Meromorphic traveling wave solutions of the complex cubic-quintic Ginzburg-Landau equation*

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Abstract

We look for singlevalued solutions of the squared modulus M of the traveling wave reduction of the complex cubic-quintic Ginzburg-Landau equation. Using Clunie's lemma, we first prove that any meromorphic solution M is necessarily elliptic or degenerate elliptic. We then give the two canonical decompositions of the new elliptic solution recently obtained by the subequation method.

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1 Introduction. The CGL5 and CGL3 equations

When a system is governed by an autonomous nonlinear algebraic partial differential equation (PDE), it frequently admits permanent profile structures such as fronts, pulses, sinks, etc [28], and usually these profiles are mathematically some singlevalued solution of the traveling wave reduction $(x,t) \to x - ct$ of the PDE to an ordinary differential equation (ODE).

When the field is a slowly varying complex amplitude A, the simplest equation involving time evolution, dispersion, nonlinearity and forcing is the one-dimensional complex Ginzburg-Landau equation

$$iA_t + pA_{xx} + q|A|^2A + r|A|^4A - i\gamma A = 0,\ (A,p,q,r) \in \mathbb{C},\ \gamma \in \mathbb{R}. \tag{1}$$

We only consider in this class the equations which have the worst singularity structure, the cubic one (CGL3, $r = 0$, $\mathrm{Im}(q/p) \neq 0$) and the cubic-quintic one (CGL5, $\mathrm{Im}(r/p) \neq 0$). For a summary of results, see the reviews [1, 28].

CGL5 depends on seven real parameters. Its travelling wave reduction

$$A(x,t) = \sqrt{M(\xi)}e^{i(-\omega t + \varphi(\xi))},\ \xi = x - ct,\ (c,\omega,M,\varphi) \in \mathbb{R}, \tag{2}$$

$$\frac{M''}{2M} - \frac{M'^2}{4M^2} + i\varphi'' - \varphi'^2 + i\varphi'\frac{M'}{M} - i\frac{c}{2p}\frac{M'}{M} + \frac{c}{p}\varphi' + \frac{q}{p}M + \frac{r}{p}M^2 + \frac{\omega - i\gamma}{p} = 0, \tag{3}$$

depends on eight real parameters, denoted $e_r, e_i, d_r, d_i, s_r, s_i, g_r, g_i$,

$$e_r + ie_i = \frac{r}{p},\ d_r + id_i = \frac{q}{p},\ s_r - is_i = \frac{1}{p},$$
$$g_r + ig_i = \frac{\gamma + i\omega}{p} + \frac{c^2 s_r}{4}(2s_i + is_r). \tag{4}$$

The fact of taking account of the phase invariance [28]

$$\varphi' = \psi + \mathrm{Re}\frac{c}{2p}, \tag{5}$$

further reduces them to the seven real parameters $e_r, e_i, d_r, d_i, g_r, g_i, cs_i \equiv \kappa_{\mathrm{i}}$.

This third order system (3) can be written either as a real two-component rational system in the real variables (M, ψ),

$$\begin{cases} \dfrac{M''}{2M} - \dfrac{M'^2}{4M^2} - \kappa_{\rm i}\dfrac{M'}{2M} - \psi^2 + e_r M^2 + d_r M + g_i = 0, \\ \psi' + \psi\dfrac{M'}{M} - \kappa_{\rm i}\psi + e_i M^2 + d_i M - g_r = 0, \end{cases} \tag{6}$$

or, by elimination of ψ, as a real third order second degree ODE in M [20]

$$\psi = \frac{2\kappa_{\rm i}G - G'}{2M^2(e_i M^2 + d_i M - g_r)},\ \psi^2 = \frac{G}{M^2}, \tag{7}$$

$$(G' - 2\kappa_{\rm i}G)^2 - 4GM^2(e_i M^2 + d_i M - g_r)^2 = 0, \tag{8}$$

$$G = \frac{1}{2}MM'' - \frac{1}{4}M'^2 - \frac{\kappa_{\rm i}}{2}MM' + e_r M^4 + d_r M^3 + g_i M^2. \tag{9}$$

The purpose of this work is to show that, for all values of the seven parameters, all meromorphic particular solutions M of CGL3 and CGL5 belong to class W (like Weierstrass), defined as consisting of elliptic functions and their successive degeneracies, i.e.: elliptic functions (meromorphic doubly periodic), rational functions of one exponential $e^{k\xi}$, $k \in \mathbb{C}$, rational functions of ξ. The assumption M meromorphic implies the same property for the variables $M'/M, \psi$ and the logarithmic derivative of the complex amplitude $Ae^{i\omega t}$,

$$B := \frac{\mathrm{d}}{\mathrm{d}\xi}\log\left(Ae^{i\omega t - ics_r\xi/2}\right) = \frac{M'}{2M} + i\psi. \tag{10}$$

In case $q\kappa_{\rm i} = 0$, the real system (6) displays a parity invariance,

$$q = 0: \quad (M, \psi, \xi) \to (-M, \psi, \xi), \tag{11}$$

$$\kappa_{\rm i} = 0: \quad (M, \psi, \xi) \to (M, -\psi, -\xi). \tag{12}$$

This paper presents results complementary to those of [8]. Section 2 recalls the singularities of M and ψ. In section 3, we prove that, for all values of the seven real parameters, any meromorphic solution M of (6) is in class W. In section 4, we recall a method to obtain all the elliptic or degenerate elliptic solutions M, and present the first order ODE for $M(\xi)$ characterizing the recently obtained elliptic solution [8]. Finally, section 5 is devoted to the construction of canonical expressions to represent this elliptic solution.

