Yves Meyer's program and vision

Analyse Harmonique non-lineaire :
Au-dela du programme de Calderon-Zygmund

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On the occasion of the Abel Prize award to Yves Meyer.

It is a privilege to be able to relate ideas, explorations , and visions that Yves, his collaborators and students, developed over the last 40 years, for some of which I was an active participant and observer.
We have had a lot of fun and excitement in this adventure, continuing the Calderon-Zygmund vision, exploring and discovering beauty and structure.

The best reference for this lecture is the paper by Yves Meyer :
*" **Complex Analysis and Operator Theory in Alberto P. Calderon's work**" in Selected papers of Alberto P. Calderon with commentary . AMS QA300.C252 , 2008.*
I apologize in advance for **many, many omissions** , as the full list of contributions , would exceed a volume. Only a few of the simplest illustrations will be addressed.

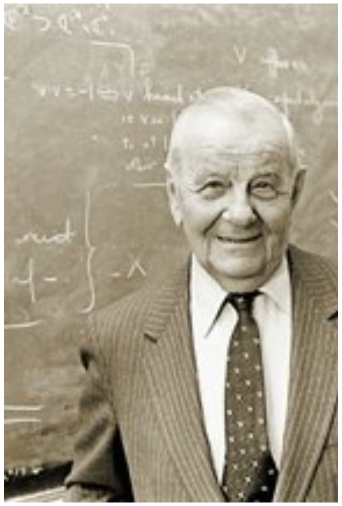
Yves Meyer's work in Harmonic Analysis continues and builds on the Calderon-Zygmund vision and program

It was Zygmund's view that Harmonic Analysis provides the infrastructure linking all areas of analysis, from complex analysis to partial differential equations to probability and geometry .

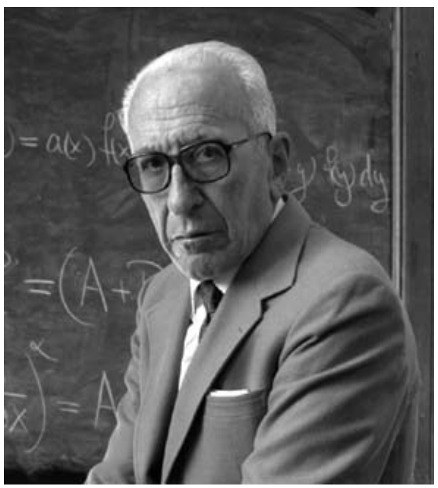
In particular he pushed forward the idea that the remarkable tools of complex analysis , which include , contour integration , conformal mappings , factorization. Tools which were used to provide miraculous proofs in real analysis , should be **deciphered** and converted to real variable tools , so that they can be "understood" and extended to other contexts.

Together with Calderon , they bucked the trend for abstraction, prevalent at the time, and formed a school pushing forward this interplay between real and complex analysis .

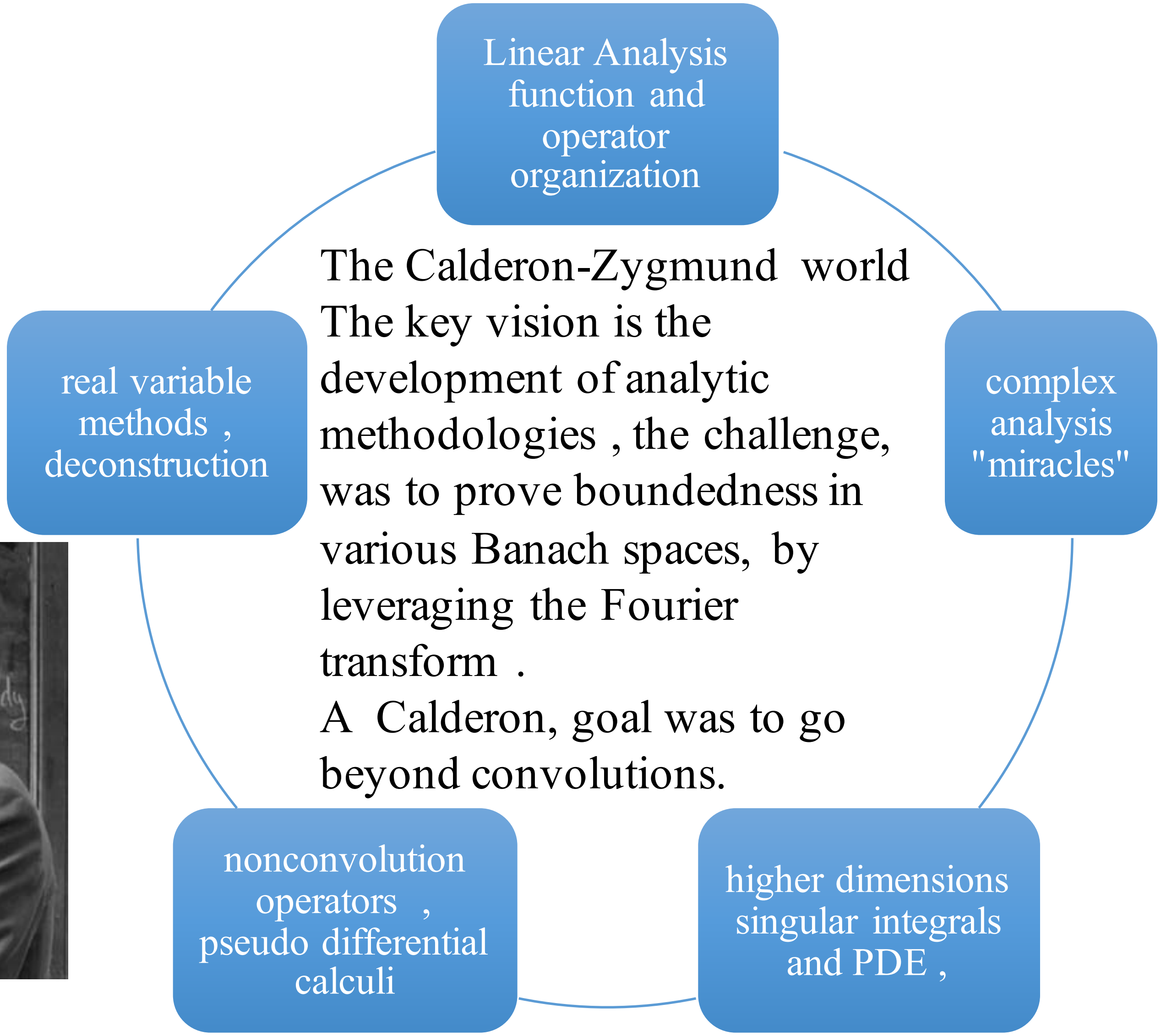
A principal bridge was provided by real variable methods, Multiscale Analysis, Littlewood Paley theory and related Calderon representation formulas , later rediscovered by Morlet



