Detection and construction of an elliptic solution to the complex cubic-quintic Ginzburg-Landau equation*

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Abstract

In evolution equations for a complex amplitude, the phase obeys a much more intricate equation than the amplitude. Nevertheless, general methods should be applicable to both variables. On the example of the traveling wave reduction of the complex cubic-quintic Ginzburg-Landau equation (CGL5), we explain how to overcome the difficulties arising in two such methods: (i) the criterium that the sum of residues of an elliptic solution should be zero, (ii) the construction of a first order differential equation admitting the given equation as a differential consequence (subequation method).

1 Introduction. Modulus *vs.* phase in amplitude equations

The time evolution equation $A_t + \cdots = 0$ for a complex amplitude $A(x,t)$ is usually, from physical requirements, invariant under an arbitrary shift of the phase $\varphi = \arg A$, in which M and φ denote the modulus and phase, $A = Me^{i\varphi}$. As a consequence, in the coupled partial differential system for (M,φ), the variable φ only contributes by its derivatives. Then, under a reduction to an ordinary differential equation (ODE) such as the travelling wave reduction

$$M \to \tilde{M}(\xi),\ \varphi \to -i\omega t + \tilde{\varphi}(\xi),\ \xi = x - ct, \tag{1}$$

in the coupled ODE system for $(\tilde{M}, \psi = \tilde{\varphi}')$, the highest derivation order for ψ will be one less than the derivation order for M. Consequently, the ODE for ψ obtained by the elimination of M will be much more complicated (by both its volume and its structure of singularities) than the ODE for M obtained by the elimination of ψ.

Let us take as an example the one-dimensional cubic-quintic complex Ginzburg-Landau equation (CGL5),

$$iA_t + pA_{xx} + q|A|^2A + r|A|^4A - i\gamma A = 0,\ (A,p,q,r) \in \mathcal{C},\ pr \neq 0,\ \Im(r/p) \neq 0. \tag{2}$$

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which depends on seven real parameters since γ can be chosen real. For a summary of results on CGL5, see the reviews [1, 16]. Its travelling wave reduction

$$A(x,t) = \sqrt{M(\xi)}e^{i(-\omega t + \varphi(\xi))},\ \xi = x - ct,\ (c,\omega,M,\varphi) \in \mathcal{R}. \tag{3}$$

$$\frac{M''}{2M} - \frac{M'^2}{4M^2} + i\varphi'' - \varphi'^2 + i\varphi'\frac{M'}{M} - i\frac{c}{2p}\frac{M'}{M} + \frac{c}{p}\varphi' + \frac{q}{p}M + \frac{r}{p}M^2 + \frac{\omega - i\gamma}{p} = 0, \tag{4}$$

introduces two additional real constants (c,ω), but the total number of real parameters (nine) can be lowered to seven by a translation of φ' and by noticing [16] that the velocity c and the imaginary part of $1/p$ only contribute by their product. Indeed, denoting the eight real parameters as $e_r, e_i, d_r, d_i, s_r, s_i, g_r, g_i$,

$$e_r + ie_i = \frac{r}{p},\ d_r + id_i = \frac{q}{p},\ s_r - is_i = \frac{1}{p},\ g_r + ig_i = \frac{\gamma + i\omega}{p} + \frac{c^2 s_r}{4}(2s_i + is_r), \tag{5}$$

and performing the translation

$$\varphi' = \frac{cs_r}{2} + \psi, \tag{6}$$

the system only depends on the seven real parameters $e_r, e_i, d_r, d_i, g_r, g_i, cs_i \equiv \kappa_{\rm i}$.

The coupled two-component system in the real variables (M,ψ),

$$\left\{\begin{array}{l} \dfrac{M''}{2M} - \dfrac{M'^2}{4M^2} - \kappa_{\rm i}\dfrac{M'}{2M} - \psi^2 + e_r M^2 + d_r M + g_i = 0, \\ \psi' + \psi\dfrac{M'}{M} - \kappa_{\rm i}\psi + e_i M^2 + d_i M - g_r = 0, \end{array}\right. \tag{7}$$

contains as highest derivatives M'' and ψ'. The following parity invariances of (7)

$$q = 0: \quad (M,\psi,\xi) \to (-M,\psi,\xi), \tag{8}$$

$$\kappa_{\rm i} = 0: \quad (M,\psi,\xi) \to (M,-\psi,-\xi), \tag{9}$$

will be used later on.

