MEYER SETS AND RELATED PROBLEMS

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FOURIER QUASICRYSTALS

We consider measures in $\mathbb{R}^n$ with discrete support Λ and spectrum S. Let

$$\mu = \sum_{\Lambda} \mu(\lambda)\, \delta_\lambda$$

Assume that the Fourier transform $\hat{\mu}$ (in sense of distributions) is also a measure:

$$\hat{\mu} = \sum_{S} \hat{\mu}(s)\, \delta_s$$

Measures with discrete support and spectrum often are called "Fourier quasicrystals"

POISSON SUMMATION FORMULA

In $\mathbb{R}$: Let φ be a Schwartz function on $\mathbb{R}$. Then:

$$\sum_{\lambda \in \mathbb{Z}} f(\lambda) = \sum_{s \in \mathbb{Z}} \hat{f}(s)$$

$$\hat{f}(x) = \int_{\mathbb{R}} f(t) e^{-2\pi i x t} dt$$

Equivalently:

$$\widehat{\sum_{\lambda \in \mathbb{Z}} \delta_\lambda} = \sum_{s \in \mathbb{Z}} \delta_s$$

In $\mathbb{R}^n$: Given a lattice $\Gamma = T(\mathbb{Z}^n)$, consider

$$\mu = \sum_{\gamma \in \Gamma} \delta_\gamma$$

The Fourier transform

$$\hat{\mu} = \frac{1}{\det \Gamma} \sum_{s \in \Gamma^*} \delta_s$$

$$\Gamma^* := (T^*)^{-1} (\mathbb{Z}^n)$$

POISSON SUMMATION FORMULA

- Poisson formula in crystallography
- Dirac combs
- Diffraction pattern

POISSON SUMMATION FORMULA

Problem

Which other measures with discrete support & spectrum do exist?

- J.-P. Kahane & S. Mandelbrot (1958);
- A.-P. Guinand (1959)

CUT AND PROJECT

Yves Meyer " Algebraic numbers and harmonic analysis", 1972

CUT AND PROJECT

Let Γ be a lattice in $\mathbb{R}^2$ (in general position),
Ω -an interval on the axes y. Consider the horizontal strip $P := \mathbb{R} \times \Omega$.
Define the "model set":

$$M := Proj_x \Gamma \cap P$$

General models: $M(\mathbb{R}^n \times \mathbb{R}^m, \Gamma, \Omega)$

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General models: $M(\mathbb{R}^n \times \mathbb{R}^m, \Gamma, \Omega)$
Arithmetics of the spectrum and almost periodicity
Models and algebraic numbers

FIBONACCI SET

$$x_n = n + (\tau - 1)\left[\frac{n}{\tau}\right]$$

$$\tau = \frac{1}{2}\left(\sqrt{5} + 1\right)$$

Substitution $0 \mapsto 01\,,\,1 \mapsto 0$

0
01
010
01001
01001010

Non-periodic, however "not random".

MEYER's MODEL SETS

1. M is a uniformly discrete set:

$$\left|x - x'\right| > d > 0$$

2. Uniform density $D(\Lambda)$: $\mathrm{card}\,(\Lambda \cap I) = D \cdot |I| + o(|I|)$

$$\text{Claim: } D(M) = \frac{|\Omega|}{\det \Gamma}$$

3. Tiling

MEYER's MODEL SETS

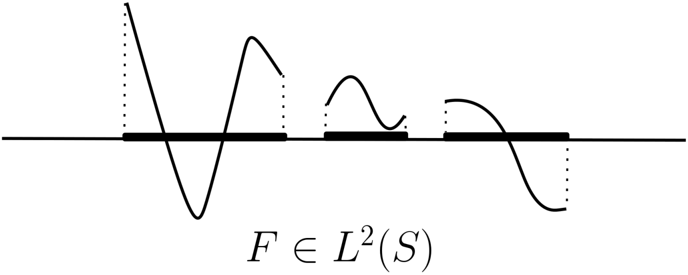
Keskuskatu, Helsinki

MEYER's MODEL SETS

A model tiles the space (by "Voronoi cells").
In particular, Penrose tiling can be obtained by a projection of 5-dim lattice onto the plane (de Bruijn).

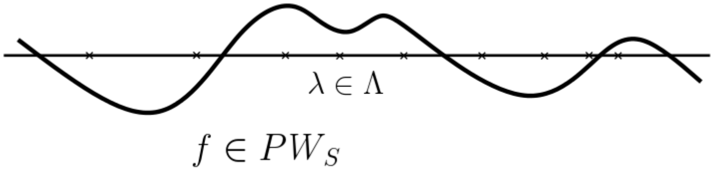
RECONSTRUCTION OF SIGNALS

S bounded set



$f := \hat{F}$

$PW_S = \{f\}$ Paley-Wiener space with spectrum S



STABLE SAMPLING

When can one reconstruct f from $f|_\Lambda$?

Definition

Λ is a set of stable sampling if

$$\|f\|^2 \leq K \cdot \sum_{\lambda \in \Lambda} |f(\lambda)|^2 \quad , \quad \forall f \in PW_S$$

Classical case $S = (-\sigma, \sigma)$
Beurling: $D(\Lambda) > |S|$ implies Λ is a set of stable sampling.

STABLE SAMPLING

Disconnected spectrum
Landau: $D(\Lambda) \geq |S|$ is necessary for stable sampling.
No sufficient condition in terms of density.
Arithmetic comes into play.

UNIVERSAL SAMPLING

Is it possible to define Λ which serves for any S of given measure independently of its structure and localization?

