Analytical and numerical computations for high frequency MEMS

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Abstract.
We consider a Micro-Electro-Mechanical Systems (MEMS) device vibrating at high frequency. We present (theoretical and numerical) computations allowing to assess the damping of the gas trapped in the channel of the device.

Semi-explicit solutions for the transient and permanent regimes associated to the linearized BGK equations with Maxwellian boundary conditions and a periodic forcing are established.

Then, different numerical methods are briefly described for the treatment of these equations.

Keywords: MEMS , Linearized Boltzmann equation, BGK model, Abramovitz functions
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INTRODUCTION

Our study is motivated by the increasing use of Micro-Electro-Mechanical Systems (MEMS) devices vibrating at high frequencies (cf. [10]) [for low frequency MEMS devices used at small pressure, we refer e.g. to [7], [4]].

We proposed in [5] the study of resonances for the acoustic waves in the channel of such a device when the gas is rarefied and modeled by a linearized BGK (or ES) equation, thanks to the use of finite differences numerical computations for the transient regime, and exact or semi-exact computations for the permanent regime. This study enables to treat very rarefied regimes which cannot be attained using an approach based on the Navier-Stokes equation like in [10].

In this paper, we keep the framework of [5], and present some new mathematical and numerical developments for the considered problem. After briefly recalling in Section 1 the principles of the modeling by linearized BGK equations of the vibrating device, we introduce in Section 2 a computation enabling to obtain a semi-explicit formula (integral equation) for the transient problem, and we deduce from it another semi-explicit formula for the permanent regime. The summation of the series appearing in this last formula can be done in the case of pure specular reflexion and pure diffuse reflexion. For pure diffuse reflexion, we recover the expression given in [5]. Finally, Section 3 is devoted to the presentation of alternative numerical approaches to the finite difference method used in [5], and a brief comparison of those approaches.

MODELING

Nonlinear Boltzmann equation

We consider a rarefied gas whose density in the phase space $f(t,x,v)$ (of molecules which at time t and point $x=(x_1,x_2,x_3)$ have velocity $v=(v_1,v_2,v_3)$) is assumed to satisfy the Boltzmann equation (cf. [3])

$$\partial_t f + v\cdot\nabla_x f = Q(f),$$

where the collision kernel Q is defined (for some cross section B) by

$$Q(f)(v) = \int_{v_*\in\mathbb{R}^3}\int_{\sigma\in S^2}\Big\{ f(v'_*)\, f(v') - f(v)\, f(v_*)\Big\} B\left(|v-v_*|, \frac{v-v_*}{|v-v_*|}\cdot\sigma\right) d\sigma\, dv_*,$$

where pre- and post-collisional velocities are related by

$$v' = \frac{v+v_*}{2} + \frac{|v-v_*|}{2}\sigma, \qquad v'_* = \frac{v+v_*}{2} - \frac{|v-v_*|}{2}\sigma, \tag{1}$$

and σ belongs to the unit sphere S^2. We are interested in the situation when this gas is confined in a spatial domain $\Omega := \Omega(t)\subset\mathbb{R}^3$ moving with time. For a point $x\in\partial\Omega(t)$, we denote by $U_w(t,x)$ its velocity.

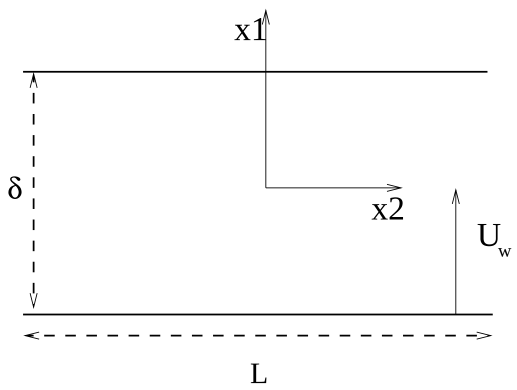
In such a domain, the Maxwellian boundary condition writes (cf. [3])

$$\forall x,v\in(\partial\Omega(t)\times\mathbb{R}^3)^+, \qquad f(t,x,v) = (1-\alpha)\, f(t,x,v-2\,((v-U_w(t,x))\cdot n(x))\, n(x))$$

$$+\alpha\,\frac{e^{-(v-U_w(t,x))^2}}{\pi}\int_{(w-U_w(t,x))\cdot n(x)\geq 0} 2\,(w-U_w(t,x))\cdot n(x)\, f(t,x,w)\, dw,$$

where $(\partial\Omega(t)\times\mathbb{R}^3)^+$ is constituted of the points $(x,v)\in\partial\Omega(t)\times\mathbb{R}^3$ such that $(v-U_w(t,x))\cdot n(x)<0$, where $n(x)$ is the outward normal unit vector at point $x\in\partial\Omega(t)$, and where $\alpha\in[0,1]$ is the proportion of diffuse reflexions at the wall.