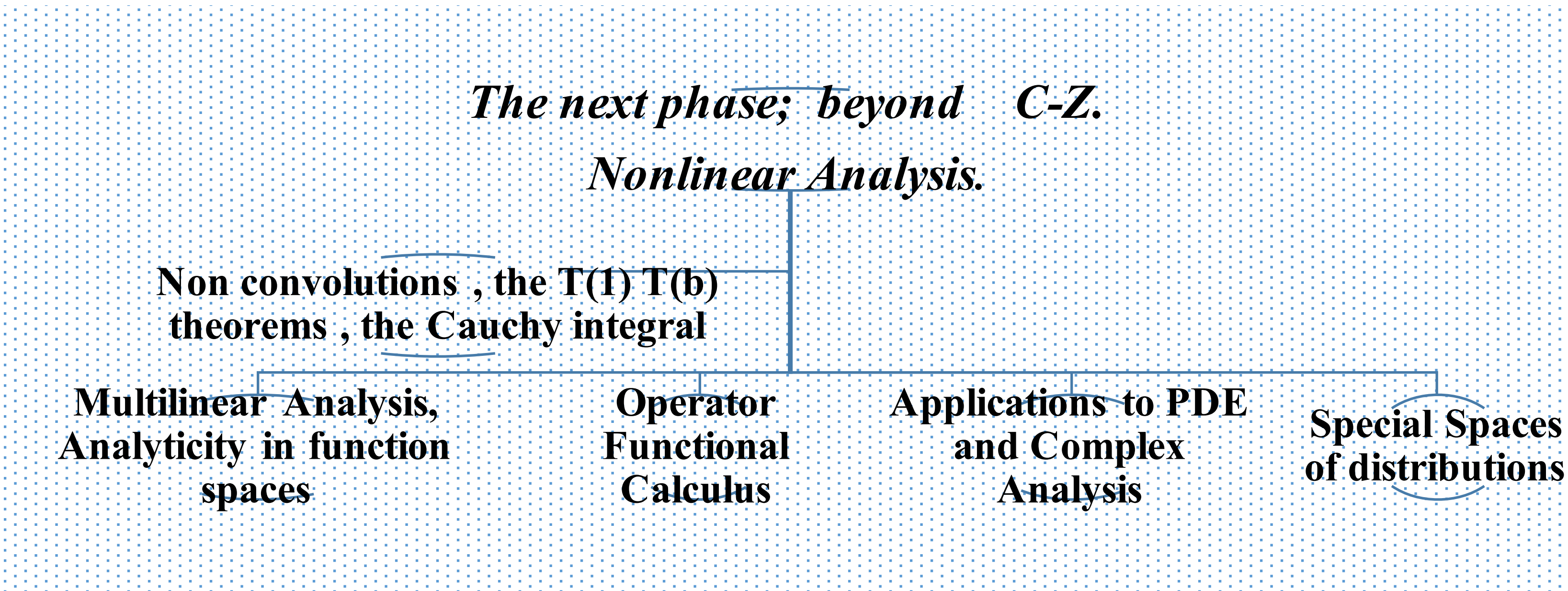
Anthony Zygmund



Alberto Calderon



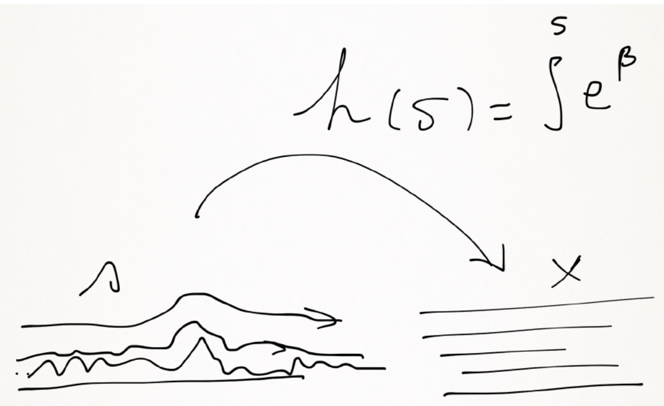
Meyer and Company



We will illustrate many of these points on one fundamental seminal example , linking them all.

The Cauchy transform and its multiple incarnations.

Natural examples of nonlinear analytic dependence, where serious mathematical challenges are posed .

