Anisotropic soap bubbles

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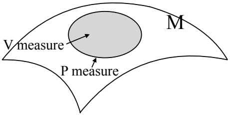
Outline

Introduction to soap bubble clusters

Minimal partition problem

M smooth manifold of dimension n, endowed with:

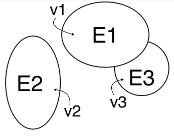
- V *volume* measure $\to$ "n-dimensional meas",
- P *perimeter* measure $\to$ "$(n-1)$-dimensional meas",



Minimal partition problem

Given $v_1, \dots, v_m > 0$, let

$$\mathcal{C}(v_1, \dots, v_m) = \left\{ E \subset M : \ E = \bigcup_{i=1}^{m} E_i, \ \mathrm{V}(E_i) = v_i, \ i = 1, \dots, m \right\}$$



where $E_1, \dots, E_m$ are pairwise disjoint.

Problem: $\inf \left\{ \mathcal{P}_P(E) : E \in \mathcal{C}(v_1, \dots v_m) \right\},$

where

$$\mathcal{P}_P(E) = \frac{1}{2} \left(\sum_{i=1}^{m} P(E_i) + P(M \setminus E) \right)$$

Solutions are called *m-minimal clusters*.

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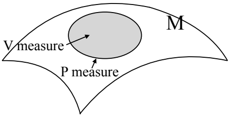

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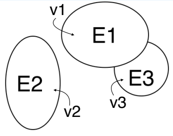

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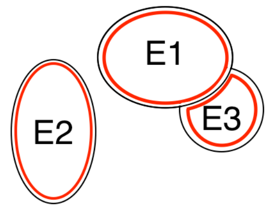

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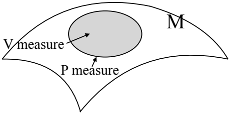

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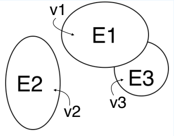

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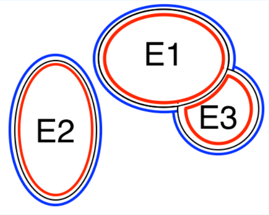

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Minimal cluster problem: Euclidean case

Example 1

$M = \mathbb{R}^n$, P = *De Giorgi perimeter*, V = *Lebesgue measure*

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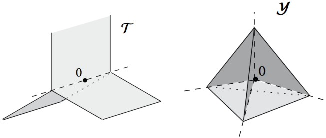
E minimal cluster $\implies$ ∂E is the union of smooth surfaces meeting in threes along an edge (Plateau border), at an angle of 120 degrees. These Plateau borders, in turn, meet in fours at a vertex at the *tetrahedral angle*.



Images from [Maggi (2012)]

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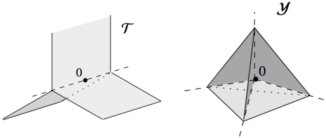

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- *Existence of solutions* and *structure of singularities* for $n = 2$: [Morgan (1994)] (singularities are only at 120 degrees).

Isoperimetric problem

(m=1)

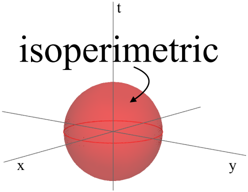
Isoperimetric Problem

Given $v > 0$, consider

$$\inf\{P(E) : E \subset M,\ V(E) = v\}.$$

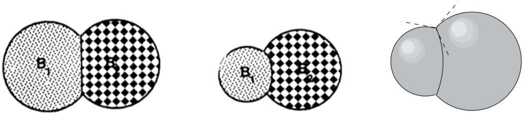
Solutions are called *isoperimetric sets*.

Isoperimetric sets are balls [De Giorgi (1958)]



Minimal cluster problem: Euclidean case

(m = 2) Double bubble problem. Solutions are standard double bubbles:



Images from [Foisy & al. (1993)] and [Maggi (2012)]

These are given by *three* $(n-1)$-dimensional spherical cups intersecting in a $(n-2)$-dimensional sphere at an angle of 120 *degrees*.