2 Movable singularities of CGL3 and CGL5

Our interest is to count the number of distinct Laurent series for M and ψ. The results, obtained in [3] for CGL3 and in [23, 8] for CGL5, are the following.

A first set of poles $\chi_1 = \xi - \xi_1 \to 0$ for both M and ψ arises by balancing A_{xx} and the highest nonlinearity ($|A|^2A$ for CGL3, $|A|^4A$ for CGL5),

$$\text{CGL3}: \ A \sim A_0\chi_1^{-1+i\alpha}, \ \overline{A} \sim A_0\chi_1^{-1-i\alpha}, \ (1-i\alpha)(2-i\alpha)p + A_0^2 q = 0, \quad (13)$$

$$\text{CGL5}: \ A \sim A_0\chi_1^{-\frac{1}{2}+i\alpha}, \ \overline{A} \sim A_0\chi_1^{-\frac{1}{2}-i\alpha}, \ (\frac{1}{2}-i\alpha)(\frac{3}{2}-i\alpha)p + A_0^4 r = 0, \quad (14)$$

they define two real values α and two real values A_0^2 (CGL3) and A_0^4 (CGL5),

$$\text{CGL3}: \ d_i\alpha^2 - 3d_r - 2d_i = 0, \ A_0^2 = 3\alpha/d_i, \quad (15)$$

$$\text{CGL5}: \ 4e_i\alpha^2 - 8e_r - 3e_i = 0, \ A_0^4 = 2\alpha/e_i, \quad (16)$$

and the Fuchs indices are $-1, 0$ and two irrational numbers

$$\text{CGL3}: \ \text{indices} = -1, 0, (7 \pm \sqrt{1-24\alpha^2})/2, \quad (17)$$

$$\text{CGL5}: \ \text{indices} = -1, 0, (5 \pm \sqrt{1-32\alpha^2})/2. \quad (18)$$

For CGL3, M presents double poles and ψ simple poles,

$$M = m_0\chi_1^{-2}\left[1 + \frac{\kappa_{\rm i}}{3}\chi_1 + \mathcal{O}(\chi_1^2)\right], \ m_0 = A_0^2, \quad (19)$$

$$\psi = \frac{d_i m_0}{3}\chi_1^{-1}\left[1 + \frac{\kappa_{\rm i}}{6}\chi_1 + \mathcal{O}(\chi_1^2)\right], \quad (20)$$

and for CGL5 M and ψ present simple poles,

$$M = m_0\chi_1^{-1}\left[1 + \left(\frac{\kappa_{\rm i}}{4} + \frac{2d_r m_0 - 2e_i d_i m_0^3}{4(1+e_i^2 m_0^4)}\right)\chi_1 + \mathcal{O}(\chi_1^2)\right], \ m_0 = A_0^2, \quad (21)$$

$$\psi = \frac{e_i m_0^2}{2}\chi_1^{-1} + \frac{e_i m_0^2}{8}\kappa_{\rm i} + m_0\frac{4d_i + 5e_i d_r m_0^2 - e_i^2 d_i m_0^4}{4(1+e_i^2 m_0^4)} + \mathcal{O}(\chi_1), \quad (22)$$

in which both invariances (11)–(12) require changing m_0 to $-m_0$. The number of distinct Laurent series M near χ_1 is two (CGL3) or four (CGL5), and the number of series ψ is two (CGL3), four (CGL5 $q \neq 0$) or two (CGL5 $q = 0$).

A second set of singularities is easier to compute from the system (6),

$$\frac{1}{M} = \frac{1}{M_0}\chi_2^{-1}\left[1 + M_1\chi_2 + \left\{M_1^2 + \kappa_{\rm i}M_1 - \frac{j}{3}g_r + \frac{2}{3}g_i\right\}\chi_2^2 + \mathcal{O}(\chi_2^3)\right], \tag{23}$$

$$\begin{aligned}\psi = \frac{j}{2}\chi_2^{-1}\Bigg[1 + (\kappa_{\rm i} + M_1)\,\chi_2 + \left\{M_1^2 + 2\kappa_{\rm i}M_1 + \frac{2}{3}g_i - \frac{4j}{3}g_r + \frac{5}{6}\kappa_{\rm i}^2\right\}\chi_2^2 \\ + \frac{1}{2}\Bigg\{(g_r + jg_i)\kappa_{\rm i} + \frac{3j\kappa_{\rm i}^3}{4} - \frac{(3d_i - jd_r)M_0}{4} + \frac{(11j\kappa_{\rm i}^2 + 4g_r + 4jg_i)M_1}{4} \\ + 3j\kappa_{\rm i}M_1^2 + jM_1^3\Bigg\}\chi_2^3 + \mathcal{O}(\chi_2^4)\Bigg],\end{aligned} \tag{24}$$

in which M_0, M_1 are arbitrary constants, and $j^2 = -1$. Invariances (11)–(12) require changing M_0 to $-M_0$, with $M_1 = 0$ when $\kappa_{\rm i} = 0$. This defines either $2N$ (when $q \neq 0$) or N (when $q = 0$) simple poles of ψ, with N an undetermined integer. A direct study [8] of the third order ODEs for M and ψ shows that neither M nor ψ admit other movable poles.

For CGL3 (resp. CGL5), M and ψM admit two (resp. four) Laurent series.

3 Results from Clunie's lemma

For convenience, ξ will be denoted as z in this section only. We shall prove

Theorem 1 *For all values of the constants $p, q, r, \gamma, c, \omega$, all meromorphic traveling wave solutions M of CGL3 and CGL5 equations belong to class W.*

The method we use is a refinement of Eremenko's method developed in [10] as well as [11, 12, 7], based on the local singularity analysis of the solutions of the given differential equation and on the zero distribution and growth rate of their meromorphic solutions by using Nevanlinna theory.