The elimination of ψ yields a one-line real third order second degree ODE for M [12]

$$\psi = \frac{2\kappa_{\rm i}G - G'}{2M^2(e_i M^2 + d_i M - g_r)},\ \psi^2 = \frac{G}{M^2}, \tag{10}$$

$$(G' - 2\kappa_{\rm i}G)^2 - 4GM^2(e_i M^2 + d_i M - g_r)^2 = 0, \tag{11}$$

$$G = \frac{1}{2}MM'' - \frac{1}{4}M'^2 - \frac{\kappa_{\rm i}}{2}MM' + e_r M^4 + d_r M^3 + g_i M^2, \tag{12}$$

while the elimination of M yields a third order fourth degree ODE for ψ which contains 11053 terms, whose dominant ones (in the sense of singularities, as developed in section 2) are,

$$\begin{aligned} e_i^6\psi^4 \Big\{ &\Big[25e_i^3\psi^2\psi''' - 10e_i^2(3e_i\psi' + 4e_r\psi^2)\psi\psi_2 - 6\psi'^3 e_i^3 - 24e_r e_i^2\psi'^2\psi^2 \\ &+ e_i(80e_i^2 - 112e_r^2)\psi^4\psi' - 32e_r(5e_i^2 + e_r^2)\psi^6\Big]^2 \\ &- 4\left[-3e_i^2\psi'^2 + 12e_r e_i\psi'\psi^2 + 4(5e_i^2 + 2e_r^2)\psi^4\right]^2 \\ &\Big[10e_i^2\psi\psi_2 + e_i^2\psi'^2 + 16e_r e_i\psi'\psi^2 + 4(5e_i^2 + e_r^2)\psi^4\Big]\Big\}^2 + \text{subdominant terms} = 0. \end{aligned} \tag{13}$$

As a by-product of the elimination process, the rational expression (10) of ψ in terms of M, M', M'' is quite short, while the rational expression of M in terms of ψ also involves the third derivative ψ''' and is quite lengthy. This is why the phase of the complex amplitude A is qualified as a "slave" variable [16], because it allows one to easily compute φ from M but not *vice versa*.

The purpose of this work is to explain on the above example how to overcome the difficulties created by the consideration of ψ in two specific methods.

The paper is organized as follows. In section 2, as a prerequisite study, we investigate the detailed structure of movable singularities of M and ψ, some features of which had been previously overlooked.
In section 3, we apply a first method successively to M and ψ in order to build necessary conditions for M or ψ to be elliptic. This proves easy for M and sets up additional questions for ψ, whose solution is provided.
In section 4, we indicate how to correctly apply a second method (the subequation method [14, 5, 6]) to M and ψ, in order to build a first order ODE sharing elliptic or degenerate elliptic solutions with the above third order ODEs (11) and (13).
In section 5, we simply present the solution of CGL5 in which the square modulus M is elliptic, whose obtention by the subequation method is detailed elsewhere [8].

2 Movable singularities of CGL5

In this section, we enumerate all the movable poles of either M or ψ, excluding those which represent a singular solution[1] of either the third order ODE (11) for M or the third order ODE (13) for ψ.

The structure of singularities of (2) has been studied in [13]. A first type of singularity $\chi = \xi - \xi_0 \to 0$ is obtained by balancing the terms A_{xx} and $|A|^4 A$ in (2),

$$A \sim A_0 \chi^{(-1/2+i\alpha)},\ \overline{A} \sim A_0 \chi^{(-1/2-i\alpha)},\ (-1/2+i\alpha)(-3/2+i\alpha)p + A_0^4 r = 0, \tag{14}$$

these algebraic branch points of A define four values of A_0^2 and two of α,

$$(e_i A_0^4)^2 - 4e_r A_0^4 - 3 = 0,\ \alpha = \frac{e_i}{2} A_0^4, \tag{15}$$

$$A_0^2 = \varepsilon_2 \sqrt{\frac{2e_r + \varepsilon_1 \Delta}{e_i^2}},\ \alpha = \frac{2e_r + \varepsilon_1 \Delta}{2e_i},\ \Delta = \sqrt{4e_r^2 + 3e_i^2},\ \varepsilon_1^2 = \varepsilon_2^2 = 1. \tag{16}$$

At these singularities, the square modulus $M = |A|^2$ displays four simple poles

$$M \sim m_0 \chi^{-1},\ m_0 = A_0^2, \tag{17}$$

and a direct investigation of the third order ODE (11) shows that M admits no other movable poles[2].

[1] A singular solution [4] of an ODE is any solution which cannot be obtained from the general solution. Such a solution must cancel an odd multiplicity factor of the discriminant of the ODE. For instance the ODE (11), whose discriminant is $GM^2(e_i M^2 + d_i M - g_r)^2$, admits as singular solutions all those of $G = 0$, which must be rejected since they are not solutions of the system (7).

[2] The Laurent series $M = \pm(3/(4e_r))^{1/2}\chi^{-1} + \cdots$ must be discarded since it cancels G and thus represents a singular solution.

