Poisson summation formulae

and Huygens' principle.

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- The ceremony took place on the 25th of November 2016 at Universidad Autónoma de Madrid.
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En homenaje a
Magdalena Walias
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Measures with locally finite
support and spectrum
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Departamento de Matemáticas
Sala 520 11:30
DEPARTAMENTO DE
MATEMÁTICAS
UAM
UNIVERSIDAD AUTONOMA
DE MADRID

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- In the present contribution a third approach is proposed. As it was guessed by Ildefonso Diaz, Guinand's work follows from Huygens' principle for the three dimensional torus.
- If the initial velocity is a Dirac mass at the origin, the solution is Guinand's distribution.
- Using this new approach one can construct a large family of initial velocities which give rise to crystalline measures generalizing Guinand's solution.
- Huygens' principle holds on a large class of homogeneous spaces derived from the Coxeter group (so, for example, the Weyl groups of simple Lie algebras). This approach will yield new crystalline measures.

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Jesús Ildefonso Díaz and Yves Meyer, Poisson summation formulae and the wave equation with a finitely supported measure as initial velocity. African Diaspora Journal of Mathematics, Volume 20, Number 1, pp. 113 (2017).

Definition of crystalline measures

Definition

The Fourier transform $\mathcal{F}(f) = \widehat{f}$ of a function f is defined by $\widehat{f}(y) = \int_{\mathbb{R}^n} \exp(-2\pi\, ix \cdot y) f(x)\, dx$.

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A set of points $\Lambda \subset \mathbb{R}^n$ is uniformly discrete if

$$\inf_{\{\lambda,\, \lambda' \in \Lambda,\, \lambda' \neq \lambda\}} |\lambda' - \lambda| = \beta > 0. \tag{1}$$

Definition of crystalline measures

Definition

A crystalline measure is an atomic measure μ on $\mathbb{R}^n$ which satisfies the conflicting but fortunately compatible properties:

(*a*) μ is supported by a locally finite set

(*b*) μ is a tempered distribution

(*c*) the distributional Fourier transform $\widehat{\mu}$ of μ is also an atomic measure supported by a locally finite set.

Definition of crystalline measures

- Let Λ be the support of a crystalline measure μ and let S be its spectrum, i.e. the support of $\widehat{\mu}$. We then have

(2) $$\mu = \sum_{\lambda \in \Lambda} a_\lambda \delta_\lambda, \quad \widehat{\mu} = \sum_{y \in S} b_y \delta_y.$$

- It yields the following *generalized Poisson summation formula*:

(3) $$\sum_{\lambda \in \Lambda} a_\lambda \widehat{f}(\lambda) = \sum_{y \in S} b_y f(y), \quad \forall f \in \mathcal{S}(\mathbb{R}^n).$$

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Definition of crystalline measures

- A well known example is given by the standard Poisson summation formula where Λ is a lattice. A lattice $\Gamma \subset \mathbb{R}^n$ is defined by $\Gamma = A\mathbb{Z}^n$ where $A \in GL(n, \mathbb{R})$.
- A *Dirac comb* is a sum $\mu = \sum_{\gamma \in \Gamma} \delta_\gamma$ of Dirac masses δ_γ on a lattice Γ. The Fourier transform of the Dirac comb on a lattice Γ is (up to a constant factor) the Dirac comb on the dual lattice Γ^*.
- This is the *standard Poisson summation formula* which plays a seminal role in X-ray crystallography and molecular biology.
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Definition of crystalline measures

- The collection of crystalline measures is a vector space. It is not a Banach space. If μ is a crystalline measure and if P is a finite trigonometric sum then $P\mu$ is also a crystalline measure. These two remarks are used in the following definition:

Definition

Let σ_j be a Dirac comb supported by a coset $a_j + \Gamma_j$ of a lattice $\Gamma_j \subset \mathbb{R}^n$, $1 \leq j \leq N$. Let g_j be a finite trigonometric sum and $\mu_j = g_j\,\sigma_j$. Then $\mu = \mu_1 + \cdots + \mu_N$ will be called a generalized Dirac comb.

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A crystalline measure μ which is not a generalized Dirac comb is called an exotic crystalline measure.

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- It is the case if the support Λ of μ is not contained in a finite union $\bigcup_1^N (a_j + \Gamma_j)$ of co-sets of lattices.