Several partial results have been previously obtained [24, 17, 29] for finding solutions of (8), but they are incomplete and Theorem 1 settles the question. These previous results are the following.

1. For CGL3, when $d_r \neq 0$, all solutions belonging to class W have been found [24]: there are six distinct solutions which are rational functions in one exponential function and there is no elliptic solution.

2. For CGL3, when $\kappa_{\rm i} \neq 0$, there exists no elliptic solution [17].

3. For CGL5, when $\kappa_{\rm i} = 0$, there exists exactly one elliptic solution [29].

Let us recall a definition. For a differential polynomial of f,

$$P(z,f) = \sum a_j f^{j_0}(f')^{j_1}\cdots(f^{(k)})^{j_k},$$

where $j = (j_0, ..., j_k)$ is a multi-index and f and a_j are meromorphic functions, the sum $j_0+...+j_k$ is called the *degree* of the monomial $a_j f^{j_0}(f')^{j_1}\cdots(f^{(k)})^{j_k}$. The *total degree* of $P(z,f)$ is defined as the maximum of the degrees of its monomials.

We shall assume the readers are familiar with the terminology and results of Nevanlinna theory [15, 21, 25] (see [11] for a quick introduction). Here, we recall some basic notations of Nevanlinna theory. Let f be a non-constant meromorphic function on the open disc $D(r) = \{z : |z| < r\}$ where r. Denote the number of poles of f on the closed disc $\overline{D}(r)$ by $n(r,f)$, counting multiplicity. Define the *integrated counting function* $N(r,f)$ by $N(r,f) = n(0,f)\log r + \int_0^r [n(t,f) - n(0,f)]\,\frac{dt}{t}$ and the *proximity function* $m(r,f)$ by $m(r,f) = \int_0^{2\pi} \log^+ f(re^{i\theta})\frac{d\theta}{2\pi}$, where $\log^+ x = \max\{0, \log x\}$. Finally, the *Nevanlinna characteristic function* $T(r,f)$ is defined by $T(r,f) = m(r,f) + N(r,f)$ and we let $S(r,f)$ be a term such that $\frac{S(r,f)}{T(r,f)} \to 0$, as $r \to +\infty$.

To prove Theorem 1, we make use of the well known Clunie's Lemma,

Lemma 1 *[21, 2.4.2] Let f be a transcendental meromorphic solution of*

$$f^n P(z,f) = Q(z,f), \tag{25}$$

with n a positive integer, P and Q differential polynomials of f with meromorphic coefficients a_λ such that $m(r,a_\lambda) = S(r,f)$. If the total degree of Q is less than or equal to n, then

$$m(r, P(z,f)) = S(r,f). \tag{26}$$

Actually, all we need is the following corollary of Clunie's Lemma.

Corollary 1 *Let f be a transcendental meromorphic solution of the ODE*

$$f^{n+1} = Q(z,f), \tag{27}$$

with n a positive integer and Q a differential polynomial of f with meromorphic coefficients a_λ such that $m(r,a_\lambda) = S(r,f)$. If the total degree of Q is less than or equal to n, then f must have infinitely many poles.

Proof of Corollary 1.

Taking $P(z,f) = f$ in Lemma 1, we conclude that $m(r,f) = S(r,f)$, and therefore $(1-o(1))T(r,f) = N(r,f)$. Assume that f has finitely many poles. Then $N(r,f) = O(\log r)$, and therefore $T(r,f) = O(\log r)$, which is impossible since f is transcendental. QED.

Proof of Theorem 1

Let M be a solution of (8) which is meromorphic in the complex plane. If M is rational, then we are done. So suppose M is transcendental and let us prove that M has infinitely many poles. We first rewrite the second equation of (6) as

$$(\psi M)' - \kappa_{\rm i}(\psi M) + e_i M^3 + d_i M^2 - g_r M = 0. \tag{28}$$

It follows easily that if ψ has infinitely many poles, then so does M.

We first show that if ψ is transcendental, then ψ has infinitely many poles and hence so does M. One can build an ODE for $\psi(z)$ via the elimination of M between the system (6). This third order ODE is given as follows,

$$\text{CGL5} \ : \ e_i^2(5e_i^2 + e_r^2)^2\psi^{20} = Q(z,\psi), \tag{29}$$

$$\text{CGL3} \ : \ d_i(3d_i^2 + d_r^2)\psi^{10} = Q(z,\psi), \tag{30}$$

where the differential polynomial $Q(z,\psi)$ has the total degree 19 (CGL5 case) or 9 (CGL3 case). Applying Corollary 1 to (29) and (30), we conclude that ψ and therefore M must have infinitely many poles.

Now suppose ψ is rational, then it is well known that $T(r,\psi) = O(\log r)$ and $T(r,\psi') = O(\log r)$, and therefore $m(r,\psi) = S(r,f)$ and so is $m(r,\psi')$ since M is transcendental. Now (28) can be written as $e_i M^3 = -\psi' M - \psi M + (\kappa_{\rm i}\psi)M - d_i M^2 + g_r M$ and applying Corollary 1 to it, we again conclude that M has infinitely many poles.

Secondly, knowing that the transcendental meromorphic solution M has infinitely many poles, let us prove that it is a periodic function. By the local singularity analysis (section 2), if z_0 is a pole of M, CGL3 (resp. CGL5) admits exactly two (resp. four) Laurent series M with poles at $z = z_0$ obeying the ODE (8). Now let $z_j, j = 1,2,3,\cdots$ be the poles of $M(z)$; the functions $w_j(z) = M(z + z_j - z_0)$ are then meromorphic solutions of the ODE (8) with a pole at z_0, therefore some of them must be equal. Consequently, M is a periodic function.