$\rightarrow$ $n = 2$ [Foisy et. al. (1993)]; $n = 3$ [Hutchings, Morgan, Ritoré, Ros (2002)]; $n \geq 4$ [Reichardt (2008)].

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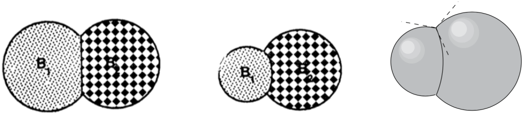

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(m = 3) If $n = 2$, solutions are the standard triple bubbles:

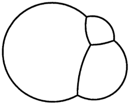


Image from [Wichiramala (2004)]

given by *six* arcs of circle intersecting at an angle of 120 *degrees*. [Wichiramala (2004)]

Minimal cluster problem: Euclidean case

(m $\geq$ 4) OPEN.

. . .

("m $= \infty$") Honeycomb theorem [Hales (2001)] : a regular hexagonal grid is the best way to tassellate the plane into regions of equal area with the least total perimeter.

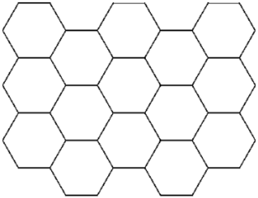


Image from Morgan's book

Minimal cluster problem: Riemannian case

Example 2

$M =$ Riemannian manifold, $P =$ *Riemannian perimeter*, $V =$ *Riemannian measure*

- *Existence of solutions* and *structure of singularities* from [Almgren (1986)]

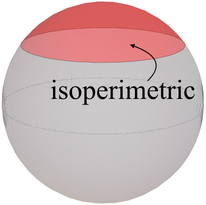
Minimal cluster problem: Riemannian case

Example 2

M = Riemannian manifold, P = *Riemannian perimeter*, V = *Riemannian measure*

- *Existence of solutions* and *structure of singularities* from [Almgren (1986)]

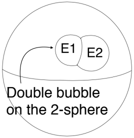
(m = 1) M = n-Sphere or the Hyperbolic space: solutions *metric balls*.

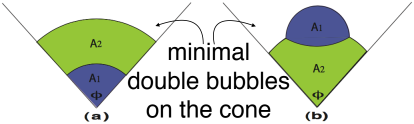


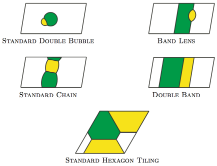
Minimal cluster problem: Riemannian case

(m = 2)

- $M = \mathbb{S}^n$: if $n = 2$ solutions are standard double bubbles [Masters (1996)]. If $n \geq 3$ only partial results are available.
- $M =$ 2-dimensional boundary of the cone in $\mathbb{R}^3$: 2 types of minimizers (two concentric circles or a circle lens) [Lopez & Baker (2006)].
- $M =$ flat 2-torus: 5 types of minimizers [Corneli et. al.(2004)] .







Anisotropic planar case

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Want to study:

$$\inf\left\{\mathcal{P}_P(E) : E = \bigcup_{i=1}^{m} E_i \subset \mathbb{R}^2,\ V(E_i) = v_i, i = 1 \dots m\right\}$$

- $n = 2$. Denote points as $(x, y) \in \mathbb{R}^2$;
- V anisotropic volume: Given $\beta \in \mathbb{R}$, and a Borel set $E \subset \mathbb{R}^2$, we consider its volume to be defined as

$$V_\beta(E) = \int_E |x|^\beta \, dxdy;$$

- P anisotropic perimeter: Given $\alpha > 0$, and $E \subset \mathbb{R}^2$, $\mathcal{L}^2$-measurable we consider its perimeter to be defined as

$$P_\alpha(E) = \sup\left\{ \int_E \partial_x\varphi_1 + |x|^\alpha \partial_y\varphi_2 \, dp : \varphi = (\varphi_1, \varphi_2) \in C_c^\infty(\mathbb{R}^2; \mathbb{R}^2),\ \|\varphi\|_\infty \le 1 \right\}.$$

This is also called the Grushin perimeter.