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Definition of crystalline measures

- This observation will apply to Guinand's measure. Our goal is the construction of exotic crystalline measures. Two methods are proposed. The first one (sketched now but detailed in the Appendix) uses Guinand's mysterious ideas.
- Huygens' principle yields a second construction.

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- Huygens' principle yields a second construction.

Guinand's construction

- Let $r_3(n),\ n \in \mathbb{N}$, be the number of points $k \in \mathbb{Z}^3$ such that $|k|^2 = n$. Guinand's distribution $\sigma \in \mathcal{S}'(\mathbb{R})$ is defined by

$$(4). \qquad \sigma = -2\frac{d}{dx}\delta_0 + \sum_1^\infty r_3(n) n^{-1/2}(\delta_{\sqrt{n}} - \delta_{-\sqrt{n}})$$

Guinand's distribution is odd.

- We have $\sum_0^N r_3(n) n^{-1/2} = 2\pi N + O(N^{1/4})$ which implies that σ is a tempered distribution. Guinand proved the following:

Theorem

[Guinand] *The distributional Fourier transform of σ is $-i\sigma$.*

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Guinand-Y's measure

- For $\alpha \in (0, 1)$ we define

$$(5) \qquad \tau_\alpha(x) = (\alpha^2 + \frac{1}{\alpha})\sigma(x) - \alpha\sigma(\alpha x) - \sigma(x/\alpha)$$

- Then the derivative of the Dirac mass at 0 disappears from this linear combination.
- Guinand's theorem implies:

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- Fix $\alpha = 1/2$ in the preceding construction. Define $\chi(n) = -1/2$ if $n \in \mathbb{N} \setminus 4\,\mathbb{N}$, $\chi(n) = 4$ if $n \in 4\,\mathbb{N} \setminus 16\,\mathbb{N}$, and $\chi(n) = 0$ if $n \in 16\,\mathbb{N}$.
- Let $\tau = \tau_{1/2}$. Then we have

Theorem

[Guinand-Y.] *The Fourier transform of the measure*

$$\tau = \sum_{1}^{\infty} \chi(n) r_3(n) n^{-1/2} (\delta_{\sqrt{n}/2} - \delta_{-\sqrt{n}/2}) \tag{6}$$

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Guinand-Y's measure

- Guinand-Y's theorem will also follow from Diaz' approach.
- The support of τ is the set $\Lambda = \{\pm\frac{\sqrt{m}}{2},\ m \neq 4^j(8k+7), j, k \in \mathbb{N}\}$.
- Therefore the density of Λ is infinite and τ is an exotic crystalline measure.

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The following theorem will be proved now:

Theorem

[Ildefonso-Y.] *Let V be a three dimensional torus (viewed as a Riemannian manifold). Let ν be a finitely supported measure on V such that $\int_V d\,\nu = 0$. Let $u : V \times \mathbb{R} \mapsto \mathbb{R}$ be the solution of the Cauchy problem*

(i) $\frac{\partial^2}{\partial t^2} u(x,t) = \Delta\, u(x,t)$

(ii) $u(x,0) = 0,\ \frac{\partial}{\partial t} u(x,0) = \nu.$

This solution $u(x,t)$ is extended to $t \leq 0$ by $u(x,-t) = -u(x,t)$.

Then $t \mapsto u(x_0,t)$ is a crystalline measure for every $x_0 \in V$ which does not belong to the support of ν.

Guinand's distribution revisited

Guinand-Y's theorem is now a direct corollary. Guinand's theorem is not needed. It suffices to define ν by the following four conditions: ν is supported by $\{k/4,\ k \in \mathbb{Z}^3\}$, ν does not charge $\mathbb{Z}^3$, the mass of ν on each point in $(\mathbb{Z}^3/2) \setminus \mathbb{Z}^3$ is $1/2$, and the charge of ν on each point in $(\mathbb{Z}^3/4) \setminus (\mathbb{Z}^3/2)$ is $-1/16$.

Well-known facts:

Lemma

Let $E = \mathcal{D}'(\mathbb{T}^3)$ *denotes the space of Schwartz distributions on* $\mathbb{T}^3$. *Then for every* $u_1(x) \in E$ *there exists a unique solution* $u(x,t) \in \mathcal{C}^\infty([0,\infty), E)$ *of the Cauchy problem*

(*i*) $\frac{\partial^2}{\partial t^2} u(x,t) = \Delta\, u(x,t)$

(*ii*) $u(x,0) = 0,\ \frac{\partial}{\partial t} u(x,0) = u_1(x)$.