Without loss of generality, we assume that M has a period of $2\pi i$. Let $D = \{z : 0 \le {\rm Im} z < 2\pi\}$. If M has more than three (CGL3) or five (CGL5)

poles in D, then by the previous argument we conclude that M is periodic in D and therefore is indeed an elliptic function and we are done.

Now suppose M has at most two (CGL3) or four (CGL5) poles in D. Since M is a periodic function with period $2\pi i$, we have $N(r, M) = O(r)$, as $r \to \infty$. It follows from $(1 - o(1))T(r, M) = N(r, M)$ that $T(r, M) = O(r)$. By Nevanlinna's First Fundamental Theorem, we know that for any $a \in \mathbb{C}$, $N(r, 1/(M - a)) = O(r)$ as $r \to \infty$. By the periodicity of M, we conclude that M takes each a finitely many times in D. Hence, the function $R(z) = M(\ln z)$ is a single-valued analytic function in the punctured plane $\{z : 0 < |z| < \infty\}$ and takes each a finitely many times. It follows that 0 is a removable singularity of R, and R must then be a rational function. Therefore, $M(z) = R(e^z)$ belongs to class W. QED.

4 A method to determine all solutions in class W

Consider an N-th order autonomous algebraic ODE,

$$E(u^{(N)}, ..., u', u) = 0, \ ' = \mathrm{d}/\mathrm{d}x, \tag{31}$$

admitting at least one Laurent series

$$u = \chi^p \sum_{j=0}^{+\infty} u_j \chi^j, \ \chi = x - x_0. \tag{32}$$

There exists an algorithm [24] to find in closed form all its elliptic or degenerate elliptic solutions. Its successive steps are [6, 5]:

1. Find the structure of movable singularities (e.g., 4 families of simple poles). For each subset of families (e.g. 2 families of simple poles) deduce the elliptic orders m, n (e.g. $m = 2, n = 4$) of u, u' and perform the next steps.

2. Compute slightly more than $(m + 1)^2$ terms in the Laurent series.

3. Define the first order m-th degree subequation

$$F(u, u') \equiv \sum_{k=0}^{m} \sum_{j=0}^{2m-2k} a_{j,k} u^j u'^k = 0, \ a_{0,m} \neq 0. \tag{33}$$

According to results of Briot-Bouquet and Painlevé [6], any solution of (31) in class W *must* obey such an ODE, called a "subequation" because (next step) it admits (31) as a differential consequence.

4. Require each Laurent series (32) to obey $F(u,u')=0$,

$$F \equiv \chi^{m(p-1)} \left(\sum_{j=0}^{J} F_j \chi^j + \mathcal{O}(\chi^{J+1}) \right), \ \forall j \ : \ F_j = 0. \tag{34}$$

and solve this **linear overdetermined** system for $a_{j,k}$.

5. Integrate each resulting ODE $F(u,u')=0$.

A similar method has later been developed [9], which also takes advantage of the Laurent series and directly searches for a canonical closed form representation of the elliptic solutions and their degeneracies.

Theorem 1 implies that the subequation method is indeed able to find all the meromorphic traveling wave solutions of the CGL3 and CGL5 equations.

For CGL5, the subequation method has produced a new elliptic solution [8], characterized by the first order, fourth degree, genus one ODE

$$F_4 \equiv M'^4 - 2\kappa_{\rm i} M M'^3 + \frac{72}{e_i} e_1 M'^2 (e_i M^2 - 12 e_0) + \frac{2^4 3^8 e_1^4}{e_i^2}$$
$$+ \frac{648 e_1^2}{e_i^2} \left(288 e_0^2 + 24 e_i e_0 M^2 - e_i^2 M^4 \right) - \frac{1}{3^4 e_i} M^2 \left(e_i M^2 - 48 e_0 \right)^3 = 0, \tag{35}$$

$$\kappa_{\rm i}^2 = 48 e_1, \ g_r = 36 e_0, \ e_r = d_r = d_i = 0, \ g_i = -\frac{3}{16} \kappa_{\rm i}^2. \tag{36}$$

5 Integration of subequation (35)

Let us first recall the differential equations of Weierstrass

$$\wp'^2 = 4\wp^3 - g_2 \wp - g_3, \ \zeta' = -\wp, \ (\log \sigma)' = \zeta. \tag{37}$$

Apart the representation as a rational function of $\wp$ and $\wp'$,

$$\frac{\text{polynomial}(\wp) + \text{polynomial}(\wp)\wp'}{\text{polynomial}(\wp)}, \tag{38}$$

elliptic functions have two main decompositions, either as a sum

$$C+\sum_{j=1}^{N}\left(r_j\ \zeta(\xi-a_j)+\sum_{k=0}^{M}c_{j,k}\wp^{(k)}(\xi-a_j)\right),\ \sum_{j=1}^{N}r_j=0, \qquad (39)$$

in which $C, r_j, a_j, c_{j,k}$ are complex constants (a_j distinct), or as a quotient of two products of an equal number of entire functions σ,

$$\text{constant}\ \prod_{j=1}^{P}\frac{\sigma(\xi-\alpha_j)}{\sigma(\xi-\beta_j)},\ \sum_{j=1}^{P}\alpha_j-\beta_j=0, \qquad (40)$$

in which α_j, β_j are not necessarily distinct complex constants.

To obtain the complex amplitude A, which is not elliptic, one can either compute the couple (M,ψ) then perform the quadrature $\int\psi\ \mathrm{d}\xi$, or compute the logarithmic derivative B, Eq. (10), then perform the quadrature $\int B\ \mathrm{d}\xi$.