Let us now count the poles of ψ by considering the real system (7). A first set of poles of ψ arises from the simple poles of M,

$$M = m_0 \chi_1^{-1} \left[1 + \left(\frac{\kappa_{\rm i}}{4} + \frac{2 d_r m_0 - 2 e_i d_i m_0^3}{4(1 + e_i^2 m_0^4)} \right) \chi_1 + \mathcal{O}(\chi_1^2) \right], \tag{18}$$

$$\psi = \frac{e_i m_0^2}{2} \chi_1^{-1} + \frac{e_i m_0^2}{8} \kappa_{\rm i} + m_0 \frac{4 d_i + 5 e_i d_r m_0^2 - e_i^2 d_i m_0^4}{4(1 + e_i^2 m_0^4)} + \mathcal{O}(\chi_1), \tag{19}$$

in which both invariances (8)–(9) require changing m_0 to $-m_0$. This first set defines four different simple poles of ψ when q is nonzero and only two when it vanishes (in which case M^2 obeys an algebraic equation admitting two double poles).

A second set of poles of ψ, not considered in [13], arises from the movable simple zeroes of M, and this set is best computed from the system (7). This is

$$\frac{1}{M} = \frac{1}{M_0} \chi_2^{-1} \left[1 + M_1 \chi_2 + \left\{ M_1^2 + \kappa_{\rm i} M_1 - \frac{j}{3} g_r + \frac{2}{3} g_i \right\} \chi_2^2 + \mathcal{O}(\chi_2^3) \right], \tag{20}$$

$$\begin{aligned} \psi = \frac{j}{2} \chi_2^{-1} \Bigg[& 1 + (\kappa_{\rm i} + M_1) \chi_2 + \left\{ M_1^2 + 2 \kappa_{\rm i} M_1 + \frac{2}{3} g_i - \frac{4j}{3} g_r + \frac{5}{6} \kappa_{\rm i}^2 \right\} \chi_2^2 \\ & + \frac{1}{2} \Bigg\{ (g_r + j g_i) \kappa_{\rm i} + \frac{3 j \kappa_{\rm i}^3}{4} - \frac{(3 d_i - j d_r) M_0}{4} + \frac{(11 j \kappa_{\rm i}^2 + 4 g_r + 4 j g_i) M_1}{4} \\ & + 3 j \kappa_{\rm i} M_1^2 + j M_1^3 \Bigg\} \chi_2^3 + \mathcal{O}(\chi_2^4) \Bigg], \end{aligned} \tag{21}$$

in which M_0 and M_1 are arbitrary constants, and j is any square root of -1. Again, both invariances (8)–(9) require changing M_0 to $-M_0$, with the additional requirement $M_1 = 0$ when $\kappa_{\rm i} = 0$. This second set defines $2N$ simple zeroes of M, and either $2N$ (when $q \neq 0$) or N (when $q = 0$) simple poles of ψ, with N an undetermined integer.

Remark 1 *In the nonlinear Schrödinger equation (p real, q real, $r = \gamma = 0$), the first integral $M\psi =$ constant implies that the only poles of ψ arise from the zeroes of M.*

We have searched for possible additional poles of ψ by directly investigating the third order ODE (13) for $\psi(\xi)$. One thus finds three kinds of families of movable simple poles, namely the two previous kinds (19) and (21) plus the following third kind comprising two families

$$\psi = p_0 \chi^{-1} \left[1 + \frac{\kappa_{\rm i}}{4} + \mathcal{O}(\chi) \right], \ 4(e_r^2 + 5 e_i^2) p_0^2 - 16 e_r e_i p_0 + 21 e_i^2 = 0. \tag{22}$$

However, this series cancels a factor of the discriminant of (13) having an odd multiplicity, therefore it is a singular solution of (13) and it must be discarded because it is not a solution of the system (7).

Therefore another distinction between M and ψ is the difference of complexity of their singularity structure: exactly 4 Laurent series for M, an unknown number of series for ψ.

Remark 2 *Out of the two special values $q = 0$ and $\kappa_{\rm i} = 0$ allowing an invariance in the differential system, see (8)–(9), only the value $q = 0$ is involved in the structure of singularities. The value $\kappa_{\rm i} = 0$ will show up in next sections 3 and 5.*

3 On the characterization of nondegenerate elliptic solutions

This is a classical result that, inside a fundamental domain, the sum of the residues of any elliptic function at its poles is equal to zero. This allowed Hone [11] to take advantage of the Laurent series to generate the following necessary conditions for the solution u of an ODE to be nondegenerate elliptic,

$$\forall j \in \mathcal{N}, \forall k \in \mathcal{N} : \; C_{jk}^{u} \equiv \sum_{\text{Laurent series}} \text{residue}\left(\left(u^{(k)}\right)^{j}\right) = 0, \tag{23}$$

in which the sum extends to any subset of the set of Laurent series, and thus to isolate those parameters for which the ODE might have a nondegenerate elliptic solution.

If one assumes that M is elliptic, it follows from (10) that ψ is also elliptic, so one can *a priori* use the Laurent series of either M or ψ to generate necessary conditions for M and ψ to be elliptic. However, as already noticed in section 2, the situation is much more complicated with ψ than with M. Indeed, some Laurent series of ψ (those near χ_2) depend on arbitrary constants, therefore one must first compute the number of different series (21), then one must also solve the generated residues conditions for these extra arbitrary constants. Let us apply the criterium successively to M and to ψ.