Preliminary remarks on volume and perimeter

$$V_\beta(E) = \int_E |x|^\beta \, dxdy; \quad P_\alpha(E) = \sup \left\{ \int_E \partial_x \varphi_1 + |x|^\alpha \partial_y \varphi_2 \, dp : \varphi \in C_c^\infty, \ \|\varphi\|_\infty \leq 1 \right\}.$$

On the anisotropic volume

- If $\beta = 0$, $V_\beta = \mathcal{L}^2$.
- It is not invariant under x-translations.

On the anisotropic perimeter

Proposition (Representation formula - Monti & Morbidelli (2004))

$E \subset \mathbb{R}^n$ bounded, ∂E Lipschitz. $N^E = (N_x^E, N_y^E)$ outer unit normal to ∂E. Then

$$P_\alpha(E) = \int_{\partial E} \sqrt{|N_x^E|^2 + |x|^{2\alpha} |N_y^E|^2} \, d\mathcal{H}^1.$$

- If $\alpha = 0$, then $P_\alpha = P$.
- It is an anisotropic perimeter, not invariant under x-translations.

A sub-Riemannian underlying geometric structure

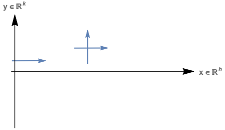
$$V_\beta(E) = \int_E |x|^\beta \, dxdy; \quad P_\alpha(E) = \sup\left\{ \int_E \partial_x \varphi_1 + |x|^\alpha \partial_y \varphi_2 \, dp : \varphi \in C_c^\infty, \ \|\varphi\|_\infty \leq 1 \right\}.$$

A sub-Riemannian underlying geometric structure

$$V_\beta(E) = \int_E |x|^\beta \, dxdy; \quad P_\alpha(E) = \sup\left\{ \int_E X\varphi_1 + Y\varphi_2 \, dp : \varphi \in C_c^\infty, \ \|\varphi\|_\infty \leq 1 \right\}.$$

Despite the fact that Y is vanishing along $\{x = 0\}$, a distance on $\mathbb{R}^2$ can be naturally associated with

$$X = \partial_x, \qquad Y(x,y) = |x|^\alpha \partial_y.$$



This is not a "Riemannian distance", but a sub-Riemannian one!

A sub-Riemannian underlying geometric structure

$$V_0(E) = \int_E dxdy; \qquad P_\alpha(E) = \sup\left\{ \int_E X\varphi_1 + Y\varphi_2 \, dp : \varphi \in C_c^\infty, \ \|\varphi\|_\infty \leq 1 \right\}.$$

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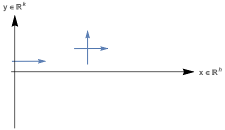

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Remark: For $\beta = 0$, P_α is then associated with such geometry, and $V_0 = \mathcal{L}^2$, i.e.,

$$P_\alpha(E) = \sup\left\{ \int_E \mathrm{div}_{\mathcal{L}^2}(\varphi_1 X + \varphi_2 Y) \, dp : \varphi \in C_c^\infty, \ \|\varphi\|_\infty \leq 1 \right\}.$$

$\rightarrow$ One can consider the *sub-Riemannian perimeter* defined above in any sub-Riemannian structure: several open problems (*Pansu's conjecture, lack of regularity theory, ...*)

$(\mathbf{m=1})$: Grushin isoperimetric problem: ✓

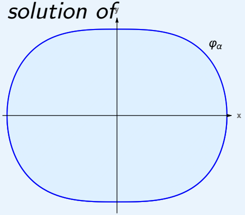
Theorem ([Monti & Morbidelli (2004)])