By elimination with (7), one first deduces the real subequation for ψ,

$$\begin{aligned}
&\kappa_{\mathrm{i}}\psi'^4-4\kappa_{\mathrm{i}}\psi'^3\left(\kappa_{\mathrm{i}}\psi+24e_0\right)\\
&+8\psi'^2\Big(-\kappa_{\mathrm{i}}(27e_1^2-324e_0^2)+1440e_1e_0\psi+27\kappa_{\mathrm{i}}e_1\psi^2+16e_0\psi^3+\frac{1}{3}\kappa_{\mathrm{i}}\psi^4\Big)\\
&+16\Big(-\frac{1}{3}\kappa_{\mathrm{i}}\psi^8-\frac{32}{3}e_0\psi^7-26\kappa_{\mathrm{i}}e_1\psi^6-1632e_1e_0\psi^5-\left(477e_1^2+552e_0^2\right)\psi^4\\
&-288\left(165e_1^2+4e_0^2\right)\psi^3+\kappa_{\mathrm{i}}\left(2106e_1^2-31320e_0^2\right)\psi^2\\
&+2^7 3^6\left(e_1^2-4e_0^2\right)e_1e_0\psi+243\left(-9e_1^4+56e_1^2e_0^2-144e_0^4\right)\Big)=0,
\end{aligned} \qquad (41)$$

then the complex subequation for B as defined Eq. (10),

$$\begin{aligned}
&\left(2B'+\kappa_{\mathrm{i}}B+24ie_0\right)\left(B'-\kappa_{\mathrm{i}}B-24ie_0\right)^2\\
&+2^{-11}\left(16(4B^3-3\kappa_{\mathrm{i}}B^2)-9(\kappa_{\mathrm{i}}^2+64ie_0)(4B+\kappa_{\mathrm{i}})\right)^2=0.
\end{aligned} \qquad (42)$$

The degree of subequation (41) drops from four to two when $\kappa_{\mathrm{i}}=0$. As to (42), it has degree three and therefore belongs to the so-called trinomial type integrated by Briot and Bouquet [2, §250–251 p. 395].

Let us derive decompositions (39) or (40) for the solution of genus one equations (35) for M, (41) for ψ or (42) for B. Three steps are required.

– The *first step* is to represent the solution as a rational function of $\wp(\xi-\xi_0)$ and $\wp'(\xi-\xi_0)$, in which ξ_0 is arbitrary, and to write it in the canonical

form (38). Because of the existence of an addition formula for $\wp$,

$$\forall x_1, x_2 : \ \wp(x_1 + x_2) + \wp(x_1) + \wp(x_2) = \frac{1}{4}\left(\frac{\wp'(x_1) - \wp'(x_2)}{\wp(x_1) - \wp(x_2)}\right)^2 \quad (43)$$

such a canonical form (38) is not unique, and general algorithms may yield messy expressions by performing a shift on ξ_0. For instance, with the (otherwise powerful) command `Weierstrassform` [16] of the computer algebra language Maple [22], which applies to any genus one equation, the ODE

$$u'^2 = u^4 - u^3 + u^2 + u + 7 \quad (44)$$

is integrated as the second degree rational function

$$u = 3\frac{12\wp + 24\sqrt{7}\wp'}{144\wp^2 - 24\wp - 251},\ g_2 = \frac{22}{3},\ g_3 = \frac{277}{432}, \quad (45)$$

while a first degree rational function $c_0 + c_1/(\wp - c_2)$ is sufficient. The same occurs with the algorithm of Briot and Bouquet [2, §250–251 p. 395] to integrate binomial or trinomial equations: with (42), instead of yielding a second degree rational function rational in the fixed constants (see (50) below), it yields a third degree rational function algebraic in the fixed constants. Consequently, the practical method used here is to determine the smallest degrees of the three polynomials in (38), then their coefficients by identification. One thus finds for the solution M of (35),

$$\left\{\begin{array}{l} M = \dfrac{8N_0(3e_1 + 4je_0)(\wp - e_1)[3e_1\wp^2 + 4(3e_1^2 + 4e_0^2)\wp + 4e_1(3e_1^2 + 5e_0^2)]}{24(3e_1 + 4je_0)(\wp - e_1)(\wp^2 - 2e_1\wp - 8e_1^2 - 12e_0^2) + \kappa_i P_2^M \wp'}, \\ P_2^M = \wp^2 + 4(e_1 + je_0)\wp + 4e_1^2 - 4je_1e_0 + 12e_0^2,\ j^2 = -1, \\ N_0^2 = -\dfrac{324je_1^2}{e_i(3e_1 + 4je_0)}, \\ \wp := \wp(\xi - \xi_0^M, g_2, g_3), \\ \wp'^2 = 4(\wp - e_1)(\wp^2 + e_1\wp + 7e_1^2 + 12e_0^2), \\ g_2 = -24(e_1^2 + 2e_0^2),\ g_3 = 4(7e_1^2 + 12e_0^2)e_1. \end{array}\right. \quad (46)$$