3.1 Criterium of residues applied to M

The variable M presents two advantages over ψ: it admits exactly four Laurent series (18), and no arbitrary coefficient enters these series. The sum (23) can include one, two, three or four Laurent series. However, with one or three series, the condition (23) applied to $\left(M^{(0)}\right)^{1}$ generates $m_0 = 0$, which is forbidden. With two series, because of the invariance $m_0 \to -m_0$ of (15), the generated conditions are identical to those with four series, therefore only one case remains to study, that with four series in the sum (23). After computing the first seven terms of each series (18), one generates ten conditions $C_{jk}^{M} = 0$, $jk = 01$, 02, 03, 04, 05, 06, 07, 12, 13, 22,

$$\begin{aligned} &C_{01}^{M} = 0, \; C_{02}^{M} \equiv \kappa_{\mathrm{i}} e_r = 0, \; C_{03}^{M} \equiv \kappa_{\mathrm{i}} \left(19 e_r d_i (e i^2 + 16 e_r^2) - e_i d_r (9 e i^2 + 6 e_r^2)\right) = 0, \\ &C_{04}^{M} \equiv \kappa_{\mathrm{i}} P(e_r, e_i, d_r, d_i, g_r, g_i, \kappa_{\mathrm{i}}^2) = 0, \; C_{12}^{M} \equiv \kappa_{\mathrm{i}} P(e_r, e_i, d_r, d_i, g_r, g_i, \kappa_{\mathrm{i}}^2) = 0, \; \cdots \end{aligned} \tag{24}$$

in which the two P's are polynomials containing respectively 18 and 16 terms. In the case $\kappa_{\mathrm{i}} \neq 0$, the generated constraints are

$$\kappa_{\mathrm{i}} \neq 0 : \; C_{02}^{M} \equiv e_r = 0, \; C_{03}^{M} \equiv d_r = 0, \; C_{04}^{M} \equiv 16 g_i + 3 \kappa_{\mathrm{i}}^2 = 0, \; C_{12}^{M} \equiv d_i = 0. \tag{25}$$

We will see in section 5 that these necessary conditions are sufficient because there does exist an elliptic solution when they are obeyed.

In the case $\kappa_{\mathrm{i}} = 0$, the parity invariance (9) selects very few nonidentically zero expressions $C_{jk}^{M} = 0$, the first few ones being: $C_{13}^{M} = P_{2,2,5,5,15,15}(g_r, g_i, d_r, d_i, e_r, e_i)$[3] (140 terms, requires 6 terms in the series), C_{32}^{M} (not computed, 8 terms in the series), C_{15}^{M} (not computed, 10 terms in the series), etc, thus requiring the computation of many terms in the series to generate the necessary conditions.

Fortunately, the unpleasant feature of having to deal with the second set of poles χ_2, whose number is unknown and which adds arbitrary coefficients to the system of necessary conditions,

[3]The indices denote the degree of P in its arguments.

can be avoided by associating to M a "subdominant" contribution of the "slave variable" ψ in the following way. Any product $\left(\psi^{(k_1)}\right)^{j_1}\left(M^{(k_2)}\right)^{j_2}$ which is holomorphic near χ_2 enjoys the same properties as M, i.e. : to have only four Laurent series at χ_1, to introduce no extra arbitrary coefficients.

The case $\kappa_{\rm i}=0$ can then be settled easily, the simplest necessary conditions being,

$$\left\{\begin{array}{l}
\psi M:\ 3e_i(2e_r^2+3e_i^2)d_r-e_r(8e_r^2+11e_i^2)d_i=0,\\
\qquad\qquad \text{solved as } d_r=\dfrac{e_r(8e_r^2+11e_i^2)}{3e_i(2e_r^2+3e_i^2)},\\
\psi M^2:\ 32(2e_r^2+3e_i^2)^2e_i(3e_ig_i+4e_rg_r)-(16e_r^2+19e_i^2)(4e_r^2+3e_i^2)e_rd_i^2=0,\\
\qquad\qquad \text{solved as } g_i=-\dfrac{4e_r}{3e_i}g_r+\dfrac{(16e_r^2+19e_i^2)(4e_r^2+3e_i^2)}{96e_i^2(2e_r^2+3e_i^2)^2}e_rd_i^2,\\
\psi M^5:\ d_ie_rP_{2,4,12,14}(g_r,d_i,e_r,e_i)=0,\\
\psi^3M^3:\ d_iP_{2,4,14,16}(g_r,d_i,e_r,e_i)=0,\\
\psi M^6:\ e_rP_{3,6,18,21}(g_r,d_i,e_r,e_i)=0.
\end{array}\right. \tag{26}$$

With only seven terms in each series, this defines three sets of necessary conditions

$$\kappa_{\rm i}=0,\ e_r=\frac{3\varepsilon}{2}e_i,\ d_r=\frac{29\varepsilon}{15}d_i,\ g_r=-\frac{d_i^2}{5e_i},\ g_i=-\frac{7\varepsilon d_i^2}{e_i},\ \varepsilon^2=1, \tag{27}$$

$$\kappa_{\rm i}=0,\ d_i=d_r=g_i=g_r=0, \tag{28}$$

$$\kappa_{\rm i}=0,\ d_i=d_r=g_i=e_r=0, \tag{29}$$

the last set being the particular case $\kappa_{\rm i}=0$ of the conditions (25). The first two sets (27)–(28) could well be refined by using more terms in each series.