We now restrict ourselves to the typical geometry of a MEMS channel (cf. [10], [8]), such as described in the following figure:



that is, the considered rarefied gas is trapped between two walls of size $L\times M$ and at distance δ from one another, with $L,M >> \delta$.

We assume that one of the walls is fixed, and that the other one oscillates with a velocity $U_w(t) = U_w^0 \sin(\omega t)$, with $U_w^0\,\omega^{-1} << \delta$, that is, we put ourselves in the framework of a high-frequency oscillation.

As a consequence of these assumptions, we model the problem thanks to a distribution function f which only depends on $x_1 \in [-\delta/2,\delta/2]$ (in fact, $x_1 \in [-\delta/2 - U_w^0\,\omega^{-1}\cos(\omega t),\delta/2]$, but this interval can be replaced by $[-\delta/2,\delta/2]$ since $U_w^0\,\omega^{-1} << \delta$) and $v = (v_1,v_2,v_3) \in \mathbb{R}^3$. Note that f does not depend on x_2,x_3, that is, the walls are large enough to be assumed of infinite size. This last choice of modeling, based on the hypothesis $L,M >> \delta$ and on the high-frequency oscillation assumption, is discussed in [5], where 2D (in space) computations are presented to sustain it.

The function f satisfies then the 1D-Boltzmann equation

$$\partial_t f(t,x_1,v) + v_1\,\partial_{x_1} f(t,x_1,v) = Q(f)(t,x_1,v),$$

and (at point $x_1 = \pm\delta/2$) the boundary conditions:

$$f(t,\delta/2,v) = (1-\alpha)\,f(t,\delta/2,-v_1,v_2,v_3) + \alpha\,\frac{e^{-|v|^2}}{\pi}\int_{w_1>0} 2w_1\,f(t,\delta/2,w)\,dw, \quad (2)$$

for $v_1 < 0$, and

$$\begin{aligned} f(t,-\delta/2,v) &= (1-\alpha)\,f(t,-\delta/2,2U_w(t)-v_1,v_2,v_3) \\ &+\alpha\,\frac{e^{-|v-U_w(t)|^2}}{\pi}\int_{w_1<0}(-2)\,(w_1-U_w(t))\,f(t,-\delta/2,w)\,dw, \end{aligned} \quad (3)$$

for $v_1 > U_w(t)$.

Linearization of the Boltzmann equation

We now assume that the density f stays close to the (absolute) Maxwellian equilibrium, that is (in a suitable set of units in which this Maxwellian function is normalized)

$$f(t,x_1,v) = \frac{e^{-|v|^2}}{\pi^{3/2}}\,(1+h(t,x_1,v)),$$

where $h << 1$ (cf. [8]).

Then, the equation satisfied by h (up to terms in $O(h^2)$) writes

$$\begin{aligned} \partial_t h(t,x_1,v) + v_1\,\partial_{x_1} h(t,x_1,v) &= \int_{v_*\in\mathbb{R}^3}\int_{\sigma\in S^2} B\left(|v-v_*|,\frac{v-v_*}{|v-v_*|}\cdot\sigma\right)\frac{e^{-|v_*|^2}}{\pi^{3/2}} \\ &\times\left\{h(t,x_1,v_*') + h(t,x_1,v') - h(t,x_1,v_*) - h(t,x_1,v)\right\}d\sigma dv_*, \end{aligned}$$

where v', v'_* are defined by (1).