Moreover $t \mapsto u(x,t)$ *extended to* $\mathbb{R}$ *as an odd function of* t *belongs to* $\mathcal{C}^\infty(\mathbb{R}, E)$

Guinand's distribution revisited

- Let $u_1(x) = \sum_{k\in\mathbb{Z}^3} \alpha(k)\exp(2\pi ik\cdot x)$ be the Fourier series expansion of $u_1(x)$. Then the solution $u(x,t)$ defined by Lemma 3.1 is given by

$$(7) \qquad u(x,t) = \alpha(0)t + \sum_{k\in\mathbb{Z}^3\setminus\{0\}} \alpha(k)\frac{\sin(2\pi t|k|)}{2\pi|k|}\exp(2\pi ik\cdot x).$$

- A similar result holds for the wave equation on $\mathbb{R}^3$ where E is replaced by the Schwartz space $\mathcal{S}'$ of tempered distributions on $\mathbb{R}^3$.
- If we are given a tempered distribution $u_1(x)$ on $\mathbb{R}^3$ there exists a unique solution $u(x,t)$ of the wave equation $\frac{\partial^2}{\partial t^2}u(x,t) = \Delta\, u(x,t)$ such that $u(x,0)=0$, $\frac{\partial}{\partial t}u(x,0) = u_1(x)$. It is given by $\widehat{u}(\xi,t) = \frac{\sin(2\pi t|\xi|)}{2\pi|\xi|}\widehat{u}_1(\xi)$.

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Guinand's distribution revisited

We now introduce Guinand's distribution.

Corollary

Let $w(x,t)$ be defined on $\mathbb{T}^3 \times \mathbb{R}$ by

$$(8) \qquad w(x,t) = t + \sum_{k \in \mathbb{Z}^3 \setminus \{0\}} \frac{\sin(2\pi t|k|)}{2\pi|k|} \exp(2\pi i k \cdot x).$$

Then $w(x,t)$ is the solution to the following Cauchy problem for the wave equation on $\mathbb{T}^3 \times \mathbb{R}$

(i) $\frac{\partial^2}{\partial t^2} u(x,t) = \Delta\, u(x,t)$

(ii) $u(x,0) = 0,\ \frac{\partial}{\partial t} u(x,0) = \delta_0(x).$

Huygens' principle

- But $w(x,t)$ can also be computed by periodization of the solution of the same Cauchy problem on $\mathbb{R}^3 \times \mathbb{R}$. This scheme is detailed now.

Lemma

Let σ_t, $t \in \mathbb{R}$, be the normalized surface measure on the sphere $B_t \subset \mathbb{R}^3$ centered at 0 with radius $|t|$ (the total mass of σ_t is 1). Then $v(x,t) = t\,\sigma_t(x)$ belongs to $\mathcal{C}^\infty(\mathbb{R}, \mathcal{S}'(\mathbb{R}^3))$ and is the solution of the Cauchy problem:

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$$u(x,0) = 0,\ \frac{\partial}{\partial t} u(x,0) = \delta_0(x).$$

Huygens' principle

- But $w(x,t)$ can also be computed by periodization of the solution of the same Cauchy problem on $\mathbb{R}^3 \times \mathbb{R}$. This scheme is detailed now.

Lemma

Let σ_t, $t \in \mathbb{R}$, be the normalized surface measure on the sphere $B_t \subset \mathbb{R}^3$ centered at 0 with radius $|t|$ (the total mass of σ_t is 1). Then $v(x,t) = t\,\sigma_t(x)$ belongs to $\mathcal{C}^\infty(\mathbb{R}, \mathcal{S}'(\mathbb{R}^3))$ and is the solution of the Cauchy problem:

$$\frac{\partial^2}{\partial t^2} u(x,t) = \Delta\, u(x,t)$$

$$u(x,0) = 0,\ \frac{\partial}{\partial t} u(x,0) = \delta_0(x).$$

-

Corollary

Under the assumptions of Lemma 3.2 one has that

$$w(x,t) = \sum_{k\in\mathbb{Z}^3} t\,\sigma_t(x-k) \tag{9}$$

is the solution of the following Cauchy problem for the wave equation on the three dimensional torus:

(a) $w(x,0) = 0$

(b) $\frac{\partial}{\partial t} w(x,0) = \delta_0(x)$.