Let $v > 0$. Up to vertical translations, there exists a unique solution of

$$\inf\{P_\alpha(E) : \mathcal{L}^2(E) = v\}. \qquad (1)$$

If E solves (1), then there exists $\lambda > 0$ such that

$$E = \delta_\lambda(E_\alpha),$$



where $E_\alpha = \{(x,y) \in \mathbb{R}^2 : |y| \le \phi_\alpha(|x|),\ |x| \le 1\}$, and $\phi_\alpha : [0,1] \to [0,+\infty)$ is given by

$$\phi_\alpha(x) = \int_{\arcsin x}^{\frac{\pi}{2}} \sin^{\alpha+1}(t)\, dt, \qquad \text{for } x \in [0,1].$$

- $\delta_\lambda(x,y) = (\lambda x, \lambda^{\alpha+1} y)$ are anisotropic dilations such that $P_\alpha(\delta_\lambda(E)) = \lambda^{Q-1} P_\alpha(E)$, $\mathcal{L}^2(\delta_\lambda(E)) = \lambda^Q \mathcal{L}^2(E)$ for $Q = \alpha + 2$.
- For $\alpha = 1$, ϕ_α is the profile function of the Pansu set.

NB: Here rearrangements work!

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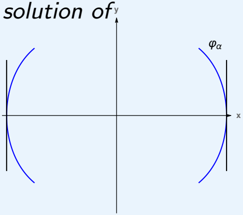

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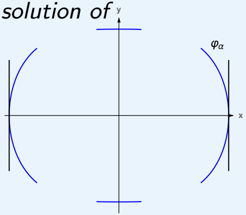

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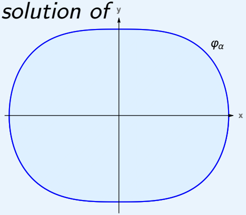

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$(\mathbf{m} \in \mathbb{N})$ Grushin minimal partition problem

Given $m \in \mathbb{N}$, $v_1, \dots, v_m > 0$ study

$$\inf \left\{ \mathcal{P}_\alpha(E) = \frac{1}{2} \left(\sum_{i=1}^{m} P_\alpha(E_i) + P_\alpha(E) \right) : E \in \mathcal{C}(v_1, \dots v_m) \right\},$$

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We address the following problems

1) Characterize solutions for $(\mathbf{m} = \mathbf{2})$ under additional assumptions: Symmetric double bubbles in the Grushin plane. (joint work with G. Stefani)

2) Prove existence of minimal clusters for any $\mathbf{m} \in \mathbb{N}$. (ongoing work with A. Pratelli, G. Stefani)

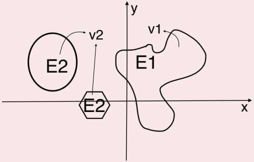
Symmetric double bubbles in the Grushin plane

Grushin double bubble problem

Want to study: Grushin double boubble problem

$$\inf\{\mathcal{P}_\alpha(E) : E = E_1 \cup E_2 \in \mathcal{C}(v_1, v_2)\}$$

$$\mathcal{P}_\alpha(E) = \frac{1}{2}(P_\alpha(E_1) + P_\alpha(E_2) + P_\alpha(\mathbb{R}^2 \setminus E)).$$



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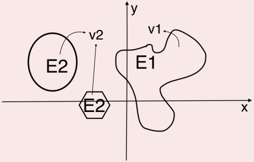


We consider the double bubble problem under more restrictive conditions:

(1) We assume $v_1 = v_2 = v \geq 0$.

(2) We assume specific structures of interfaces.