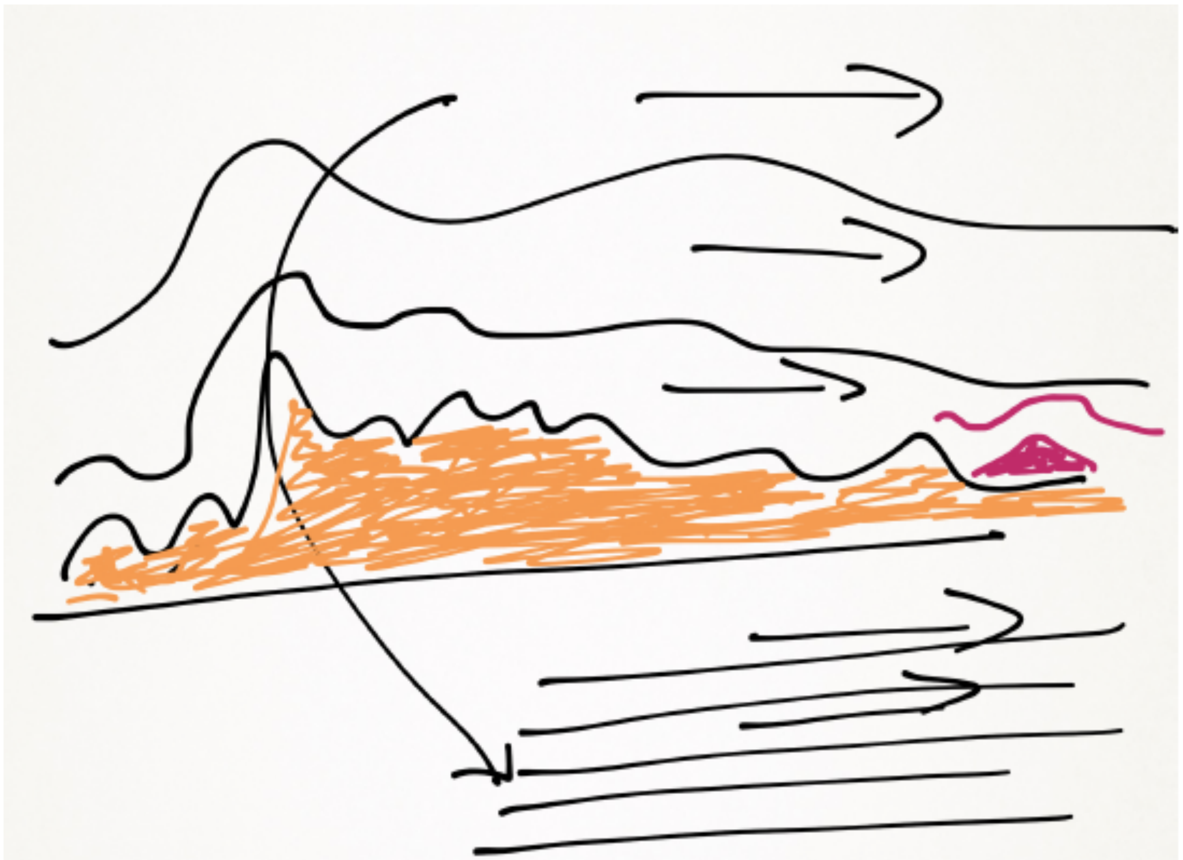


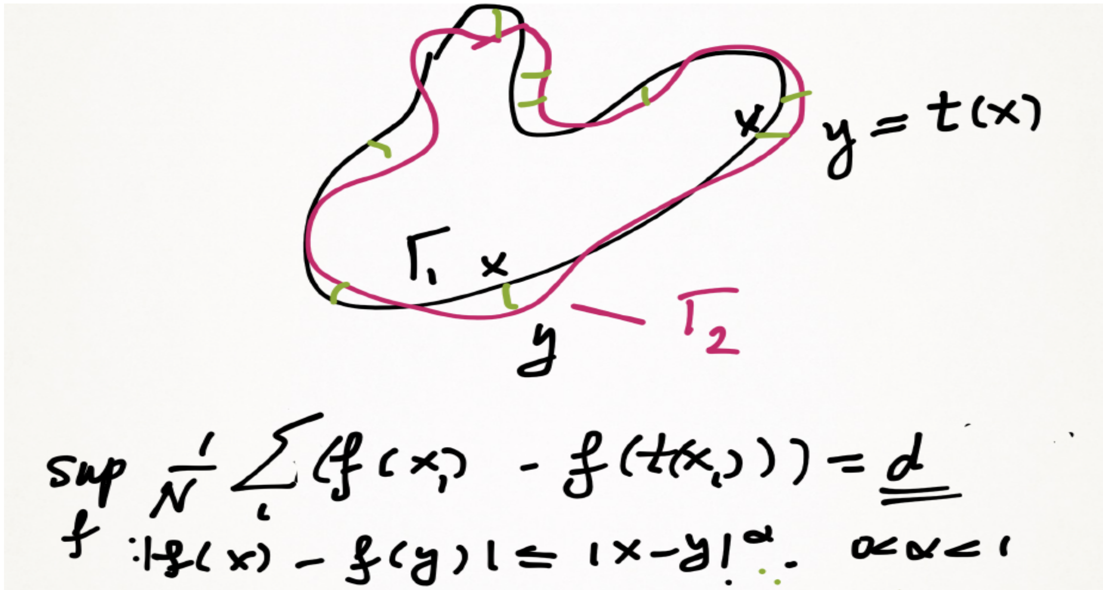
Discover the shape of flow lines of water over a rough river bed . This problem is theoretically solved by using the conformal map from the upper half plane onto the region above the river bed.

The image of the horizontal lines provides the flow lines.

Clearly the nature of the dependence of the conformal map on the shape of the river bed is of interest, can we quantify the effect on the flow of adding a bump or smoothing the bed (This could also be part of the design of an airplane wing profile).

As we will see most of these questions , can be answered through nonlinear Calderon Zygmund theory , developed by Meyer and his collaborators and students.