The two expansions of $w(x,t)$ given by (8) and (9) are equal and this is the main step to the proof of Guinand's theorem.

Lemma

With the preceding notations we have

$$(10)\quad w(x,t)=\sum_{k\in\mathbb{Z}^3} t\,\sigma_t(x-k)=t+\sum_{k\in\mathbb{Z}^3\setminus\{0\}}\frac{\sin(2\pi t|k|)}{2\pi|k|}\exp(2\pi i k\cdot x).$$

This identity holds in $\mathcal{C}^\infty(\mathbb{R},\mathcal{S}'(\mathbb{R}^3))$.

- Let us compute the trace on $x = x_0$ of the LHS and RHS of (10) as a function of t. This trace is defined as follows.

Definition

A distribution $u(x,t) \in \mathcal{S}'(\mathbb{R}^3 \times \mathbb{R})$ defines a continuous mapping from $\mathbb{R}^3$ to $\mathcal{S}'(\mathbb{R})$ if for every test function $\phi \in \mathcal{S}(\mathbb{R})$, the distribution $< u(x,\cdot), \phi(\cdot) >$ is a continuous function of $x \in \mathbb{R}^3$.

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- The RHS of (10) fulfills this requirement since $\widehat{\phi}(|k|)$ is rapidly decreasing for $\phi \in \mathcal{S}(\mathbb{R})$. Therefore the trace $w(x_0, t)$ exists for every $x_0 \in \mathbb{R}^3$ and belongs to $\mathcal{S}'(\mathbb{R})$.

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- For computing the trace of the LHS of (10) one uses the following observation:

Lemma

For every $x_0 \in \mathbb{R}^3 \setminus \{0\}$, the trace on $x = x_0$ of the tempered distribution $t\sigma_t(\cdot) \in \mathcal{S}'(\mathbb{R}^3 \times \mathbb{R})$ is $\frac{1}{4\pi|x_0|}(\delta_{|x_0|} - \delta_{-|x_0|})$.

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Lemma 3.4 implies the following

Lemma

If $x_0 \notin \mathbb{Z}^3$ the trace of $\sum_{k\in\mathbb{Z}^3} t\,\sigma_t(x-k)$ is $\sum_{k\in\mathbb{Z}^3} \frac{1}{4\pi|x_0-k|}\left(\delta_{|x_0-k|} - \delta_{-|x_0-k|}\right)$.

Guinand's distribution revisited

- We can conclude: Let $x_0 \notin \mathbb{Z}^3$. Then we have

$$\sum_{k \in \mathbb{Z}^3} \frac{1}{|x_0 - k|}(\delta_{|x_0-k|} - \delta_{-|x_0-k|}) =$$

$$4\pi t + 2 \sum_{k \in \mathbb{Z}^3 \setminus \{0\}} \frac{\sin(2\pi|k|t)}{|k|} \exp(2\pi i k \cdot x_0) \tag{11}$$

and theses two series converge in $\mathcal{S}'(\mathbb{R})$.

- If $x_0 = 0$ he RHS of (11) is the Fourier transform of the Guinand's distribution.
- This identity does not make sense if $x_0 = 0$ which is needed for recovering Guinand's theorem (this is not needed to recover the Guinand-Y's theorem). As it will be seen the divergence which occurs is responsible for the derivative of the Dirac mass in the definition of σ.

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- To settle this problem it suffices to observe that the distribution $\sum_{k\in\mathbb{Z}^3\setminus\{0\}} \frac{\sin(2\pi|k|t)}{|k|} \exp(2\pi ik\cdot x)$ is continuous on $\mathbb{R}^3$.
- We then compute $w(0,t)$ in (10) as $\lim_{x\to 0,\, x\neq 0} w(x,t)$.
- Then $\frac{1}{|x_0|}(\delta_{|x_0|} - \delta_{-|x_0|}) \to -2\frac{d}{dt}\delta_0$ as $x_0 \to 0$ which yields a new proof of Guninand's theorem.