Problem 1: **(DBV)** Double bubbles with vertical interface: $\mathcal{C}(v_1, v_2)$ replaced by

$$\mathcal{A}^V(v) = \{E \subset \mathbb{R}^2 : \mathcal{L}^2(E \cap \{x > 0\}) = \mathcal{L}^2(E \cap \{x < 0\}) = v\}.$$

Problem 2: **(DBH)** Double bubbles with horizontal interface: $\mathcal{C}(v_1, v_2)$ replaced by

$$\mathcal{A}^H(v) = \{E \subset \mathbb{R}^2 : \mathcal{L}^2(E \cap \{y > 0\}) = \mathcal{L}^2(E \cap \{y < 0\}) = v\}.$$

Main result

Theorem (F., Stefani)

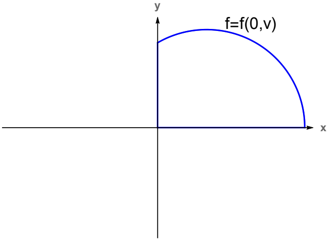
Let $v > 0$. Then solutions to problems **(DBV)**, **(DBH)** *exist. Moreover, the following statements hold.*

(DBV) *If $E^V \subset \mathbb{R}^2$ is a solution to (DBV), then, up to vertical translations, we have*

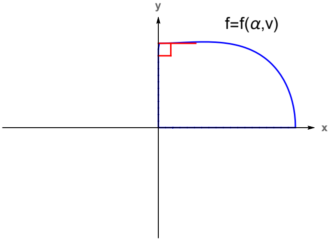
$$E^V = \{(x, y) \in \mathbb{R}^2 : |y| \leq f(|x|),\ |x| \leq r\},$$

where $f \in C([0, r]) \cap C^\infty(]0, r[)$, $r \in]0, +\infty[$, depends explicitly on v and α. In particular, if $\alpha > 0$, then $f'(0) = 0$.

Case $\alpha = 0$



Case $\alpha = 1$



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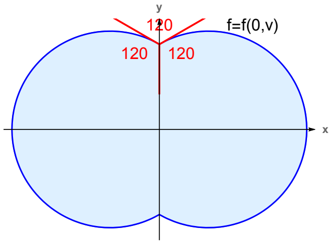

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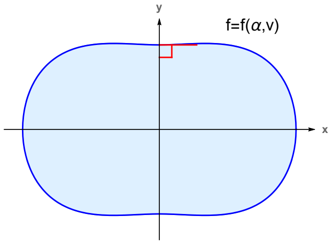

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Theorem (F., Stefani continuation)

(DBH) *If $E^H \subset \mathbb{R}^2$ is a solution to (DBH), then, up to vertical translations, we have*

$$E^H = \delta_{\frac{1}{h}} \left(\{(x,y) \in \mathbb{R}^2 : \left(x, |y| - \phi_\alpha \left(\frac{\sqrt{3}}{2} \right) \right) \in E_\alpha \} \right),$$

where $\phi_\alpha : [0,1] \to [0,+\infty[$ is the isoperimetric profile and h depends explicitly on v, α.

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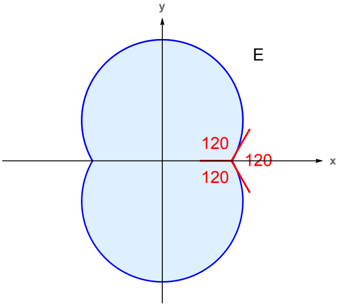

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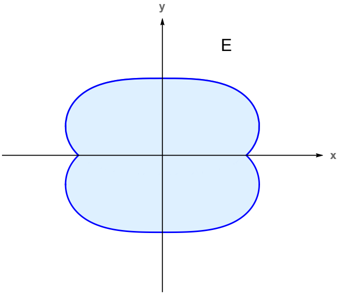

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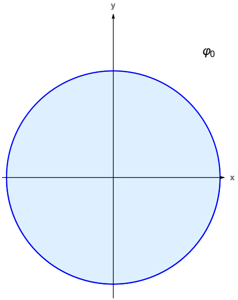