We also assume that the velocity U_w^0 is very small (with respect to the thermal velocity of the Maxwellian distribution, which is here of order 1, in other words $U_w^0 << 1$). Neglecting therefore also $(U_w^0)^2$ and $U_w^0 h$ in front of U_w^0, the boundary conditions (2), (3) becomes

$$h(t,\delta/2,v) = (1-\alpha)\,h(t,\delta/2,-v_1,v_2,v_3) + \frac{\alpha}{\pi}\int_{w_1>0} 2w_1\, e^{-|w|^2}\, h(t,\delta/2,w)\,dw, \quad (4)$$

for $v_1 < 0$, and

$$\begin{aligned} h(t,-\delta/2,v) = (1-\alpha)\,h(t,-\delta/2,-v_1,v_2,v_3) + \frac{\alpha}{\pi}\int_{w_1<0} (-2\,w_1)\, e^{-|w|^2}\, h(t,-\delta/2,w)\,dw \\ +4\,(1-\alpha)\,v_1\,U_w(t) + \alpha\,(\sqrt{\pi}+2v_1)\,U_w(t) \end{aligned} \quad (5)$$

for $v_1 > 0$.

BGK model

The linearized Boltzmann equation can be approximated by the linearized BGK equation (cf. [2]) (more precisely, a version of the linearized BGK equation in which the thermal effects are not taken into account) under conditions which are detailed in [5] (and which are sustained by numerical computations using the ES model). Those conditions amount to assume that the frequency of oscillations of the wall is large enough.

The equation can then be written (note that from now on, we replace x_1 by x and v_1 by v for the sake of readability):

$$\partial_t h(t,x,v) + v\,\partial_x h(t,x,v) = -h(t,x,v) + L_0(t,x) + 2v L_1(t,x), \quad (6)$$

where (for any $k \in \mathbb{N}$)

$$L_k(t,x) = \pi^{-1/2}\int_{w\in\mathbb{R}} e^{-w^2}\, w^k\, h(t,x,w)\,dw. \quad (7)$$

The boundary conditions (4), (5) are now rewritten using notations which enable to write computations in the sequel which are common to all possible values of α:

$$v<0, \qquad h(t,\delta/2,v) = \int_{w>0} h(t,\delta/2,w)\,\mu_v^L(dw), \quad (8)$$

$$v>0, \qquad h(t,-\delta/2,v) = \int_{w>0} h(t,-\delta/2,w)\,\mu_v^R(dw) + \beta(t,v), \quad (9)$$

with

$$\mu_v^L(w) = (1-\alpha)\,\delta_{-v}(w) + \alpha\,2we^{-w^2}; \qquad \mu_v^R(w) = (1-\alpha)\,\delta_{-v}(w) + \alpha\,(-2w)e^{-w^2},$$

$$\beta(t,v) = \tilde{\beta}(v)\,U_w(t), \qquad \tilde{\beta}(v) = (1-\alpha)\,\tilde{\beta}_1(v) + \alpha\,\tilde{\beta}_0(v),$$

$$\tilde{\beta_1}(v) = 4v; \qquad \tilde{\beta_0}(v) = \sqrt{\pi} + 2v. \tag{10}$$

In [5], eq. (6) – (10) is used to compute the resonances/antiresonances in the channel of the MEMS device, that is the values of ω for which the amplitude of the oscillations of $L_2(t, -\delta/2)$ (related to the pressure on the moving wall) in the permament regime, are maximal/minimal.

These computations involve an analytical part (semi-explicit formulas for the permanent regime of the oscillations of $L_2(t, -\delta/2)$ in the special case of diffuse boundary conditions ($\alpha = 1$)), and a numerical part (finite difference scheme).

In next section, we present a computation that holds for all α, and that enables to give a semi-explicit solution to the transient regime as well as to the permanent regime (seen here as an asymptotics of the transient regime). It enables to recover the analytical results of [5], the method used here being quite different from that of [5] (where the transient problem is not used).

SEMI-EXPLICIT SOLUTIONS

Time dependent (transient) formulas

We present here a simple computation allowing to obtain a semi-explicit formula for the solution of eq. (6) - (10).