3.2 Criterium of residues applied to ψ

The motivation of this study comes from a previous application of the criterium to ψ, in which Vernov [18] found the set of necessary conditions

$$\kappa_{\rm i}=e_r=d_r=d_i=g_ig_r=0, \tag{30}$$

which only detects the subcase $\kappa_{\rm i}=0$ of the elliptic solution (47). This result was achieved from the Laurent series of ψ^j, $j=1,2,3,4$, but only from the series (19) near χ_1, discarding the second set of poles (21) near χ_2. Let us show in this subsection that, if one takes account of both sets of poles of ψ, the single consideration of the variables $\left(\psi^{(k)}\right)^j$ allows one to recover the correct results.

Since an elliptic function possesses as many zeroes as poles, M must have exactly four simple zeroes (20). The corresponding set of poles of ψ, necessarily simple, are then the following:

1. – $(q\neq 0)$ 4 poles χ_1 plus 4 poles χ_2;
2. – $(q=0)$ 2 poles χ_1 plus 2 poles χ_2.

Let us denote the triplets (M_0,M_1,j) in (21) as either the four sets $(M_{0,1\pm},M_{1,1\pm},\pm i)$, $(M_{0,2\pm},M_{1,2\pm},\pm i)$ (case $q\neq 0$), or the two sets $(M_{0,\pm},M_{1,\pm},\pm i)$(case $q=0$).

Let us denote $C_{jk}^{\psi P}$ the sum of the four residues associated to the four poles χ_1 (19), and $C_{jk}^{\psi Z}$ the sum of two residues associated to two of the poles χ_2 (21) with opposite values of j.

The first set of sum of residues evaluates to [18]

$$\left\{\begin{array}{l} C_{01}^{\psi P} \equiv \dfrac{4e_r}{e_i},\ C_{02}^{\psi P} \equiv \kappa_{\rm i}\dfrac{8e_r^2+3e_i^2}{2e_i^2}, \\ C_{03}^{\psi P} \equiv 6\dfrac{e_r}{e_i}g_i + 4\dfrac{e_r^2}{e_i^2}g_r + \dfrac{9e_i^2+32e_r^2}{8e_i^3}e_r\kappa_{\rm i}^2 + \dfrac{520e_i^2e_r^4+256e_r^6+303e_i^4e_r^2+9e_i^6}{64e_i^3(e_r^2+e_i^2)^2}d_i^2 \\ \qquad - \dfrac{(132e_i^4+229e_i^2e_r^2+112e_r^4)}{16e_i^2(e_r^2+e_i^2)^2}e_rd_id_r + 3\dfrac{e_i^2(64e_r^4+135e_i^2e_r^2+81e_i^4)}{64e_i(e_r^2+e_i^2)^2}d_r^2, \\ C_{12}^{\psi P} \equiv \kappa_{\rm i}\Big[g_i - 2\dfrac{e_r}{e_i}g_r - \dfrac{3}{16}\kappa_{\rm i}^2 - \dfrac{112e_i^4+23e_i^2e_r^2-224e_r^4}{96e_i^2(e_r^2+e_i^2)^2}d_i^2 \\ \qquad + \dfrac{132e_i^4+229e_i^2e_r^2+112e_r^4}{32(e_r^2+e_i^2)^2}e_rd_id_r + \dfrac{19e_i^4-123e_i^2e_r^2-232e_r^4}{32e_i(e_r^2+e_i^2)^2}d_r^2\Big], \\ \ldots, \end{array}\right. \tag{31}$$

and the second set to (denoting $M_{1s} = M_{1,+} + M_{1,-}$, $M_{1d} = M_{1,+} - M_{1,-}$),