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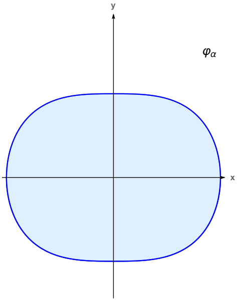

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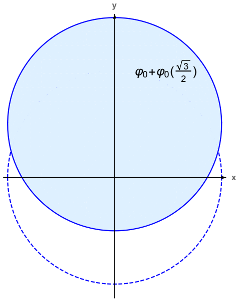

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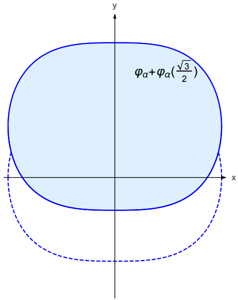

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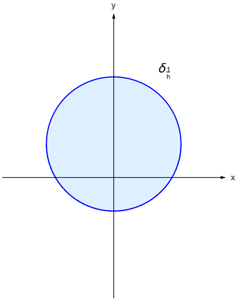

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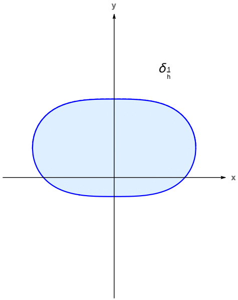

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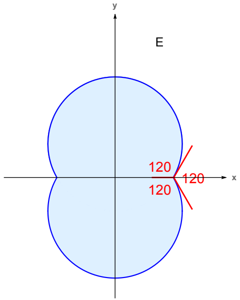

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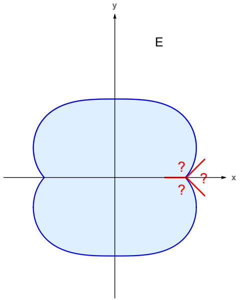
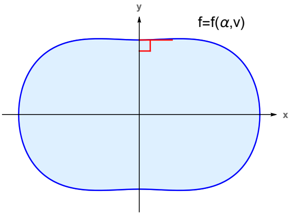


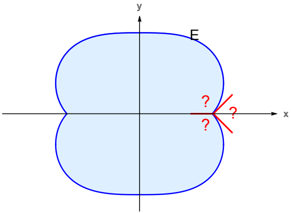
120 degrees rule.

In the Euclidean setting ($\alpha = 0$): $E \subset \mathbb{R}^2$ minimal cluster $\implies$ ∂E is the union of three arcs of circumference meeting in threes at an angle of 120 degrees.

What about 120 degrees rule in the case $\alpha > 0$?



(DBV) The angle of interface between the bubbles is flat.



(DBH) The angle of interface between the bubbles depends on α, v.

120 degrees rule.

Transformed plane [Monti & Morbidelli (2004)]

The diffeomorphism $\Psi : \mathbb{R}^2_{x,y} \to \mathbb{R}^2_{\xi,\eta}$, $\quad \Psi(x,y) = \left(\operatorname{sgn}(x)\dfrac{|x|^{\alpha+1}}{\alpha+1}, y\right)$ is such that

$$F := \Psi(E) \quad \Longrightarrow \quad P_\alpha(E) = P(F), \quad \mathcal{L}^2(E) = \mathcal{M}_\alpha(F) := \int_F |(\alpha+1)\xi|^{-\frac{\alpha}{\alpha+1}}\, d\xi\, d\eta$$

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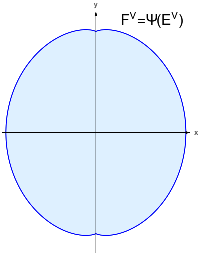
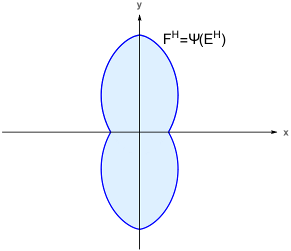
$$\Downarrow$$

E^V solution to **(DBV)**. Let $F^V = \Psi(E^V)$ $\quad$ E^H solution to **(DBH)**. Let $F^H = \Psi(E^H)$





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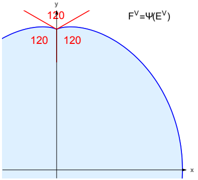
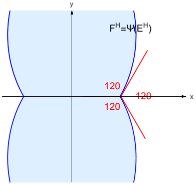




Corollary (F., Stefani)

The boundaries of the transformed bubbles meet at an angle of 120 degrees.