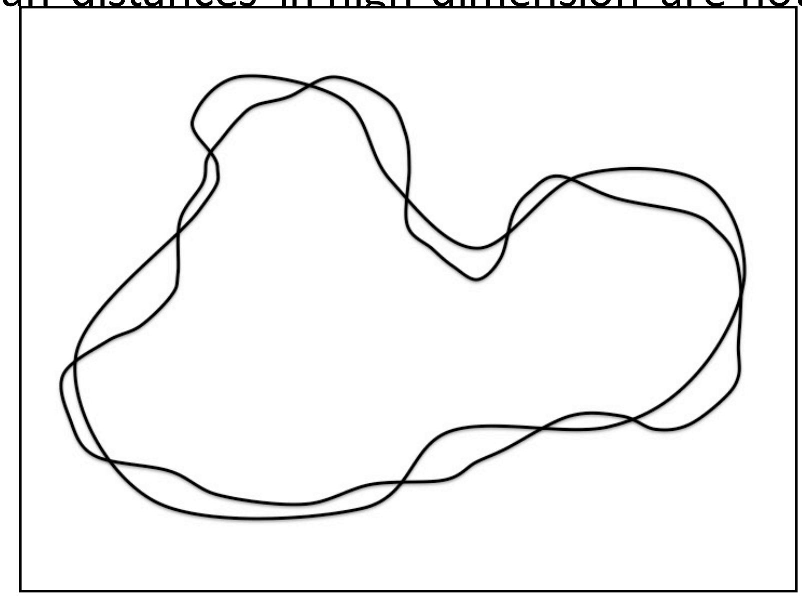
A similar but related problem arises when comparing two sets of points, like the red and black curves . How do we measure their “similarity or distance” , this issue arises, when matching hand written letters, or digits. The natural distances are obtained by distorting one curve to the other, and imposing a cost on the distortion. (Earth mover distance or ,transportation cost). These can also be measured by considering operators on functions on the set and measuring their distance in appropriate metrics , converting the geometry to analysis .

These earth mover distances are easily computed by considering each curve as a distribution in the Besov space dual to a Holder space , a space which is characterized by the sum of the scaled absolute value of wavelet coefficients of the difference between the point distribution. ***This transport metric becomes the distance in the Besov space, with the added advantage that it can be computed effectively***.

Conceptually a version of this idea appears for matching or measuring continuity of solutions of the Navier Stokes equations.

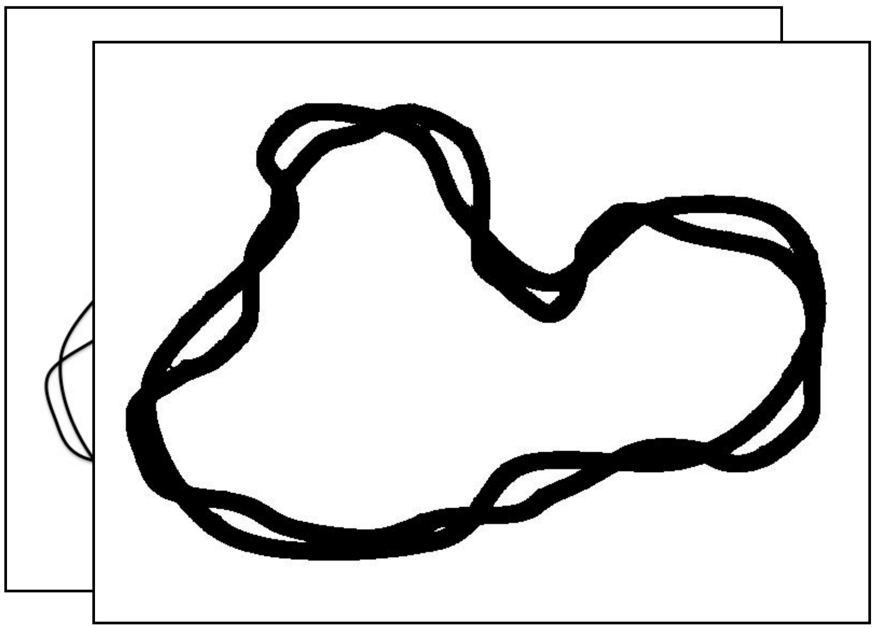
Earth Mover Distance computation, equivalent to Besov distances ,multiscale analysis .

- Coming back to comparing 2D slices (rather than full image profiles…)
- Euclidean distances in high dimension are not “informative”



Earth Mover's Distance

- Efficient implementation via "filtering": obtaining coarser and coarser views [Shirdhonkar & Jacobs, 08]



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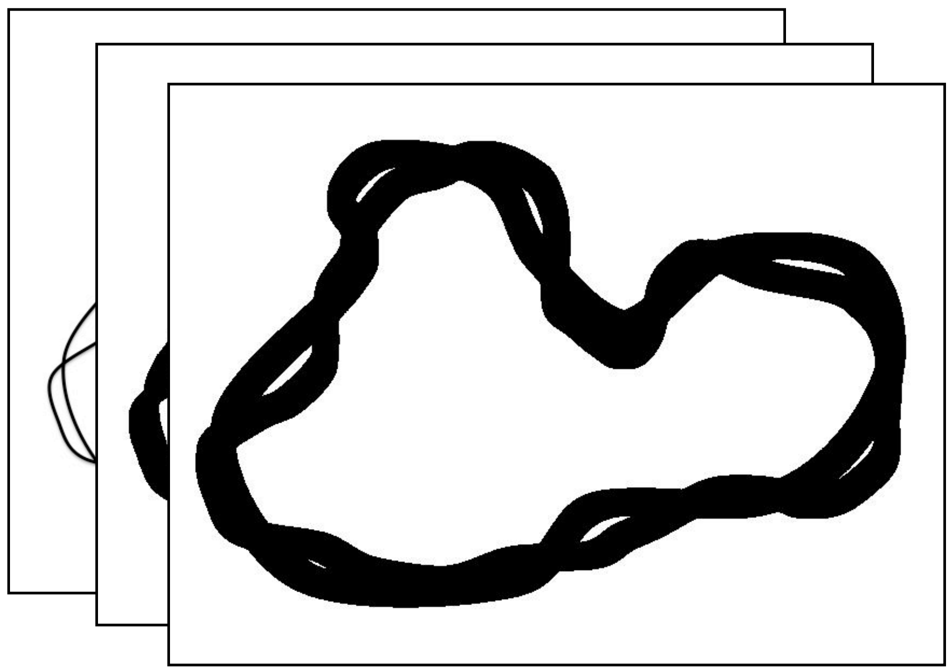


Returning to fundamental challenges in Analysis .