Recalling definition (7) and using the characteristics, we get for the solution of eq. (6) – (10) [with $h = 0$ at time $t = 0$] the following identities (for any $k \in \mathbb{N}$):

$$\begin{aligned}
\pi^{1/2} L_k(t,x) &= \int_{\mathbb{R}} \int_{t_0}^{t} e^{-(t-s)} \, (L_0 + 2v L_1)(s, x - v\,(t-s)) \, ds \, v^k e^{-v^2} \, dv \\
&\quad + \int_{v>0} e^{-(t-t_0)} \beta(t_0, v) \, v^k e^{-v^2} \, dv
\end{aligned}$$

$$+ \sum_{R=0}^{\infty} \int_{\mathbb{R}} \int_{w_1>0} .. \int_{w_{2R+1}>0} \int_{t_{2R+1}}^{t_{2R}} e^{-(t-s)} \, (L_0 - 2sgn(v)\, w_{2R+1} L_1)(s, sgn(v)\,(-\delta/2 + w_{2R+1}(t_{2R} - s)))$$

$$ds d\mu^R_{sgn(v)\, w_{2R}}(-sgn(v)\, w_{2R+1}) d\mu^L_{-sgn(v)\, w_{2R-1}}(sgn(v)\, w_{2R}) .. d\mu^R_v(-sgn(v)\, w_1)\, v^k e^{-v^2} \, dv$$

$$+ \sum_{R=1}^{\infty} \int_{\mathbb{R}} \int_{w_1>0} .. \int_{w_{2R}>0} \int_{t_{2R}}^{t_{2R-1}} e^{-(t-s)} \, (L_0 + 2sgn(v)\, w_{2R} L_1)(s, sgn(v)\,(\delta/2 - w_{2R}(t_{2R-1} - s)))$$

$$ds d\mu^L_{-sgn(v)\, w_{2R-1}}(sgn(v)\, w_{2R}) d\mu^R_{sgn(v)\, w_{2R-2}}(-sgn(v)\, w_{2R-1}) .. d\mu^R_v(-sgn(v)\, w_1)\, v^k e^{-v^2} \, dv$$

$$+ \sum_{Q=1}^{\infty} \int_{\mathbb{R}} \int_{w_1>0} .. \int_{w_Q>0} e^{-(t-t_Q)} \, \beta(t_Q, w_Q) \, 1_{sgn(v)=(-1)^Q}$$

$$d\mu^{R \text{ if } v<0; L \text{ if } v>0}_{-w_{Q-1}}(w_Q) \, d\mu^{L \text{ if } v<0; R \text{ if } v>0}_{w_{Q-2}}(-w_{Q-1}) .. d\mu^R_v(-sgn(v)\, w_1)\, v^k e^{-v^2} \, dv, \tag{11}$$

with

$$t_k = \sup\left(t - \frac{x}{v} - \frac{\delta/2}{|v|} - \delta\,[\frac{1}{w_1} + .. + \frac{1}{w_k}], 0 \right). \tag{12}$$

As can be seen, it is possible to replace the original PDE by two time-delay integral equations for the quantities L_0 and L_1. Then all the moments L_k can be recovered once L_0 and L_1 are known.

Time independent (permanent) solutions

We relate the computation of the previous section with the problem of finding the permanent regime (as stated in [5]) by using $\beta(t,v) = \text{Im}\,(e^{i\omega t}\,\tilde\beta(v))$. The large time behavior asymptotics of L_k in this case is expected to be of the form Im $(\lambda_k(x)\,e^{i\omega t})$ (that is, the permanent regime). As a consequence, we introduce the quantity $l_k(t,x)$ defined by $L_k(t,x) = \text{Im}\,(e^{i\omega t}\,l_k(t,x))$ and assume that $\lambda_k(x) := \lim_{t\to\infty} l_k(t,x)$ exists.

Then, at the formal level, we get out of the formula of the previous subsection the following integral equation for λ_k:

$$\pi^{1/2}\,\lambda_k(x) = \int_{v\in\mathbb{R}}\int_{[x,-sgn(v)\,\delta/2]} e^{-(\frac{x-y}{v})\,(1+i\,\omega)}(\lambda_0+2v\,\lambda_1)(y)\,dy\,sgn(v)\,v^{k-1}e^{-v^2}\,dv$$

$$+\sum_{Q=1}^{\infty}\int_{(-1)^Q\,v>0}\int_{w_1>0}..\int_{w_Q>0} e^{-(t-t_Q)\,(1+i\,\omega)}\,\tilde\beta(w_Q)$$

$$d\mu^{L \text{ if } Q \text{ even },R \text{ if } Q \text{ odd}}_{-w_{Q-1}}\,(w_Q)\ldots d\mu^R_v((-1)^{Q+1}w_1)\,v^k e^{-v^2}\,dv$$