This expression will simplify greatly as (60). Because of the correspondence (7), the solution ψ of (41) involves the same square root j of -1

as in (46). When $\kappa_{\rm i}$ is nonzero this is

$$\left\{\begin{array}{l}
\psi = -\dfrac{j\kappa_{\rm i}(9e_1-4je_0)}{24e_1} + \dfrac{P_2^\psi + Q_2^\psi \wp'}{12e_1(3e_1\wp+15e_1^2+16e_0^2)((\wp+2e_1)^2+3(3e_1+4je_0)^2)},\\
P_2^\psi = -j\kappa_{\rm i}(3e_1+4je_0)[(3e_1+2je_0)((9e_1-4je_0)\wp^2+2(-9e_1-44je_0)e_1\wp)\\
\qquad -945e_1^4-1434je_1^3e_0-1192e_0^2e_1^2-1440je_0^3e_1-384e_0^4],\\
Q_2^\psi = 9je_1(e_1(\wp^2+22\wp e_1+24j\wp e_0)+121e_1^3+48e_1e_0^2+192je_1^2e_0+128je_0^3),\\
\wp := \wp(\xi-\xi_0^M, G_2, G_3),\\
\wp'^2 = 4(\wp+2e_1)(\wp^2-2e_1\wp-35e_1^2-48e_0^2),\\
G_2 = 12(13e_1^2+16e_0^2),\ G_3 = 8(35e_1^2+48e_0^2)e_1,
\end{array}\right. \tag{47}$$

while for $\kappa_{\rm i}=0$ it is

$$\kappa_{\rm i}=0:\ \psi = \sqrt{6j\sqrt{3}e_0}\left(1+\frac{8j\sqrt{3}e_0}{\wp(\xi-\xi_0^M,192e_0^2,0)-4j\sqrt{3}e_0}\right), \tag{48}$$

or simply (but with yet another g_2),

$$\kappa_{\rm i}=0:\ \psi = \frac{\sqrt{3}}{2}\sqrt{\wp(\xi-\xi_0^M,-768e_0^2,0)}. \tag{49}$$

Finally, the solution B of (42) is expressed as,

$$\left\{\begin{array}{l}
B = \dfrac{\kappa_{\rm i}}{2} - \dfrac{6\kappa_{\rm i}(3e_1+4ie_0)^2+(\wp+2e_1+3(3e_1+4ie_0))\wp'}{2((\wp+2e_1)^2+3(3e_1+4ie_0)^2)},\\
\wp := \wp(\xi-\xi_0^B, G_2, G_3),
\end{array}\right. \tag{50}$$

The properties of the above three expressions are: all coefficients (except the global factor N_0) are rational in $(\kappa_{\rm i}, g_r)$, the two different Weierstrass functions $\wp(.,g_2,g_3)$ and $\wp(.,G_2,G_3)$ are linked by a Landen transformation (Appendix A), the relation (10) between B, M, ψ holds true when the square root j of -1 is equal to $+i$ and the constant origins ξ_0^M, ξ_0^B are equal.

The degeneracy $\Delta=0$ implies $g_2=g_3=0$, i.e. it directly defines the reducible subequation $(3M')^4 - e_i^2 M^8 = 0$, whose solutions are rational. Because of this, even a four-family extension of the method used in [23] would fail for CGL5.

– The *second step* is to compute the partial fraction decomposition of the rational functions (38) of the variable $\wp$, considering for a moment $\wp'$ as a parameter. The rational function (46), once converted to the canonical form (38), admits four poles for each choice of j, and we characterize their affixes $\xi^M_{j,k}$ by choosing the signs of $\wp'(\xi^M_{j,k}, g_2, g_3)$ as follows,

$$\left\{\begin{array}{l} \wp(\xi^M_{j,k}, g_2, g_3) = \left(-3 + 3(j^k + \sqrt{3}j^{1-k})\rho + (-1)^k\rho^2\right) e_1/6, \\ \wp'(\xi^M_{j,k}, g_2, g_3) = (9j^k + 3((-1)^{1+k} - j\sqrt{3})\rho + j^{2-k}\rho^2)e_1\kappa_{\rm i}\rho/36, \\ e_1\rho^2 = 3j\sqrt{3}(3e_1 + 4je_0),\ j = \pm i,\ k = 1,2,3,4. \end{array}\right. \tag{51}$$

The rational function (47) admits one real pole and, for each $j = \pm i$, two complex poles similarly characterized as follows,

$$\kappa_{\rm i} \neq 0 : \left\{\begin{array}{l} \wp(\xi^\psi_j, G_2, G_3) = -5e_1 - 16e_0^2/(3e_1), \\ \wp'(\xi^\psi_j, G_2, G_3) = -2j\kappa_{\rm i}e_0(9e_1^2 + 16e_0^2/(9e_1^2),\ j = \pm i, \end{array}\right. \tag{52}$$

$$\left\{\begin{array}{l} \wp(\xi^\psi_{j,k}, G_2, G_3) = -2e_1 + (-1)^k j\sqrt{3}(3e_1 + 4je_0),\ j = \pm i,\ k = 0,1, \\ \wp'(\xi^\psi_{j,k}, G_2, G_3) = (3 - (-1)^k j\sqrt{3})\kappa_{\rm i}(3e_1 + 4je_0)/2. \end{array}\right. \tag{53}$$

Finally, the two poles of (50) are just $\wp(\xi^\psi_{i,k}, G_2, G_3)$.