In order to understand some of the basic ideas and methods introduced by Meyer and to illustrate the scope of the program, we start with the basic example of pseudo calculus, introduced by Calderon, of a bilinear operator needed to extend the smooth pseudo differential calculus, to rough environments.
He managed by an analytical tour de force , to prove that the so called Calderon commutator of $\left|\frac{d}{dx}\right|$ and multiplication by A defined as;

$$C(a,f) = \left[\left|\frac{d}{dx}\right|, A\right]f = \left|\frac{d}{dx}\right|(Af) - A\left(\left|\frac{d}{dx}\right| f\right) = \int \frac{A(x)-A(y)}{(x-y)^2} f(y)dy$$

where A' =a is a bounded function , defines a bounded operator in L^2

and $\left|\frac{d}{dx}\right| f = \int e^{ix\xi} |\xi| \hat{f}(\xi) d\xi$

Calderon proved this result by considering a dual form described below reducing the result to deep complex analysis.

Let a and f be of power series type $a=\sum_{k>0} a_k^{\wedge} e^{ik\theta}$ and $f=\sum_{k>0} f_k^{\wedge} e^{ik\theta}$

define $h'(\theta)=a(\theta)f'(\theta)$, $h=\Pi(a,f)$

ie $h(\theta)=\sum_{k>0} h_k^{\wedge} e^{ik\theta} = \int_0^{\theta} a(t)f'(t)dt$ where $h_k^{\wedge} = \sum_{j=1}^{k} (j/k) a_{k-j}^{\wedge} f_j^{\wedge}$

Here we assume that f is in L^2 and a is bounded, Calderon proved that h is in L^2.
Meyer came up with following simple proof .

write $j/k = s = \int s^{i\gamma} \frac{1}{1+\gamma^2} d\gamma$, leading to the representation of h as

$h(\theta) = \int h^{\gamma} \frac{1}{1+\gamma^2} d\gamma$, where $h^{\gamma\wedge}_k = k^{-i\gamma} \sum_{j=1}^{k} a_{k-j}^{\wedge} f_j^{\wedge} j^{i\gamma}$

ie $h = \int M_{-\gamma}(aM_{\gamma}(f)) \frac{1}{1+\gamma^2} d\gamma$, where $M_{\gamma}(f)(\theta) = \sum_{k>0} k^{i\gamma} f_k^{\wedge} e^{ik\theta}$

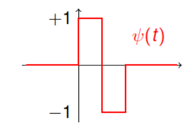
Since M is an isometry on L^2, and a is bounded , h is in L^2.

- There are several points to be made here related to this seminal example
- C(a,f) is a bilinear transformation commuting with translations and dilations of both functional arguments
- The Fourier transform , reveals the structure and enables a simple representation as a sum of simple bilinear transformations of the form A(B(a)C(f)) , where A,B,C are linear operators .

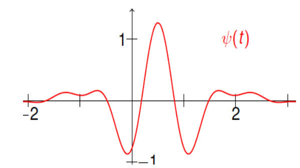
An important idea, hidden in the paraproduct $h = \Pi(a, f)$ is the weak continuity of this bilinear expression .

Observe that the product of functions is not a bilinear operation which is weakly continuous in the arguments . (consider sin(nx) which converges weakly to 0 while sin(nx).sin(nx) converges weakly to ½ .) On the other hand the product of functions of analytic type is weakly continuous.

The simplest real variable version of the Calderon paraproduct is given as



$$\mathrm{h} = \Pi(a, f) = \sum_I \mathrm{m}_I(a) \langle f, \mathrm{h}_I \rangle \mathrm{h}_I$$



where h_I is the Haar function based on the dyadic interval I and $\mathrm{m}_I(a)$ is the mean value of a on that interval

It is quite clear that for two sequences of distributions converging weakly in the sense that their mean converge to a limit on each dyadic intervals, the paraproduct converges weakly , (for example we can have a and f be two fixed functions sampled randomly from two probability distributions , and the means ,computed empirically by averaging over the samples.)

More generally it was J.M. Bony who discovered the utility of paraproducts to analyze nonlinear expressions , and developed paradifferential calculus in analyzing the propagation of singularities of nonlinear PDE. In particular if we replace the Haar wavelet by the Meyer wavelet (or others) and the averaging by a low pass filter, he obtained the following remarkable formula in which the paraproduct captures the roughness of the function.

Let F be smooth , and f Holder function with exponent $\alpha < 1/2$ then

$$F(f) = \Pi(F'(f), f) + e = \sum_I \langle F'(f, \varphi_I \rangle \langle f, \psi_I \rangle \psi_I + e$$

where e is Holder with exponent 2α,

A class of problems in nonlinear Fourier analysis , concerns the analytic dependence of operators on functional parameters, as we will see such problems are deeply connected to all aspects of Harmonic Analysis, as we shall describe on the seminal example of the Cauchy transform