Idea of the proof

1. *Rearrangement theorem.*

Theorem (F., Stefani)

$E \in \mathcal{A}^V$, $P_\alpha(E) < \infty$. Then, there exists $\tilde{E} \subset \mathbb{R}^2$ bounded and locally Lipschitz such that

$$\mathcal{P}_\alpha(\tilde{E}) \leq \mathcal{P}_\alpha(E), \quad \text{and} \quad \mathcal{L}^2(\tilde{E}) = \mathcal{L}^2(E).$$

Moreover, if E is a solution of **(DBV)**, *then*

$$\mathcal{P}_\alpha(\tilde{E}) = \mathcal{P}_\alpha(E) \implies E = \tilde{E}$$

up to a set of measure zero and a vertical translation.

It is based on a rearrangement on the half plane $\mathbb{R}^2 \cap \{x > 0\}$ that allows us to decrease the total perimeter of a set possibly increasing the trace on $\{x = 0\}$.

2. *Existence* of a bounded and Lipschitz minimizer: via direct method.
3. Lipschitz *regularity* of minimizers: via rearrangement.
4. *Characterization*: first variation of $\mathcal{P}_\alpha$ and study of the resulting differential equations.

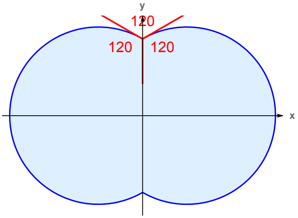
Comparison between vertical and horizontal

What can we conclude in view of the general double bubble problem?

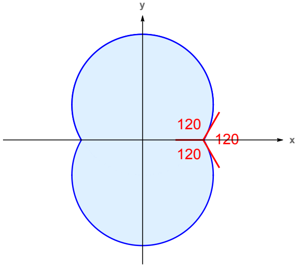
$$\inf \left\{ \mathcal{P}_\alpha(E) : E = E_1 \cup E_2,\ \mathcal{L}^2(E_i) = v \right\}$$

($\alpha = 0$): Solutions to **(DBV)** and **(DBH)** are the euclidean standard double bubbles.

Vertical interface



Horizontal interface



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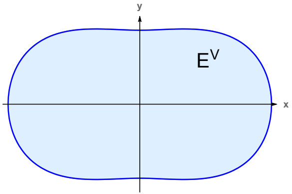
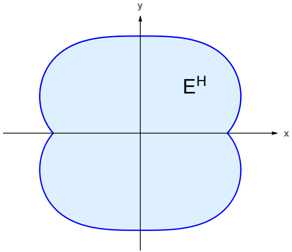
Vertical Interface $\alpha = 1$ Horizontal Interface





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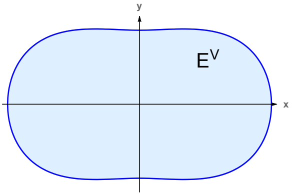

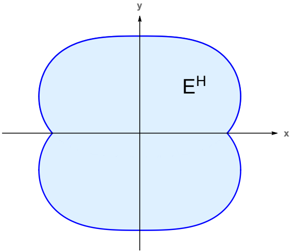

$$\mathcal{P}_1(E^V) \gneq \mathcal{P}_1(E^H)$$

(Non-)Existence of anisotropic m-clusters

A more general existence result

Theorem (F., Pratelli, Stefani)

