ [X(F), Y(G)] = 0,\ [X(F), a] = 0,\ [Y(G), a] = 0. \end{array} \right. \quad (2)$$

The largest finite-dimensional subalgebra has dimension 7

$$\left\{ \begin{array}{l} e_j = X(z^j),\ f_j = Y(\bar{z}^j),\ j = 0, 1, 2, \\ (c = 0 \text{ only})\ a = -H\partial_H + Q\partial_Q + \overline{Q}\partial_{\overline{Q}} + 2U\partial_U, \end{array} \right. \quad (3)$$

with the nonzero commutators

$$[e_0, e_1] = e_0,\ [e_0, e_2] = 2e_1,\ [e_1, e_2] = e_2,$$
$$[f_0, f_1] = f_0,\ [f_0, f_2] = 2f_1,\ [f_1, f_2] = f_2. \quad (4)$$

The (single) reduction defined by Lie point symmetries

RC, A.M. Grundland, 2016

$(e^u, H, Q, \overline{Q})(z, \bar{z}) \to (v, h, q, r)(\xi)$ (notation $g_1 = F$)

$$\left\{ \begin{array}{l} \xi = \log g_1(z) - \log g_2(\bar{z}),\ e^u = g_1^{2a_1-1} g_2^{2a_2-1} g_1' g_2' \tilde{v},\ H = g_1^{-a_1} g_2^{-a_2} \tilde{h}, \\ Q = g_1^{a_1-2} g_2^{a_2} {g_1'}^2 \tilde{q},\ \overline{Q} = g_2^{a_2-2} g_1^{a_1} {g_2'}^2 \tilde{r}, \end{array} \right.$$

to three ODEs in four variables $(v, h, q, r)(\eta = e^\xi)$, g arbitrary,

$$(1+g)c = 0 : \left\{ \begin{array}{l} \left(\eta \dfrac{v'}{v}\right)' - \dfrac{h^2 - c^2}{2\eta^2} v + 2 \dfrac{qr}{\eta^2 v} = 0, \\ -\eta v h' - 2\eta^2 q' + (1+g) v h = 0, \\ -\eta^2 v h' - 2\eta r' + 2(1-g) r = 0, \end{array} \right.$$

which admits the first integral

$$K = \left(\eta \frac{v'}{v} + g\right)^2 - \frac{(h^2 - c^2) v}{\eta} - 2(h+c) q - 2 \frac{(h-c)}{\eta^2} r - 4 \frac{qr}{\eta v}.$$

Three cases: $g^2 \neq 1$ (and $c = 0$), $g = 1$ (and $c = 0$), $g = -1$ (and c arb).

Outline

$\mathbb{R}^3$, $Q = \overline{Q}$, generic case $g^2 \neq 1$, $c = 0$. New solution

RC, A.M. Grundland, 2016

$hv/q, h'/h, q'/q$ are homographic transforms of $V(X) = \mathrm{P_{VI}}$,

$$1 + \frac{\eta hv}{2q} = \frac{2 - (1-\eta^2)\eta h'/h}{(1+g)} = 1 - \frac{(1-g)\eta^2}{(1-\eta^2)\eta q'/q - (1-g)} = \frac{V}{X},$$

$$hq = \frac{2(X-1)}{V(V-1)(V-X)} \times \left[\left(X(X-1)V' - \frac{1+g}{2}V(V-1)\right)^2 - \frac{K}{4}(V-X)^2\right],$$

$$X = \frac{1}{1-\eta^2},\ (\theta_\infty^2, \theta_0^2, \theta_1^2, \theta_X^2) = \left(\left(\frac{1+g}{2}\right)^2, \frac{K}{4}, \frac{K}{4}, \left(\frac{1-g}{2}\right)^2\right).$$

in which $V(X)$ is $\mathrm{P_{VI}}$.

Invariant under the involution.

$\mathbb{R}^3$, $Q = \overline{Q}$, nongeneric $g = 1$, $c = 0$. $= 1/H$ harmonic

q is the Hamiltonian of P_{VI},

$$h = c_{\mathrm{h}} \frac{\eta^2}{\eta^2 - 1},\ v = -\frac{q'}{h},\ q(\eta) = -\frac{8}{c_{\mathrm{h}}} X(X-1) \mathrm{H}_{\mathrm{VI}},\ X = \frac{1}{1-\eta^2}.$$

This solution, for which $1/H$ is harmonic,

$$\frac{1}{H} = \frac{1}{c_{\mathrm{h}}} \left(g_1^2(z) - g_2^2(\bar{z}) \right),$$

is the particular case $c = 0$ of the solution of Bobenko and Eitner.

$\mathbb{R}^3$, $Q=\overline{Q}$, nongeneric $g=-1$, c arb. $=1/Q$ harmonic

h is the Hamiltonian of $\mathrm{P_{VI}}$,

$$q=c_\mathrm{q}\frac{\eta^2}{\eta^2-1},\ v=\frac{4c_\mathrm{q}\eta^2}{(1-\eta^2)^2h'},\ h=-\frac{8}{c_\mathrm{q}}X(X-1)\mathrm{H_{VI}},\ X=\frac{1}{1-\eta^2},$$

$1/Q$ is then harmonic,

$$\frac{(g_1'(z))^2}{Q}=\frac{1}{c_\mathrm{q}}\left(g_1^2(z)-g_2^2(\bar{z})\right).$$

This is the solution of Bonnet (Bonnet surface) as extrapolated by BEK to c arbitrary.

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