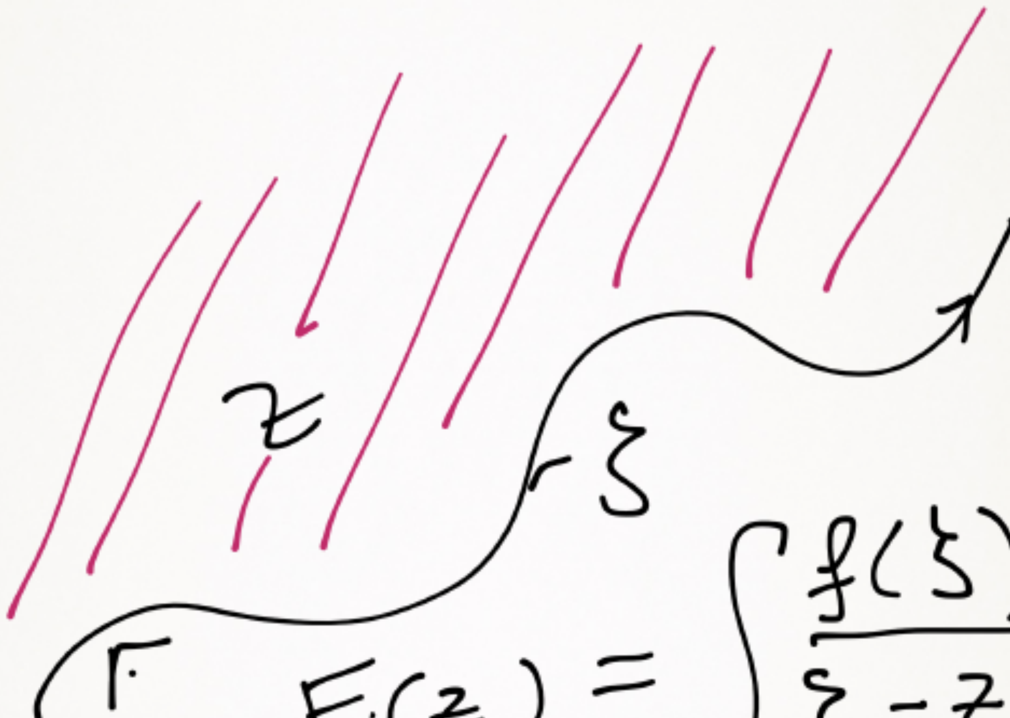
Perhaps the most common occurrence of such questions involves the dependence of solutions of a differential or partial differential equation on the coefficients of the differential operator. To be specific, let $P(D)$ denote a differential operator such as

$$a(x)\frac{d}{dx}, \quad x \in \mathbb{R}, \qquad a(z)\frac{\partial}{\partial z}, \quad z \in \mathbb{C},$$

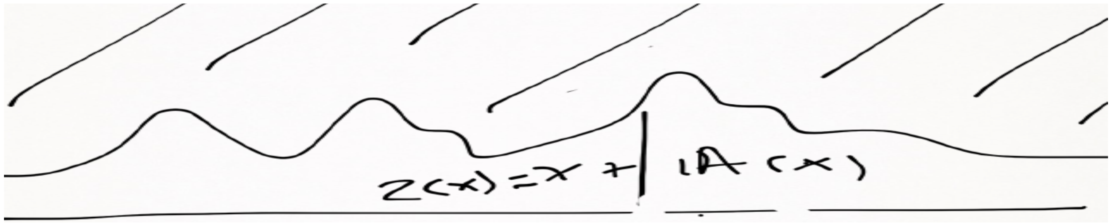
(1.1)

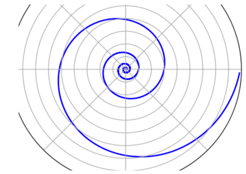
$$\sum_{i,j} \frac{\partial}{\partial x_i}\left(a_{ij}(x)\frac{\partial}{\partial x_j}\right), \quad \sum_{i,j} a_{ij}(x)\frac{\partial^2}{\partial x_i \partial x_j}, \qquad x \in \mathbb{R}^n.$$

Let F be a function. We are interested in $F(P(D))$; for example, $\operatorname{sgn}(P(D))$, $\sqrt{P(D)}$, $\exp(t(P(D)))$, $\exp(-t\sqrt{P(D)})$.



$$F_\Gamma(z) = \int_\Gamma \frac{f(\xi)}{\xi - z}$$





Consider a the graph z(x) of a function A(x) with bounded derivative a on the line

i,e z(x)= x+iA(x) and the corresponding Cauchy integral operator

$$C(a,f)(x)=\lim_{\varepsilon\to 0}\ (1/2\pi i)\int \frac{f(t)}{i\varepsilon + z(x)-z(t)} z'(t)dt \ =\lim_{\varepsilon\to 0}\ (1/2\pi i)\int \frac{f(t)}{i\varepsilon + x-t+i(A(x)-A(t))}(1+ia(t))dt$$

This is the non orthogonal projection of f , onto boundary values of holomorphic functions above the curve . By expanding as a series if |a|<1-ε we can write this integral as multilinear series .

$$C(a,f)(x)=\sum_0^\infty p.v\int \frac{i^k (A(x)-A(t))^k f(t)}{(x-t)^{k+1}}(1+ia(t))dt$$

whose first nontrivial term is the Calderon commutator .

More generally we may want to represent a Jordan curve by its arc length parameterization , ie

$z(s)=\int_0^s \exp(i\alpha(t))\,dt$, and assume that α is in BMO (bounded mean oscillation) or that the curve defined by is a Lavrentiev curve verifying the chord arc condition |s-t|< C|z(s)-z(t)|.

A common remarkable component in much of the nonlinear analysis is the space of functions of bounded mean oscillation introduced by John and Nirenberg

$\|a\|_{BMO} = \sup_I \frac{1}{|I|}\int_I |a(x) - m_I| dx$ where m_I is the mean of a on the inteval I.

as we will see this space is the natural domain for a variety of nonlinear functionals.

in particular the operator norm on L^2 of the commutator of multiplication by a and the Hilbert transform is equivalent to the BMO norm of a . The image of bounded functions under singular integrals are in BMO

for example ln|x| is unbounded but has bounded man oscillation.