$$+\int_{v>0} e^{-(t-t_0)\,(1+i\,\omega)}\,\tilde\beta(v)\,v^k e^{-v^2}\,dv$$

$$+\sum_{R=0}^{\infty}\int_{v\in\mathbb{R}}\int_{w_1>0}..\int_{w_{2R+1}>0}\int_{-\delta/2}^{\delta/2} e^{-(\frac{x}{v}+\frac{\delta}{2|v|}+\frac{\delta}{w_1}+..+\frac{\delta}{w_{2R}}+\frac{\delta/2+sgn(v)\,y}{w_{2R+1}})\,(1+i\,\omega)}$$

$$(\lambda_0-2\,sgn(v)\,w_{2R+1}\,\lambda_1)(y)\,\frac{dy}{w_{2R+1}}\,d\mu^R_{sgn(v)\,w_{2R}}(-sgn(v)\,w_{2R+1})\ldots d\mu^R_v(-sgn(v)\,w_1)\,v^k e^{-v^2}\,dv$$

$$+\sum_{R=1}^{\infty}\int_{v\in\mathbb{R}}\int_{w_1>0}..\int_{w_{2R}>0}\int_{-\delta/2}^{\delta/2} e^{-(\frac{x}{v}+\frac{\delta}{2|v|}+\frac{\delta}{w_1}+..+\frac{\delta}{w_{2R-1}}+\frac{\delta/2-sgn(v)\,y}{w_{2R}})\,(1+i\,\omega)}$$

$$(\lambda_0+2\,sgn(v)\,w_{2R}\,\lambda_1)(y)\,\frac{dy}{w_{2R}}\,d\mu^L_{-\,sgn(v)\,w_{2R-1}}(sgn(v)\,w_{2R})\ldots d\mu^R_v(-\,sgn(v)\,w_1)\,v^k e^{-v^2}\,dv.$$

This equation is quite complicated because of the presence of the series (which are already present in the formula for the transient regime). These series are related to the possibility for a molecule to bump successively on the left and right walls (or on the right and left walls) several times before being involved in a collision with another molecule of the gas.

As a consequence, usable formulas can be found only in the cases in which these series can be explicitly computed: those cases include the pure specular reflexion ($\alpha = 0$) and the pure diffuse reflexion ($\alpha = 1$) [this last case was already treated in [5]], but also an asymptotic expansion corresponding to α close to 1.

Next subsection is devoted to the presentation of the summation processes, which are different for the different values of α.

Summation of the series

We first look to the case when $\alpha = 0$. Denoting by $T_k(\rho) = \int_0^\infty v^k e^{-v^2} e^{-\rho/v} dv$ the Abramowitz function and by $U_k(\rho,\sigma) = \int_0^\infty \frac{v^k e^{-v^2} e^{-\rho/v}}{1-e^{-\sigma/v}} dv$ the modified Abramowitz function, the formula obtained in the above subsection becomes, after summation of the geometric series, for all $k \in \mathbb{N}$,

$$\pi^{1/2}\lambda_k(x) = \int_{-\delta/2}^{x} \lambda_0(y)\, T_{k-1}([1+i\,\omega]\,(x-y))\,dy$$

$$+\int_{-\delta/2}^{x} 2\,\lambda_1(y)\, T_k([1+i\,\omega]\,(x-y))\,dy + (-1)^k \int_{x}^{\delta/2} \lambda_0(y)\, T_{k-1}([-1-i\,\omega]\,(x-y))\,dy$$

$$+(-1)^{k+1}\int_{x}^{\delta/2} 2\,\lambda_1(y)\, T_k([-1-i\,\omega]\,(x-y))\,dy$$

$$+\int_{-\delta/2}^{\delta/2} \lambda_0(y)\,[U_{k-1}((1+i\,\omega)\,(2\delta+x-y),\nu) + U_{k-1}((1+i\,\omega)\,(\delta+x+y),\nu)$$

$$+(-1)^k U_{k-1}((1+i\,\omega)\,(2\delta-x+y),\nu) + (-1)^k U_{k-1}((1+i\,\omega)\,(\delta-x-y),\nu)]\,dy$$