$$\left\{\begin{array}{l} C_{01}^{\psi Z} \equiv 0,\ C_{02}^{\psi Z} \equiv -\kappa_{\rm i} - \dfrac{M_{1s}}{2},\ C_{03}^{\psi Z} \equiv -g_r - \dfrac{3iM_{1d}}{4}(M_{1s}+2\kappa_{\rm i}), \\ C_{12}^{\psi Z} \equiv \dfrac{1}{4}M_{1s}^3 + \dfrac{3}{2}\kappa_{\rm i}M_{1s}^2 + \dfrac{41}{16}\kappa_{\rm i}^2M_{1s} + \dfrac{9}{8}\kappa_{\rm i}^3 + \dfrac{3}{4}(2\kappa_{\rm i}+M_{1s})M_{1d}^2 \\ \qquad - ig_rM_{1d} + g_i(2\kappa_{\rm i}+M_{1s}) + \dfrac{d_r}{4}(M_{0,+}+M_{0,-}) + \dfrac{3id_i}{4}(M_{0,+}-M_{0,-}), \\ C_{04}^{\psi Z} \equiv \dfrac{5}{16}M_{1s}^3 + \dfrac{15}{8}\kappa_{\rm i}M_{1s}^2 + \dfrac{57}{16}\kappa_{\rm i}^2M_{1s} + \dfrac{17}{8}\kappa_{\rm i}^3 + \dfrac{15}{16}(2\kappa_{\rm i}+M_{1s})M_{1d}^2 \\ \qquad - \dfrac{5}{4}ig_rM_{1d} + \dfrac{3}{4}g_i(2\kappa_{\rm i}+M_{1s}) + \dfrac{d_r}{16}(M_{0,+}+M_{0,-}) + \dfrac{3id_i}{16}(M_{0,+}-M_{0,-}), \\ \ldots \end{array}\right. \tag{32}$$

The sets of conditions to be solved are then (the arguments of $C_{jk}^{\psi Z}(\cdots)$ describe the additional unknowns)

$$q \neq 0,\ N=2: \quad C_{jk}^{\psi P} + C_{jk}^{\psi Z}(M_{0,1\pm}, M_{1,1\pm}, \pm i) + C_{jk}^{\psi Z}(M_{0,2\pm}, M_{1,2\pm}, \pm i) = 0, \tag{33}$$

$$q = 0,\ N=1: \quad \frac{1}{2}C_{jk}^{\psi P} + C_{jk}^{\psi Z}(M_{0,\pm}, M_{1,\pm}, \pm i) = 0. \tag{34}$$

The first set of conditions (33) contains too many unknowns to be solved when only seven terms in each series are considered, we leave the computation to the interested reader.

For the second set of conditions (34), since q is zero, the Laurent series (20)–(21) must possess the invariance (8),

$$q=0:\ (M,\psi,\chi,M_0,M_1,j) \to (-M,\psi,\chi,-M_0,M_1,j), \tag{35}$$

and, when $\kappa_{\rm i}$ is also zero, the additional invariance (9),

$$q=0,\ \kappa_{\rm i}=0:\ M_1=0,\ (M,\psi,\chi,M_0,j) \to (M,-\psi,-\chi,-M_0,j). \tag{36}$$

This second set of conditions evaluates to

$$\left\{\begin{array}{l} \psi^1:\ e_r = 0, \\ \psi^2:\ 2M_{1s} + \kappa_{\rm i} = 0, \\ \psi^3:\ 8g_r + 9\kappa_{\rm i}M_{1d} = 0, \\ \psi^5:\ 128e_i(M_{0+}^2 + M_{0-}^2) - 9i\kappa_{\rm i}M_{1d}(43\kappa_{\rm i}^2 + 272g_i) = 0, \\ \psi^4:\ \kappa_{\rm i}(3\kappa_{\rm i}^2 + 16g_i) = 0, \\ \psi'^3M^3:\ M_{1d}(128ie_i(M_{0+}^2 - M_{0-}^2) + 1296g_i\kappa_{\rm i}^2 + 279\kappa_{\rm i}^4) = 0, \\ \ldots \end{array}\right. \tag{37}$$

and only admits the two solutions

$$\kappa_{\rm i} \neq 0: \ e_r = 0, \ g_i = -\frac{3}{16}\kappa_{\rm i}^2, \ M_{0\pm}^2 = \frac{\kappa_{\rm i}^2(16g_r \pm 9i\kappa_{\rm i}^2)}{64e_i}, \ M_{1\pm} = \frac{(-9\kappa_{\rm i}^2 \pm 16ig_r)}{36\kappa_{\rm i}}, \tag{38}$$

$$\kappa_{\rm i} = 0: \ e_r = 0, \ g_r = 0, \ M_{1\pm} = 0, \ M_{0+}^2 + M_{0-}^2 = 0, \tag{39}$$

therefore the subcase $\kappa_{\rm i} \neq 0$ does succeed to generate the desired constraints (25).

Remark 3 *The denominator $\kappa_{\rm i}$ in (38) corresponds to the factor $\kappa_{\rm i}$ of the M'^4 term in (48) and, when $\kappa_{\rm i}$ vanishes, the singularities χ_2 do not exist any more in (48), see (49).*

When one now requires both sets of necessary conditions to hold true (the set (25), (27)–(29) generated in subsection 3.1, and the above set (38)–(39)), then only two possibilities remain for an elliptic solution M to exist: either (25), which does define a nondegenerate elliptic solution, or $e_r = d_r = d_i = g_r = g_i = 0$, which defines the four rational solutions $M = m_0/(\xi - \xi_0), \psi = (e_i m_0^2/2)/(\xi - \xi_0)$.