Let $\alpha \geq 0$, $\beta \in \mathbb{R}$ *and* $m \in \mathbb{N}$. *Given* $v_1, \ldots, v_m > 0$, *there exists a solution to*

$$\inf \left\{ \mathcal{P}_\alpha(E) = \frac{1}{2} \left(\sum_{i=1}^{m} P_\alpha(E_i) + P_\alpha(E) \right) : E \in \mathcal{C}_\beta(v_1, \ldots v_m) \right\},$$

where

$$\mathcal{C}_\beta(v_1, \ldots, v_m) = \left\{ E \subset \mathbb{R}^2 : \; E = \bigcup_{i=1}^{m} E_i, \; V_\beta(E_i) = v_i, \; i = 1, \ldots, m \right\}.$$

if and only if $\beta \in]-1, \alpha]$.

Idea of the proof

Nonexistence: If $\beta \notin]-1, \alpha]$, we construct a sequence E_n of sets such that

$$V_\beta(E_n) = v \text{ for all } n \in \mathbb{N}, \quad \text{and} \quad P_\alpha(E_n) \to 0, \text{ as } n \to \infty.$$

In particular:

- If $\beta \leq 1$, the sets E_n are approaching $\{x = 0\}$;
- If $\beta > \alpha$, the sets E_n are escaping at infinity in the x-variable.

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Existence (Step 1): Reduction to $\beta = 0$.

Lemma

Let $\alpha \geq 0$, $\beta \in]-1, \alpha]$. Then there exist a change of coordinates $\Psi_{\alpha,\beta} : \mathbb{R}^2_{(x,y)} \to \mathbb{R}^2_{(\xi,\eta)}$ and a parameter $\tilde{\alpha} \geq 0$ such that if $F = \Psi_{\alpha,\beta}(E)$ then (up to a multiplicative constant)

$$P_\alpha(E) = P_{\tilde{\alpha}}(F), \qquad V_\beta(E) = \mathcal{L}^2(F).$$

Idea of the proof

(Step 2) We prove the following for the case $\beta = 0$

Let $(E_j)_{j\in\mathbb{N}} = (E_j^1 \cup \cdots \cup E_j^m)_{j\in\mathbb{N}}$ be a minimizing sequence for the minimal partition problem. Then there exists a sequence $\tilde{E}_j \in C(v_1, \ldots, v_m)$ such that

- $\mathcal{P}_\alpha(\tilde{E}_j) \le \mathcal{P}_\alpha(E_j)$;
- there exists $M > 0$ such that $\tilde{E}_j \subset [-M, M] \times [0, M]$ for all $j \in \mathbb{N}$.

Then we conclude by the direct method.

Strategy: 1) Project the set E_j on the x- and y-axes, obtaining the sets $\pi_x(E_j), \pi_y(E_j)$ with finite (equibounded) length

$$\max\{\mathcal{H}^1(\pi_x(E_j)), \mathcal{H}^1(\pi_y(E_j))\} \le C \sup_{j\in\mathbb{N}} \mathcal{P}_\alpha(E_j) < \infty.$$

2) Cover $\pi_x(E_j), \pi_y(E_j)$ by disjoint open intervals $\{I_j^k\}_{k\in\mathbb{N}}$ such that $\sum_k \mathcal{H}^1(I_j^k) < 2M$.
3) Vertical boundedness: "fill the holes" by moving the stripes towards the x-axis;
Horizontal boundedness: "Order the intervals from the closest to the furthest" from the y axes and "fill the holes" by moving the stripes towards the y-axis.

□

Ongoing work

* Starting from the suggestion given by the interpretation of the 120 degree rule, study the regularity theory for perimeter minimizers in the Grushin plane.
* Double bubble problem: remove the assumptions on the type of interface between the bubbles. As a first step, study CMC curves in the Grushin plane.

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Thank you for your attention!