$$+2\int_{-\delta/2}^{\delta/2} \lambda_1(y)\,[U_k((1+i\,\omega)\,(2\delta+x-y),\nu) - U_k((1+i\,\omega)\,(\delta+x+y),\nu)$$

$$+(-1)^{k+1} U_k((1+i\,\omega)\,(2\delta-x+y),\nu) + (-1)^k U_k((1+i\,\omega)\,(\delta-x-y),\nu)]\,dy$$

$$+4U_{k+1}((1+i\,\omega)\,(x+\delta/2),\nu) + 4\,(-1)^k U_{k+1}((1+i\,\omega)\,(3\delta/2-x),\nu),$$

with $\nu = (1+i\,\omega)\,(2\delta)$.

We see therefore that λ_0 and λ_1 satisfy a system of two convolution-like equations on $[-\delta/2,\delta/2]$, and that all λ_k can then be recovered once λ_0 and λ_1 are known. These convolution-like equations can be solved numerically in order to compute the permanent regime in the case $\alpha = 0$ without using a numerical scheme for the time-dependent problem.

The case when $\alpha = 1$ can also be treated thanks to a summation of the series and leads to the formula (68) p.16 of [5].

This formula can be extended to the case when α is close to 1. It leads to an expansion in powers of $1-\alpha$ which is fully detailed below.

We recall that

$$\pi^{1/2}\,\lambda_k(x) = \int_{v>0}\int_{-\delta/2}^{x} e^{-(\frac{x-y}{v})\,(1+i\,\omega)}(\lambda_0 + 2v\,\lambda_1)(y)\,dy v^{k-1} e^{-v^2}\,dv$$

$$+\int_{v<0}\int_{x}^{\delta/2} e^{-(\frac{x-y}{v})\,(1+i\,\omega)}(\lambda_0 + 2v\,\lambda_1)(y)\,dy(-v^{k-1})\,e^{-v^2}\,dv$$

$$+\sum_{Q=1,Q even}^{\infty}\int_{v>0}\int_{w_1>0}\cdot\cdot\int_{w_Q>0} e^{-(t-t_Q)\,(1+i\,\omega)}\tilde{\beta}(w_Q)$$

$$d\mu^L_{-w_{Q-1}}(w_Q)\,d\mu^R_{w_{Q-2}}(-w_{Q-1})..d\mu^R_v(-w_1)\,v^k e^{-v^2}\,dv$$

$$+\sum_{Q=1,Q odd}^{\infty}\int_{v<0}\int_{w_1>0}..\int_{w_Q>0} e^{-(t-t_Q)\,(1+i\,\omega)}\tilde{\beta}(w_Q)$$

$$d\mu^R_{-w_{Q-1}}(w_Q)\,d\mu^L_{w_{Q-2}}(-w_{Q-1})..d\mu^R_v(w_1)\,v^k e^{-v^2}\,dv$$

$$+\int_{v>0} e^{-(t-t_0)\,(1+i\,\omega)}[(1-\alpha)\,\tilde{\beta}_1+\alpha\,\tilde{\beta}_0](v)\,v^k e^{-v^2}\,dv$$

$$+\sum_{R=0}^{\infty}\int_{v>0}\int_{w_1>0}..\int_{w_{2R+1}>0}\int_{-\delta/2}^{\delta/2} e^{-(\frac{x}{v}+\frac{\delta}{2|v|}+\frac{\delta}{w_1}+..+\frac{\delta}{w_{2R}}+\frac{\delta/2+y}{w_{2R+1}})\,(1+i\,\omega)}$$

$$(\lambda_0-2w_{2R+1}\,\lambda_1)(y)\,dy\frac{dy}{w_{2R+1}}\,d\mu^R_{w_{2R}}(-w_{2R+1})\,d\mu^L_{-w_{2R-1}}(w_{2R})..d\mu^R_v(-w_1)\,v^k e^{-v^2}\,dv$$

$$+\sum_{R=1}^{\infty}\int_{v>0}\int_{w_1>0}..\int_{w_{2R}>0}\int_{-\delta/2}^{\delta/2} e^{-(\frac{x}{v}+\frac{\delta}{2|v|}+\frac{\delta}{w_1}+..+\frac{\delta}{w_{2R-1}}+\frac{\delta/2-y}{w_{2R}})\,(1+i\,\omega)}$$