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Proof of Ildefonso-Y.'s theorem

- It can be assumed that $x_0 = 0$. Then

$$(12) \qquad u(x,t) = \sum_{\gamma^* \in \Gamma^*} \widehat{\nu}(\gamma^*) \frac{\sin(2\pi t|\gamma^*|)}{2\pi|\gamma^*|} \exp(2\pi i x \cdot \gamma^*)$$

and we also have as above

$$(13) \qquad u(x,t) = \sum_{\gamma \in \Gamma} (t\,\sigma_t * \nu)(x - \gamma).$$

- By (13) $u(0,t)$ is an atomic measure and by (12) $u(0,t)$ is the Fourier transform of the atomic measure

$$(14) \qquad \mu = \sum_{\gamma^* \in \Gamma^*} \frac{i\,\widehat{\nu}(\gamma^*)}{4\pi|\gamma^*|}(\delta_{|\gamma^*|} - \delta_{-|\gamma^*|}).$$

- Let F be the support of ν. Then the support of the crystalline measure μ is the set $\Lambda = \{\pm|\gamma^*|,\ \gamma^* \in \Gamma^*,\ \gamma^* \neq 0\}$ and its spectrum is the set $S = \{\pm|x + \gamma|,\ \gamma \in \Gamma,\ x \in F\}$.

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Concluding remarks

The detour by the wave equation on the three dimensional torus provided us with a remarkable understanding of Guinand's distribution and Guinand's measure. Let us observe that the crystalline measures of I. Diaz and Y.M. are odd measures. However there exist many more odd crystalline measures than those described by I. Diaz and Y.M. . For example if α in (5) is irrational the corresponding crystalline measure τ_α cannot be described by our method. On the other hand Guinand proposed some examples of even crystalline measures in [3] without giving satisfactory proofs. These proofs were completed in [12].

Concluding remarks

- Finally an important family of even crystalline measures was constructed by D. Radchenko and M. Viazovska in [14]. They proved the following theorem:

Theorem

For every real number $\theta > 0$ there exists a sequence $a_n = a_n(\theta)$, $n \in \mathbb{N}$, such that

$$\mu_\theta = \sum_{0}^{\infty} a_n(\delta_{\sqrt{n}} + \delta_{-\sqrt{n}})$$

is a crystalline measure and its Fourier transform is

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The authors are indebted to Kristian Seip for pointing out this reference. Crystalline measures are still mysterious mathematical objects.

Guinand's construction

- Let us begin with Guinand's genuine construction as it can be found in [3]. By Legendre's theorem, an integer $n \geq 0$ can be written as a sum of three squares (0^2 being admitted) if and only if n is not of the form $4^j(8k+7)$, $j, k \in \mathbb{N}$.
- For instance 0, 1, 2, 3, 4, 5, 6 are sums of three squares but 7 is not. Let $r_3(n)$ be the number of decompositions of the integer $n \geq 1$ into a sum of three squares (with $r_3(n) = 0$ if n is not a sum of three squares).
- More precisely $r_3(n)$ is the number of points $k \in \mathbb{Z}^3$ such that $|k|^2 = n$. We have $r_3(4n) = r_3(n), \forall n \in \mathbb{N}, \; r_3(0) = 1, \, r_3(1) = 6, r_3(2) = 12, \ldots$. Then $r_3(2^j) = 6$ if j is even and 12 if j is odd.
- The behavior of $r_3(n)$ as $n \to \infty$ is erratic.

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Guinand's construction

Guinand began his seminal work [3] with a simple lemma

Lemma

For every $a > 0$ we have

$$1 + \sum_{1}^{\infty} r_3(n) \exp(-\pi na) =$$

$$a^{-3/2} + a^{-3/2} \sum_{1}^{\infty} r_3(n) \exp(-\pi n/a). \tag{14}$$

Guinand's construction

- Guinand continued as follows. Let $f_a(x) = x\exp(-\pi a x^2),\ x \in \mathbb{R},\ \ a > 0$. Then $f_a(x)$ is odd and its Fourier transform is

$$\widehat{f_a}(y) = -ia^{-3/2} y \exp(-\pi y^2/a).$$

- Now (4) can be written

$$\frac{df_a}{dx}(0) + \sum_1^{\infty} r_3(n) n^{-1/2} f_a(\sqrt{n}) =$$

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$$\sigma = -2\frac{d}{dx}\delta_0 + \sum_1^\infty r_3(n)n^{-1/2}(\delta_{\sqrt{n}} - \delta_{-\sqrt{n}}) \tag{16}$$

  which will be named Guinand's distribution.
- We have $\sum_0^N r_3(n)n^{-1/2} = 2\pi N + O(N^{1/4})$ which implies that σ is a tempered distribution. Guinand proved the following