Modulo the periods of $\wp(., G_2, G_3)$, the affixes of these poles obey

$$\xi^\psi_{j,0} + \xi^\psi_{j,1} - \xi^\psi_j : \quad \wp = -2e_1,\ \wp' = 0,\ \zeta = {\rm H}_1 \ \ \text{(half-period)} \tag{54}$$

$$\xi^\psi_{j,0} + \xi^\psi_{j,1} : \quad \wp = 7e_1,\ \wp' = -6j\kappa_{\rm i}e_0, \tag{55}$$

$$\xi^\psi_{j,0} - \xi^\psi_{j,1} : \quad \wp = -5e_1,\ \wp' = -2\sqrt{3}\kappa_{\rm i}e_0, \tag{56}$$

$$\xi^\psi_{j,k} + \xi^\psi_{-j,1-k} - \xi^\psi_j : \quad \wp = \infty,\ \wp' = \infty \ \ \text{(period)}. \tag{57}$$

The Landen transformation maps $\wp(\xi^M_{j,2}, g_2, g_3)$ and $\wp(\xi^M_{j,4}, g_2, g_3)$ to $\wp(\xi^\psi_{j,0}, G_2, G_3)$, and maps $\wp(\xi^M_{j,1}, g_2, g_3)$ and $\wp(\xi^M_{j,3}, g_2, g_3)$ to $\wp(\xi^\psi_{j,1}, G_2, G_3)$. Expressions (46), (47), (50) thus evaluate to the sum

$$\text{constant } + \sum_{j=1}^{4} \frac{\text{constant } + \text{constant } \wp'(\xi - \xi_0)}{\wp(\xi - \xi_0) - \wp(\xi_j)}. \tag{58}$$

– The *third step* is, using the classical identities

$$\forall u,v:\ \begin{cases} \zeta(u+v)+\zeta(u-v)-2\zeta(u)=\dfrac{\wp'(u)}{\wp(u)-\wp(v)}, \\ \zeta(u+v)-\zeta(u-v)-2\zeta(v)=\dfrac{-\wp'(v)}{\wp(u)-\wp(v)}, \end{cases} \tag{59}$$

to convert (58) into a finite sum of ζ functions.

The result for (46),

$$\forall \kappa_{\rm i}:\ M=\frac{3^{1/4}}{\sqrt{-e_i}}\sum_{k=1}^{4} j^{k-1}\left(\zeta(\xi-\xi^M_{j,k},g_2,g_3)+\zeta(\xi^M_{j,k},g_2,g_3)\right),\ j^2=-1 \tag{60}$$

clearly displays the four simple poles.

As to the three simple pole variables $B,\psi,M'/M$, their decompositions evaluate to (we abbreviate $\zeta(.,G_2,G_3)$ to $\zeta(.)$)

$$\begin{cases} \forall \kappa_{\rm i}:\ \dfrac{\rm d}{{\rm d}\xi}\log\left(Ae^{i\omega t-i\frac{cs_r}{2}\xi}\right)=\dfrac{\kappa_{\rm i}}{2}+\zeta(\xi)+\left(\dfrac{-1+i\sqrt{3}}{2}\right)\left(\zeta(\xi-\xi^\psi_{i,0})+\zeta(\xi^\psi_{i,0})\right) \\ \qquad\qquad +\left(\dfrac{-1-i\sqrt{3}}{2}\right)\left(\zeta(\xi-\xi^\psi_{i,1})+\zeta(\xi^\psi_{i,1})\right), \\ \kappa_{\rm i}\neq 0:\ \psi=-j\dfrac{9e_1-4je_0}{24e_1}\kappa_{\rm i}+\dfrac{j}{2}\left(\zeta(\xi-\xi^\psi_j)+\zeta(\xi^\psi_j)-\zeta(\xi)\right) \\ \qquad\qquad +\dfrac{\sqrt{3}}{2}\left(\zeta(\xi-\xi^\psi_{j,0})+\zeta(\xi^\psi_{j,0})-\zeta(\xi-\xi^\psi_{j,1})-\zeta(\xi^\psi_{j,1})\right), \\ \kappa_{\rm i}\neq 0:\ \dfrac{M'}{M}=\dfrac{3e_1+4je_0}{12e_1}\kappa_{\rm i}+\zeta(\xi-\xi^\psi_j)+\zeta(\xi^\psi_j)+\zeta(\xi) \\ \qquad\qquad -\left(\zeta(\xi-\xi^\psi_{j,0})+\zeta(\xi^\psi_{j,0})+\zeta(\xi-\xi^\psi_{j,1})+\zeta(\xi^\psi_{j,1})\right). \end{cases} \tag{61}$$

The choice $j=+i$ must be made for (10) to hold true, while the choice $j=-i$ corresponds to the relation

$$\frac{\rm d}{{\rm d}\xi}\log\left(\overline{A}e^{-i\omega t+i\frac{cs_r}{2}\xi}\right)=\frac{M'}{2M}-i\psi. \tag{62}$$

Before taking the quadrature of the above three expressions (61), let us recall the definition of the *élément simple* [14, vol. II, p. 506] introduced by

Hermite for integrating the Lamé equation,

$$\mathrm{H}(\xi,q,k)=\frac{\sigma(\xi+q)}{\sigma(\xi)\sigma(q)}e^{(k-\zeta(q))\xi},\ (k,q)\text{ constants.} \tag{63}$$

Its only singularity is a simple pole with residue unity at the origin.

Equations $(61)_1$ and $(61)_3$ then integrate as

$$\forall\kappa_{\mathrm{i}}:\ A=K_0e^{-i\omega t+i\frac{c\xi}{2p}}\,\mathrm{H}(\xi,-\xi^{\psi}_{i,0},0)^{(-1+i\sqrt{3})/2}\,\mathrm{H}(\xi,-\xi^{\psi}_{i,1},0)^{(-1-i\sqrt{3})/2}, \tag{64}$$

$$\kappa_{\mathrm{i}}\neq 0:\ M=K_1e^{\frac{3e_1+4je_0}{12e_1}\kappa_{\mathrm{i}}\xi}\,\mathrm{H}(\xi,-\xi^{\psi}_{j},0)\,\mathrm{H}(\xi,-\xi^{\psi}_{j,0},0)^{-1}\,\mathrm{H}(\xi,-\xi^{\psi}_{j,1},0)^{-1} \tag{65}$$

The integration constants K_0, K_1 are determined by requiring that, near the simple pole $\chi_1=\xi-\xi^{\psi}_{j,0}\to 0$ of M, the variables A and M admit the principal parts $A\sim A_0\chi_1^{(-1+i\sqrt{3})/2}$, $M\sim A_0^2\chi_1^{-1}$, $A_0^8=3/e_i^2$, see (14) and (21).