(A.O., A. Ulanovskii, 2006): There exists a set Λ with $D(\Lambda) = 1$ which is a set of stable sampling for every compact S, $|S| < 1$.

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(B. Matei, Y. Meyer, 2008): $M(\mathbb{R} \times \mathbb{R}, \Gamma, \Omega)$ is a set of stable sampling for every compact S, $|S| < D(M)$.

MEASURES ON MODEL SETS

Theorem (Y. Meyer, 1970): Given a model set M in $\mathbb{R}^n$ there is a measure, supported on M, whose spectrum is a countable set.

Take $\varphi \in \mathscr{S}(\mathbb{R})$, *compactly supported*.

$$\mu := \sum_{(x,y)\in\Gamma} \varphi(y)\, \delta_x$$

Then

$$\hat{\mu} = c \sum_{(u,v)\in\Gamma^*} \hat{\varphi}(v)\, \delta_u$$

QUASICRYSTALS

Dan Shechtman

QUASICRYSTALS

- There exists non-periodic atomic structures whose diffraction patterns consist of "spots" (1983).
- Nobel prize (2011)

UNIFORMLY DISCRETE QUASICRYSTALS

J. Lagarias (2000): "Mathematical quasicrystals and problem of diffraction".

Conjecture

If the support Λ and the spectrum S of a positive-definite measure μ both are *Uniformly Discrete* (u.d.) sets, then Λ is a periodic set.

We proved this conjecture in collaboration with Nir Lev.

PERIODICITY OF U.D. QUASICRYSTALS

Theorem (N. Lev , A.O., Inventiones Math., 2015)

1. If the support and the spectrum of a measure μ in $\mathbb{R}$ are u.d. then μ is a finite sum of Dirac combs, translated and modulated:

$$\mu = \sum_{j\in[1,N]} \sum_{\lambda\in\Lambda} P_j(\lambda)\,\delta_{\lambda+\theta_j}$$

   Here Λ is a lattice, P_j- trigonometric polynomials.

2. The same result is true in $\mathbb{R}^n$, under extra assumption that μ is positive (or positive-definite) measure.

The proof is based on an interaction of Harmonic Analysis and Discrete Geometry.

DISCRETE GEOMETRY

Λ is a Delone set if it is u.d. and relatively dense.

Definition

Λ is a Meyer set if it is a Delone set and $\Lambda - \Lambda \subset \Lambda + F$, F a finite set.

(Y. Meyer, 1972): Every Meyer set can be covered by a finite union of translates of a model set.

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The result remains true if

$$card(\Lambda - \Lambda) \cap B(x, 1) < C$$

SPECTRAL GAPS

Definition

A ball $B(x,a)$ is called a spectral gap of μ if $\hat{\mu}|_B = 0$.

$n = 1$: If μ has a spectral gap then the asymptotic density

$$D_{\#}(\Lambda) := \liminf_{r\to\infty} \frac{\#(\Lambda \cap B(0,r))}{|B|}$$

is positive.

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Not true for $n > 1$. However:

If μ in $\mathbb{R}^n$ has an "isolated spectral atom" then $D_{\#}(\Lambda) > c(a) > 0$.

PROOF OF PERIODICITY CONJECTURE

$$h \in \Lambda - \Lambda$$

$$\Lambda_h = \Lambda \cap (\Lambda + h)$$

$$\mu_h = \sum \mu(\lambda) \overline{\mu(\lambda + h)} \delta_\lambda$$

- μ_h have a common spectral gap
- If $\mu > 0$ then there is a common "isolated spectral atom"
- $D_{\#}(\Lambda_h) > c$
- $\Lambda - \Lambda$ has bounded upper density
- $\Lambda \subset M + F$
- M is a lattice
- μ *has a periodic structure*

NON-SYMMETRIC SITUATION

(N. Lev, A.O., to appear in Adv. Math.): Let μ is a positive-definite measure with u.d. support Λ and discrete closed spectrum S.
Then μ has the periodic structure.

APPLICATION TO HOF's DIFFRACTION

A. Hof "On diffraction by aperiodic structures", Comm. Math. Phys,1995

$$\mu := \sum_{\lambda \in \Lambda} \delta_\lambda$$

Let $\Lambda - \Lambda$ be a discrete closed set (=Λ has "finite local complexity")
Set:

$$\gamma_R := \frac{1}{R^n} \sum \delta_{\lambda - \lambda'} \quad , \quad \gamma(\Lambda) = \text{weak} \lim_{R \to \infty} \gamma_R$$

γ is the auto-correlation measure of Λ.

Corollary

If the diffraction spectrum S (the support of $\widehat{\gamma}$) is u.d. it is periodic.

NON-CLASSIC POISSON SUMMATION FORMULAS

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1. Λ and S are both discrete closed sets;
2. Λ contains only finitely many elements of any arithmetic progression.

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1. Λ and S are both discrete closed sets;
2. Λ contains only finitely many elements of any arithmetic progression.

$$\mu := \sum_{(x,y)\in\Gamma} f(y)\,\delta_x$$

f is a special non-compactly supported function in $\mathscr{S}(\mathbb{R})$.

NON-CLASSIC POISSON SUMMATION FORMULAS

Guinand's nodes: $\pm\left(n+\frac{1}{9}\right)^{1/2}$

New examples (Y. Meyer, 2016)

Let $\alpha \in \mathbb{R}^3 \backslash \mathbb{Z}^3$ Then:

$$\mu := \sum_{k\in\mathbb{Z}^3} \frac{1}{|k+a|} e^{2\pi i k\alpha} \left(\delta_{|k+\alpha|} - \delta_{-|k+\alpha|}\right),$$

satisfies $\hat{\mu} = c\mu$.