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- The oscillating behavior at infinity of Guinand's distribution follows from Guinand's theorem.
- Since the Fourier transform of $\delta_{\sqrt{n}} - \delta_{-\sqrt{n}}$ is $-2i\sin(2\pi\sqrt{n}x)$, the Fourier transform of the tempered distribution

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- Then Guinand's theorem can be written equivalently $\sigma = \widetilde{\sigma}$. This remark will play a seminal role in Section 4.
- The terminology of signal processing is used in the following corollary.
- It happens that a signal can be decomposed into the sum between a *trend* and some fluctuation around this trend.
- A trend indicates the large scale evolution of the signal. An obvious example is given by the stock market.

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Corollary

Guinand's distribution is the sum of the trend $4\pi t$ and a fluctuation which is an almost periodic distribution. ıMore precisely we have

$$\sigma(t) = 4\pi t + 2\sum_{1}^{\infty} r_3(n)n^{-1/2}\sin(2\pi\sqrt{n}t). \tag{17}$$

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- Let us observe that σ is not an almost periodic distribution.
- We recall that a tempered distribution τ is almost periodic if for every test function ϕ in the Schwartz class the convolution product $\tau * \phi$ is an almost periodic function in the sense of Bohr.
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- Let us observe that σ is not an almost periodic distribution.
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- We return to Guinand's theorem. We need to prove $\langle \sigma, \widehat{\phi} \rangle = -i\langle \sigma, \phi \rangle$ for every test function ϕ.
- But (5) can be rewritten as $\langle \sigma, f_a \rangle = i\langle \sigma, \widehat{f_a} \rangle$ or $\langle \sigma, f_a \rangle = i\langle \widehat{\sigma}, f_a \rangle$.
- The collection f_a, $a > 0$, of odd functions is total in the subspace of odd functions of the Schwartz class and σ is a tempered distribution. By continuity it implies $\langle \sigma, f \rangle = i\langle \widehat{\sigma}, f \rangle$ for every odd function in the Schwartz class.
- For even functions ϕ the identity $\langle \sigma, \widehat{\phi} \rangle = -i\langle \sigma, \phi \rangle$ is trivial since σ is odd and $\langle \sigma, \widehat{\phi} \rangle = -i\langle \sigma, \phi \rangle = 0$. Every function in the Schwartz class is the sum of an even one and of an odd one which ends the proof.

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This proof belongs to Guinand. We now move one small step beyond Guinand's work and extract what we call Guinand's measure from Guinand's distribution.

Guinand's construction

- If χ was erased from (9) τ would no longer be a crystalline measure. The cancellations provided by χ are playing a key role.
- The measure τ is not an almost periodic measure.
- A Borel measure μ is almost periodic if for every compactly supported continuous function f the convolution product $g = \mu * f$ is an almost periodic function in the sense of Bohr.
- An almost periodic measure is translation bounded, which is not the case for τ. Indeed $|\tau|([x, x+1]) \to \infty$, $x \to \infty$. But τ is an almost periodic distribution.

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- If μ is a crystalline measure and if $\widehat{\mu} = \lambda\mu$ then $\lambda \in \{1, -1, i, -i\}$.
- Conversely for each of these four eigenvalues there exists a crystalline measure μ such that $\widehat{\mu} = \lambda\mu$. This will be proved in a forthcoming paper.

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The wave equation

- Since the Fourier transform of $\delta_{\sqrt{n}} - \delta_{-\sqrt{n}}$ is $-2i\sin(2\pi\sqrt{n}t)$, the Fourier transform of the tempered distribution

$$\sigma = -2\frac{d}{dt}\delta_0 + \sum_1^\infty r_3(n)n^{-1/2}(\delta_{\sqrt{n}} - \delta_{-\sqrt{n}})$$

  is $-i\widetilde{\sigma}$ where

$$\widetilde{\sigma} = 4\pi t + 2\sum_1^\infty r_3(n)n^{-1/2}\sin(2\pi\sqrt{n}t).$$

- This was already observed in Section 3. Using the variable t here is intentional.
- Then Guinand's theorem can be written equivalently

$$\sigma = \widetilde{\sigma} \tag{18}$$

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