In order to check that the product of the complex amplitude A (64) by its complex conjugate is equal to the decomposition (65), one must take account of (57) and remember that the origin of ξ, not displayed in the above formulae, depends on j and is therefore different for A and its complex conjugate.

The restriction $\kappa_{\mathrm{i}}\neq 0$ in (65) is removed by taking into account the relation

$$\zeta(\xi^{\psi}_{j,0})+\zeta(\xi^{\psi}_{j,1})-\zeta(\xi^{\psi}_{j})=\zeta(\xi^{\psi}_{j,0}+\xi^{\psi}_{j,1}-\xi^{\psi}_{j})+\kappa_{\mathrm{i}}/4+j\kappa_{\mathrm{i}}e_0/(3e_1), \tag{66}$$

and using the definition (54), yielding

$$M=-K_1e^{-\mathrm{H}_1\xi}\frac{\sigma(\xi-\xi^{\psi}_{j})\sigma(\xi)}{\sigma(\xi-\xi^{\psi}_{j,0})\sigma(\xi-\xi^{\psi}_{j,1})}\frac{\sigma(\xi^{\psi}_{j,0})\sigma(\xi^{\psi}_{j,1})}{\sigma(\xi^{\psi}_{j})}. \tag{67}$$

In order to check the equality of the two decompositions of M as the sum (60) and the product (67), it is sufficient to convert the elliptic function (67) to a rational function of $\wp(\xi,G_2,G_3)$ and its derivative, then to identify it to (46) *modulo* the Landen transformation (Appendix A).

Numerical simulations with periodic boundary conditions [27, Fig. 4] do display solutions M having a real period (similar features are observed in CGL3 [4, Fig. 7]), these could well correspond to the present elliptic solution.

Remark. The elliptic (hence singlevalued) nature of $\mathrm{d}\log(Ae^{i\omega t})/\mathrm{d}\xi$ explains the so-called "ad hoc Hirota method" [26] in which A is essentially assumed to be a product of powers of entire functions, the powers being those of the singularity structure, here $(-1 \pm i\sqrt{3})/2$. In order to recover our result (64), two upgrades to this method are needed: (i) to assume A to be a product of powers of Hermite's simple elements (63), not of Weierstrass σ functions or Jacobi θ functions, so as to ensure that the logarithmic derivative of A is elliptic; (ii) to allow arbitrary shifts ξ_j in the arguments of the entire functions, not only half periods like with the choice $\theta_j(\xi), j = 0, 1, 2, 3$ in the Jacobi notation.

Appendix A. Landen transformation

We are indebted to the grateful indications of Yuri Brezhnev for this appendix.

The Landen or Gauss transformation consists in halving only one of the two periods, it is naturally defined [18] [19, p. 384] in the notation of $\wp$ displaying the two periods $2\omega, 2\omega'$,

$$\wp(x|\omega, 2\omega') = \wp(x|2\omega, 2\omega') + \wp(x-\omega|2\omega, 2\omega') - \wp(\omega|2\omega, 2\omega'). \tag{68}$$

In the other usual notation

$$\left\{ \begin{array}{l} \wp(x, G_2, G_3) = \wp(x|\omega, 2\omega'),\ \ \wp_1(x, g_2, g_3) = \wp(x|2\omega, 2\omega'), \\ \wp'^2 = 4(\wp^3 - g_2\wp - g_3) = 4(\wp - e_1)(\wp - e_2)(\wp - e_3), \\ \wp_1'^{\,2} = 4(\wp_1^3 - G_2\wp_1 - G_3) = 4(\wp_1 - E_1)(\wp_1 - E_2)(\wp_1 - E_3), \end{array} \right. \tag{69}$$

the expression of $\wp(x|\omega, 2\omega')$ as a rational function of $\wp(x|2\omega, 2\omega')$ is

$$\wp(x, G_2, G_3) = \wp(x, g_2, g_3) - \frac{g_2 - 12e_1^2}{4(\wp(x, g_2, g_3) - e_1)}, \tag{70}$$

and similarly at the ζ and σ levels [19, Eqs. (16b), (17b)]

$$\zeta(x, G_2, G_3) = \zeta(x, g_2, g_3) + \zeta(x-\omega, g_2, g_3) - e_1 x + \zeta(\omega, g_2, g_3), \tag{71}$$

$$\sigma(x, G_2, G_3) = e^{e_1 x^2/2 - \zeta(\omega, g_2, g_3)x}\ \frac{\sigma(x, g_2, g_3)\sigma(x+\omega, g_2, g_3)}{\sigma(\omega, g_2, g_3)}. \tag{72}$$

Between e_j, E_j (and g_k, G_k), there exist two algebraic relations

$$\left\{ \begin{array}{l} E_1 = -2e_1,\ (E_2 - E_3)^2 = 36e_1^2 - 4(e_2 - e_3)^2, \\ -32g_2g_3 + 22g_3G_2 + 11g_2G_3 - G_2G_3 = 0, \\ 196g_2^3 + 49g_2^2G_2 - 7260g_3^2 + 660g_3G_3 - 15G_3^2 = 0. \end{array} \right. \tag{73}$$

The ratio -2 of the two zeros $-2e_1$ of ${\wp_1'}^2$ in (47) and e_1 of ${\wp'}^2$ in (46) is the signature of such a Landen transformation.

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