$$(\lambda_0+2w_{2R}\,\lambda_1)(y)\,dy\frac{dy}{w_{2R}}\,d\mu^L_{-w_{2R-1}}(w_{2R})\,d\mu^R_{w_{2R-2}}(-w_{2R-1})..d\mu^R_v(-w_1)\,v^k e^{-v^2}\,dv$$

$$+\sum_{R=0}^{\infty}\int_{v<0}\int_{w_1>0}..\int_{w_{2R+1}>0}\int_{-\delta/2}^{\delta/2} e^{-(\frac{x}{v}+\frac{\delta}{2|v|}+\frac{\delta}{w_1}+..+\frac{\delta}{w_{2R}}+\frac{\delta/2-y}{w_{2R+1}})\,(1+i\,\omega)}$$

$$(\lambda_0+2w_{2R+1}\,\lambda_1)(y)\,dy\frac{dy}{w_{2R+1}}\,d\mu^R_{-w_{2R}}(w_{2R+1})\,d\mu^L_{w_{2R-1}}(-w_{2R})..d\mu^R_v(w_1)\,v^k e^{-v^2}\,dv$$

$$+\sum_{R=1}^{\infty}\int_{v<0}\int_{w_1>0}..\int_{w_{2R}>0}\int_{-\delta/2}^{\delta/2} e^{-(\frac{x}{v}+\frac{\delta}{2|v|}+\frac{\delta}{w_1}+..+\frac{\delta}{w_{2R-1}}+\frac{\delta/2+y}{w_{2R}})\,(1+i\,\omega)}$$

$$(\lambda_0-2w_{2R}\,\lambda_1)(y)\,dy\frac{dy}{w_{2R}}\,d\mu^L_{w_{2R-1}}(-w_{2R})\,d\mu^R_{-w_{2R-2}}(w_{2R-1})..d\mu^R_v(w_1)\,v^k e^{-v^2}\,dv$$

$$:= a+b+c+d+e+f_1+f_2+f_3+f_4.$$

We now detail all the formulas corresponding to those terms:

$$a=\int_{-\delta/2}^{x}[\lambda_0(y)\,T_{k-1}((1+i\omega)\,(x-y))+2\,\lambda_1(y)\,T_k((1+i\omega)\,(x-y))]\,dy,$$

$$b=(-1)^k\int_{x}^{\delta/2}[\lambda_0(y)\,T_{k-1}((-1-i\omega)\,(x-y))-2\,\lambda_1(y)\,T_k((-1-i\omega)\,(x-y))]\,dy,$$

$$e=4(1-\alpha)\,T_{k+1}((x+\delta/2)(1+i\omega))+\alpha\,[\sqrt{\pi}\,T_k((x+\delta/2)(1+i\omega))+2T_{k+1}((x+\delta/2)(1+i\omega))],$$

$$f_1=O((1-\alpha)^2)+\sum_{R=0}^{\infty}\alpha^{2R+1}T_k((x+\delta/2)\,(1+i\,\omega))\,[2T_1(\delta\,(1+i\,\omega))]^{2R}$$

$$\times\int_{-\delta/2}^{\delta/2}[2\lambda_0(y)\,T_0((y+\delta/2)\,(1+i\,\omega))-4\lambda_1(y)\,T_1((y+\delta/2)\,(1+i\,\omega))]dy$$

$$+\sum_{R=1}^{\infty}(2R-1)\,\alpha^{2R}\,(1-\alpha)\,T_k((x+\delta/2)\,(1+i\,\omega))\,[2T_1(2\delta\,(1+i\,\omega))]\,[2T_1(\delta\,(1+i\,\omega))]^{2R-2}$$

$$\times\int_{-\delta/2}^{\delta/2}[2\lambda_0(y)\,T_0((y+\delta/2)\,(1+i\,\omega))-4\lambda_1(y)\,T_1((y+\delta/2)\,(1+i\,\omega))]dy$$

$$+\sum_{R=1}^{\infty}\alpha^{2R}\,(1-\alpha)\,T_k((x+\delta/2)\,(1+i\,\omega))\,[2T_1(\delta\,(1+i\,\omega))]^{2R-1}$$

$$\times\int_{-\delta/2}^{\delta/2}[2\lambda_0(y)\,T_0((y+3\delta/2)\,(1+i\,\omega))-4\lambda_1(y)\,T_1((y+3\delta/2)\,(1+i\,\omega))]dy$$