4 Method to determine all elliptic and degenerate elliptic solutions

Consider an N-th order autonomous algebraic ODE (40)

$$E(u^{(N)}, ..., u', u) = 0, \ ' = \frac{\mathrm{d}}{\mathrm{d}x}, \tag{40}$$

admitting at least one Laurent series

$$u = \chi^p \sum_{j=0}^{+\infty} u_j \chi^j, \ \chi = x - x_0. \tag{41}$$

There exists an algorithm [14] to find in closed form all its elliptic or degenerate elliptic solutions. Its successive steps are [6, 5]:

1. Find the analytic structure of movable singularities (e.g., 4 families of simple poles, 2 of double poles). For each subset of families (e.g. 2 families of simple poles) deduce the elliptic orders m, n (e.g. $m = 2, n = 4$) of u, u' and perform the next steps.

2. Compute slightly more than $(m+1)^2$ terms in the Laurent series.

3. Define the first order m-th degree subequation $F(u, u') = 0$ (it contains at most $(m+1)^2$ coefficients $a_{j,k}$),

$$F(u, u') \equiv \sum_{k=0}^{m} \sum_{j=0}^{2m-2k} a_{j,k} u^j u'^k = 0, \ a_{0,m} \neq 0. \tag{42}$$

   According to classical results of Briot-Bouquet and Painlevé (see details in [6]), any elliptic or degenerate elliptic solution of (40) *must* obey such an ODE, which is called "subequation" of (40) because it is required (in next step) to admit (40) as a differential consequence.

4. Require each Laurent series (41) to obey $F(u,u')=0$,

$$F \equiv \chi^{m(p-1)} \left(\sum_{j=0}^{J} F_j \chi^j + \mathcal{O}(\chi^{J+1}) \right), \ \forall j \ : \ F_j = 0. \tag{43}$$

   and solve this **linear overdetermined** system for $a_{j,k}$.

5. Integrate each resulting ODE $F(u,u')=0$.

The structure of singularities of M and ψ has been established in section 2, and the result of concern to us is: the movable poles of M and ψ are all simple, and the number of distinct Laurent series at these simple poles is equal to: 4 in the case of M, $4+2N$ in the case of ψ or M'/M if $q \neq 0$, and $2+N$ in the case of ψ or M'/M if $q=0$, with N an undetermined integer.

Let us start with M. In order to find all elliptic and degenerate elliptic solutions M by the subequation method, at step 1 the various subsets of families to be considered are made of one, two, three or four series (18), defining subequations $F=0$ whose degrees in (M',M) are respectively $(1,2)$, $(2,4)$, $(3,6)$, $(4,8)$. The computation presents no other difficulties than technical ones and its detailed results can be found in [8]. The main new result is a nondegenerate elliptic solution presented in section 5.

Applying the subequation method with ψ presents several difficulties.

1. The main one is not to forget the movable singularities of type χ_2 (movable simple zeroes (20) for M, movable simple poles (21) for ψ). This is the reason why a previous investigation [18] could only find a particular case of the elliptic solution presented in section 5.

2. The number of Laurent series (21) for ψ is undetermined, thus failing to set an upper bound to the degree of the subequation F. This point has already been settled in section 3.2, where we have established that the number of distinct Laurent series ψ is either eight ($q \neq 0$, four series near χ_1 plus four series near χ_2) or four ($q=0$, two series near χ_1 plus two series near χ_2).

3. The arbitrary coefficients in the series (21) must be determined by the subequation method and therefore require the computation of many more terms in each Laurent series.

4. Another difficulty, undetectable *a priori*, is the infinite value of $M_{1\pm}$ at $\kappa_{\rm i}=0$ in (38). Setting $\kappa_{\rm i}=0$ (for some reason) while looking for subequations being obeyed by four series will fail. In such a case ($q=0$ and $\kappa_{\rm i}=0$), ψ admits two distinct series near χ_1 and no series near χ_2, and the correct assumption for F is

$$F \equiv \psi'^2 + (a_{01}+a_{11}\psi+a_{21}\psi^2)\psi' + a_{00}+a_{10}\psi+a_{20}\psi^2+a_{30}\psi^3+a_{40}\psi^4 = 0, \tag{44}$$

   which is further reduced by a classical theorem (no first degree term [2, §181] in ψ' since ψ is assumed elliptic) and by the invariance (9) to the canonical Briot-Bouquet type

$$F \equiv \psi'^2 + a_{00} + a_{20}\psi^2 + a_{40}\psi^4 = 0, \tag{45}$$

   for which a solution was indeed found [18], see (49).

For all those reasons, the choice of M is by far the best one for amplitude equations such as CGL3, CGL5 or others.