The relation with operator functional calculus , the Kato square root problem was revealed to us by **Alan McIntosh**It was the basis of the remarkable proof by Meyer of the boundedness of the Cauchy Transform. This and the relation to complex analysis becomes transparent if we consider perturbations of d/dx obtained through conjugations by changes of variable . As described below;

Alan McIntosh

h(x)= $\int_0^x (1+a(t))dt$ or more generally h(x) =$\int_0^x \exp(\beta(t))dt$ where |a|<1-ε and β is in BMO.

h is monotone and defines a change of variable on the line which can be used to perturb d/dx by conjugation giving rise to the operator 1/(1+a) d/dx a simple calculation shows that sgn[d/dx] is converted to

$\int \frac{f(t)}{\text{x-t+(A(x)-A(t))}} dt$ = sgn[1/(1+a) d/dx]f , thus the Cauchy transform is an analytic continuation in a

of this operator , and the boundedness in L^2 would imply an analytic functional calculus of perturbations of d/dx.

Remarkably if we let $U_h f = (f \circ h)h'^{1/2}$ then U is unitary on L^2 and the conjugation of the Hilbert transform with U is analytic in the functional parameter $\beta = \ln h'$, viewed as an operator valued functional, on the space BMO. Moreover the following result is valid

$\| U_h H U_h^{-1} - H \| \approx \|\ln h'\|_{BMO}$ formally this operator becomes the Cauchy transform when ln(h') is imaginary.

and $U_h H U_h^{-1} f = \sum_k \Lambda_k(\beta) f$, Where Λ_k is a k multilinear operator valued functional of $(\beta, \beta, \ldots \beta)$

and $\|\Lambda_k\| < c^k (\|\beta\|_{BMO})^k$

Several remarkable features appear here :

- The operator valued functional is a convergent power series on BMO and is the Cauchy transform on Chord arc curves .

- The norm on BMO is the operator norm defined by the first linear term . It defines the space of analyticity of the nonlinear transform .

- The geometry of the curve is equivalent to the property of the operator carried by the curve, as shown by Guy David , S. Semmes , Peter Jones also in higher dimension for Calderon Zygmund operators.

- A beautiful example; the Riemann Mapping functional mapping one side of a Chord-arc curve onto the upper half plane is real analytic on the Manifold of Chord-arc curves parameterized by BMO into the group of changes of variables (preserving BMO) itself parameterized by BMO .
In fact BMO is characterized as the **largest Banach space for which we have analyticity.**

- This theme of discovery of the natural space is reoccurring in Meyer's work for the Navier Stokes, and other nonlinear PDE . Where the appropriate Banach space structure is defined by the problem .

- As H=|d/dx| is the square root of the Laplace operator , one can compute it and its perturbations through a functional calculus using the resolvent of the perturbations.

- The Kato conjecture which concerns the domain of divergence form accretive Laplace operators was ` solved by Meyer and Dong Gao Deng for small perturbations and in full generality by Steven Hofmann, Pascal Auscher, Michael Lacey, John Lewis, Alan McIntosh and Philippe Tchamitchian.

- The multilinear operators arising in the powers series , can be analyzed and decomposed directly , using Fourier or other transforms, this approach provides insight and could enable efficient numerical implementations.

- The Cauchy integral generalizes directly to higher dimensions ,for example as double layer potential operator , or more generally as the restriction to a submanifold of a Calderon Zygmund operator of the appropriate homogeneity.

- The long standing problem of short time existence of water waves in 2 or 3 dimensions, was solved by Sijue Wu using these higher dimensional extensions.

Multilinear operators commuting with translations , and Fourier transforms.

consider the bilinear Calderon commutator written, as

$$C(a,f)=\int\frac{\int_x^y a(t)dt}{(x-y)^2}f(y)dy=\int\exp(ix(\xi+\eta))\hat{f}(\xi)\hat{a}(\eta)\sigma(\xi,\eta)d\xi d\eta \ ,$$

sigma is the bilinear symbol.

More generally this expression which defines bilinear operators commuting with translations extends to multilinear operators.

$$\Lambda(a_1,a_2,...a_k)(x)=\iiint\exp(ix(\xi_1+\xi_2+...+\xi_k))\hat{a}_1(\xi_1)\hat{a}_2(\xi_2)...\hat{a}_k(\xi_k)\sigma(\xi_1,\xi_2,...\xi_k)d\xi_1 d\xi_2...d\xi_k$$

$$\sigma(\xi_1,\xi_2,...\xi_k)=\exp(-ix(\xi_1+\xi_2+...+\xi_k))\Lambda(e_1,e_2,...e_k)(x),\quad \text{where } e_k=\exp(ix(\xi_k)).$$

Such expressions arise naturally , when considering power series in function space , or analytic dependence of natural operators on functional parameters . The Fourier representation enables partitioning Fourier space into simple components where the symbol is easily expressible as a product. The challenge is the reassembly.

The following remarkable theorem of G.David and J.L Journe , gives a simple necessary and sufficient condition for boundedness, and just as important enables dealing with k multilinear operators , by induction on k.

THEOREM. *Let* $T(f) = \int k(x, y) f(y)\, dy$, *where* k *is locally integrable in* $x \neq y$ *such that*

$$|x - y|\,|\nabla_x k| + |x - y|\,|\nabla_y k| + |k| \leqslant |x - y|^{-n}. \tag{2.1}$$

Moreover, assume that T *has a weak cancellation property (see* [7] *or Y. Meyer in these proceedings). Then* T *is a bounded operator on* L^2 *iff* $T(1)$ *and* $T^*(1)$ *are in* B.M.O. *Moreover,*

$$c\left(1 + \|T(1)\|_* + \|T^*(1)\|_*\right) < \|T\|_{L^2} < C\left(1 + \|T(1)\|_* + \|T^*(1)\|_*\right),$$

and T *maps* L^∞ *into* B.M.O.

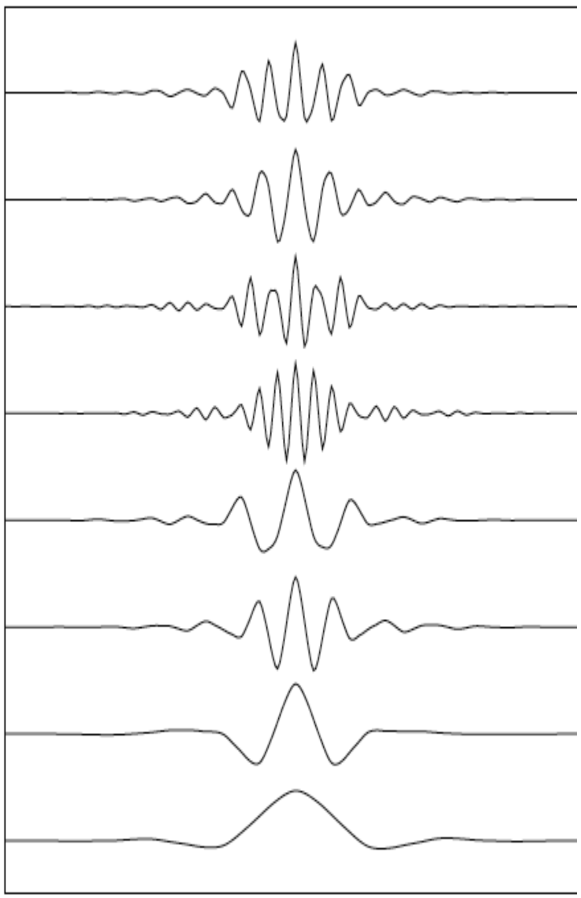
The next example goes beyond the preceding ideas , and requires much more subtle book keeping to analyze

Another bilinear operator much harder thatn the commutator , is the bilinear Hilbert transform defined as

B(f,g) =p.v $\int f(x-\alpha t)g(x+t)dt/t$ ξ_1

here the bilinear symbol is $\sigma(\xi_1,\xi_2) = \text{sgn}\ (\ \alpha\xi_1 - \xi_2)$ is deceivingly simple and easy to represents by partitioning the frequencies away from the singularity .The trouble is estimating the sum as being in some space .

This work by Lacey and Thiele, corresponds to oscillating paraproducts in space frequency , or time frquency atomic decompositions in terms of wavelet packets, it creates a common method to explain the Carleson's theorem for Fourier series as done by Ch.Fefferman , and the Bilinear Hilbert transform .



wavelet packets

The waveforms on the left are ***wavelet packets*** which provide better frequency localization, they can be viewed as musical notes , having a pitch, duration, localization and amplitude, so that the description of a signal or sound, becomes more like musical notation .

The decomposition , or orchestration of complex transformations on functions is a main tool of linear harmonic analysis , and much more so in the nonlinear context , where interactions are difficult to track.

The Navier Stokes equation and adapted functional spaces .

Yves Meyer, Marco Cannone* and Fabrice Planchon*:
Prove existence of global Kato solution to Navier-Stokes
when initial conditions are oscillating.

$$v(t) = S(t)v_0 + \mathbb{P}\int_0^t S(t-\tau)\sum_1^3 \partial_j(v_j v)(\tau)d\tau \qquad v = h + \mathcal{B}(v,v)$$

H. Koch and D. Tataru used a function space X describing the expected behavior of a solution $u(x,t)$ when the initial data belongs to $\mathcal{K}_\infty$. The definition of X is driven by the behavior of the linear evolution. In the H. Koch and D. Tataru theorem, the Banach space X in which the iteration scheme is applied is defined as follows.

f belongs to the Koch&Tataru space if and only if

$$\sum_{Q\subset R} 2^{-2j}|\alpha_Q|^2 \le C|R| \qquad \textit{wavelet coefficients } \alpha_Q = \langle f, \psi_Q \rangle$$

The condition $\sum_{Q\subset R} |\alpha_Q|^2 \le C|R|$ is equivalent to f being in BMO, and the Bilinear operator , can be estimated using Littlewood Paley theory , and many of the tools discussed above .

Where are we heading , and future challenges.

Problems of nonlinear analysis and weak convergence , are "everywhere dense", for example;

- In Stochastic differential equations where the noise variance is depending on the solution

The work of M. Hairer , M. Gubinelli, T.J. Lyons and others are all centered around definitions of spaces of distributions, which are in the domain of weak continuity of nonlinear expressions.

- In statistics , say for random matrix theory , in what sense do the resolvent operators converge to a limit as the size of the random matrix grows . (Matrix central limit theorems)

- In homogenization theory , Meyer , Sijue Wu , on weak limits of Green operators.

- Given two sets of points in space, we can view each as a sum of dirac measures , the Besov distance viewed as the distance in the dual space of Holder , is an earth mover transportation distance between the clouds of points . This is a fundamental problem is statistical machine learning , and empirical geometry , the adaptation of the Geometry to the structure of the data, is a main problem in ***Mathematical Empirical Modeling.*** Currently local Euclidean geometry reigns, on the other hand geometries of data are controlled by the processes that generate them, biology, chemistry physics , all different . The tools described are adaptable to this rich setting

- **"Harmonic Hard Analysis"** can be tuned , and adapted to the future of data/knowledge processing. At the moment , many of the adaptive waveform analysis tools , are being replaced by machine learning optimizations , lacking fundamental estimates and error bars . A **challenge is to merge the tools of signal processing with various machine learning methodologies.**

- **Compressed sensing ,** has become a tool for dimensionality reduction, it also provides universal representations , beyond Fourier ,????

We see a rapid evolution, in the machine learning community to generate data driven descriptions of the world around us , at the moment the world of artificial intelligence is showing spectacular progress on problems which have challenged engineers and scientists.

These methods for automated data driven tabulations are in the process of being integrated with mathematics, for example in the context of the Mallat scattering transforms, or integrated with the analysis of transformations on data clouds, in the same sense that the Cauchy transform on a curve reflected the geometric properties of the curve, and enabled quantifications of distances between shapes.

Although this seems farfetched , the real variable combinatorial geometric multiscale methods , are essential for the digital computational world. If nothing else the conceptual mathematical affinity between the different fields of analysis explored and developed by Meyer and his group is translated into the world of data-empirical modeling by computation.

Having fun in 1989