5 The elliptic solution of CGL5

By "elliptic solution of CGL5", we mean a solution of CGL5 in which the square modulus M of the traveling wave reduction is elliptic.
The subequation method yields a unique genus-one subequation for M [8], and it requires exactly the constraints (25). Introducing the shorthand notation,

$$e_1 = \frac{\kappa_{\rm i}^2}{48},\ e_0 = \frac{g_r}{36}, \tag{46}$$

this fourth degree subequation is

$$\left\{ \begin{array}{l} M'^4 - 2\kappa_{\rm i} M M'^3 + \dfrac{72}{e_i} e_1 M'^2 (e_i M^2 - 12 e_0) + \dfrac{2^4 3^8 e_1^4}{e_i^2} \\ \qquad + \dfrac{648 e_1^2}{e_i^2}\left(288 e_0^2 + 24 e_i e_0 M^2 - e_i^2 M^4\right) - \dfrac{1}{3^4 e_i} M^2 \left(e_i M^2 - 48 e_0\right)^3 = 0. \end{array} \right. \tag{47}$$

Because of the previously mentioned numerous difficulties with ψ, the corresponding subequation for ψ is not determined by the subequation method, but by elimination with the correspondence (10), resulting in

$$\left\{ \begin{array}{l} \kappa_{\rm i} \psi'^4 - 4\kappa_{\rm i} \psi'^3 (\kappa_{\rm i} \psi + 24 e_0) \\ \quad + 8\psi'^2 \left(-\kappa_{\rm i}(27 e_1^2 - 324 e_0^2) + 1440 e_1 e_0 \psi + 27 \kappa_{\rm i} e_1 \psi^2 + 16 e_0 \psi^3 + \dfrac{1}{3}\kappa_{\rm i} \psi^4\right) \\ \quad + 16\Big(- \dfrac{1}{3}\kappa_{\rm i} \psi^8 - \dfrac{32}{3} e_0 \psi^7 - 26 \kappa_{\rm i} e_1 \psi^6 - 1632 e_1 e_0 \psi^5 - \left(477 e_1^2 + 552 e_0^2\right) \psi^4 \\ \quad - 288 \left(165 e_1^2 + 4 e_0^2\right) \psi^3 + \kappa_{\rm i} \left(2106 e_1^2 - 31320 e_0^2\right) \psi^2 \\ \quad + 2^7 3^6 \left(e_1^2 - 4 e_0^2\right) e_1 e_0 \psi + 243 \left(-9 e_1^4 + 56 e_1^2 e_0^2 - 144 e_0^4\right) \Big) = 0. \end{array} \right. \tag{48}$$

The singularity $\kappa_{\rm i} = 0$ already uncovered in (38) is displayed as the factor $\kappa_{\rm i}$ in front of the ψ'^4 term in (48), with the consequence that the degree of subequation (48) drops from four to two when $\kappa_{\rm i} = 0$. One then recovers the result of Vernov [18],

$$q = 0,\ e_r = 0,\ g_i = 0,\ \kappa_{\rm i} = 0, \left\{ \begin{array}{l} e_i (3M')^4 - M^2 \left(3 e_i M^2 - 4 g_r\right)^3 = 0, \\ 9\psi'^2 - 12\psi^4 - g_r^2 = 0, \end{array} \right. \tag{49}$$

in which both subequations for M and ψ belong to the list of five canonical equations of Briot and Bouquet.

Full details on the integration of (47) and (48) can be found in [9]. The final result for the complex amplitude A is,

$$\forall \kappa_{\rm i}:\ A = \text{constant}\ e^{-i\omega t + i\frac{c\xi}{2p}}\ {\rm H}(\xi, -\xi_+^B, 0)^{(-1+i\sqrt{3})/2}\, {\rm H}(\xi, -\xi_-^B, 0)^{(-1-i\sqrt{3})/2}, \tag{50}$$

in which ${\rm H}(\xi, q, k)$ is the *élément simple* defined by Hermite [10, vol. II, p. 506] for integrating the Lamé equation,

$${\rm H}(\xi, q, k) = \frac{\sigma(\xi + q)}{\sigma(\xi)\sigma(q)} e^{(k - \zeta(q))\xi}, \tag{51}$$

and the fixed constants $\xi_\pm^B$ in (50) are defined by

$$\left\{\begin{array}{l} \wp(\xi_\pm^B, G_2, G_3) = -2e_1 \pm i\sqrt{3}(3e_1 + 4ie_0),\ \wp'(\xi_\pm^B, G_2, G_3) = \dfrac{3 \mp i\sqrt{3}}{2}(3e_1 + 4ie_0), \\ G_2 = 12(13e_1^2 + 16e_0^2),\ G_3 = 8(35e_1^2 + 48e_0^2)e_1, \\ \wp'^2 = 4(\wp + 2e_1)(\wp^2 - 2e_1\wp - 35e_1^2 - 48e_0^2). \end{array}\right. \tag{52}$$

Numerical simulations with periodic boundary conditions [15, Fig. 4] do display solutions M having a real period (similar features are observed in CGL3 [3, Fig. 7]), these could well correspond to the present elliptic solution.

6 Conclusion

The traps described in this article should be kept in mind when looking for all the elliptic or degenerate elliptic solutions of other amplitude equations, such as the complex Swift-Hohenberg equation [17].

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