$$+\sum_{R=1}^{\infty}\alpha^{2R}\,(1-\alpha)\,T_k((x+3\delta/2)\,(1+i\,\omega))\,[2T_1(\delta\,(1+i\,\omega))]^{2R-1}$$

$$\times\int_{-\delta/2}^{\delta/2}[2\lambda_0(y)\,T_0((y+\delta/2)\,(1+i\,\omega))-4\lambda_1(y)\,T_1((y+\delta/2)\,(1+i\,\omega))]dy$$

$$+(1-\alpha)\int_{-\delta/2}^{\delta/2}[\lambda_0(y)\,T_{k-1}((x+y+\delta)\,(1+i\,\omega))-2\lambda_1(y)\,T_k((x+y+\delta)\,(1+i\,\omega))]dy$$

$$=O((1-\alpha)^2)+T_k((x+\delta/2)\,(1+i\,\omega))\,\frac{\alpha}{1-\alpha^2\,[2T_1(\delta\,(1+i\,\omega))]^2}$$

$$\times\int_{-\delta/2}^{\delta/2}[2\lambda_0(y)\,T_0((y+\delta/2)\,(1+i\,\omega))-4\lambda_1(y)\,T_1((y+\delta/2)\,(1+i\,\omega))]dy$$

$$+(1-\alpha)\,\alpha^2\,T_k((x+\delta/2)\,(1+i\,\omega))\,[2T_1(2\delta\,(1+i\,\omega))]\,\frac{1+\alpha^2\,[2T_1(\delta\,(1+i\,\omega))]^2}{(1-\alpha^2\,[2T_1(\delta\,(1+i\,\omega))]^2)^2}$$

$$\times\int_{-\delta/2}^{\delta/2}[2\lambda_0(y)\,T_0((y+\delta/2)\,(1+i\,\omega))-4\lambda_1(y)\,T_1((y+\delta/2)\,(1+i\,\omega))]dy$$

$$+(1-\alpha)\,\alpha^2\,T_k((x+\delta/2)\,(1+i\,\omega))\,\frac{2T_1(\delta\,(1+i\,\omega))}{1-\alpha^2\,[2T_1(\delta\,(1+i\,\omega))]^2}$$

$$\times\int_{-\delta/2}^{\delta/2}[2\lambda_0(y)\,T_0((y+3\delta/2)\,(1+i\,\omega))-4\lambda_1(y)\,T_1((y+3\delta/2)\,(1+i\,\omega))]dy$$

$$+(1-\alpha)\,\alpha^2\,T_k((x+3\delta/2)\,(1+i\,\omega))\,\frac{2T_1(\delta\,(1+i\,\omega))}{1-\alpha^2\,[2T_1(\delta\,(1+i\,\omega))]^2}$$

$$\times\int_{-\delta/2}^{\delta/2}[2\lambda_0(y)\,T_0((y+\delta/2)\,(1+i\,\omega))-4\lambda_1(y)\,T_1((y+\delta/2)\,(1+i\,\omega))]dy$$

$$+(1-\alpha)\int_{-\delta/2}^{\delta/2}[\lambda_0(y)\,T_{k-1}((x+y+\delta)\,(1+i\,\omega))-2\lambda_1(y)\,T_k((x+y+\delta)\,(1+i\,\omega))]dy;$$

$$f_2=O((1-\alpha)^2)+\sum_{R=1}^{\infty}\alpha^{2R}T_k((x+\delta/2)\,(1+i\,\omega))\,[2T_1(\delta\,(1+i\,\omega))]^{2R-1}$$

$$\times \int_{-\delta/2}^{\delta/2} [2\lambda_0(y)\, T_0((\delta/2-y)\,(1+i\,\omega)) + 4\lambda_1(y)\, T_1((\delta/2-y)\,(1+i\,\omega))] dy$$

$$+\sum_{R=2}^{\infty} (2R-2)\,\alpha^{2R-1}\,(1-\alpha)\, T_k((x+\delta/2)\,(1+i\,\omega))\,[2T_1(2\delta\,(1+i\,\omega))]\,[2T_1(\delta\,(1+i\,\omega))]^{2R-3}$$

$$\times \int_{-\delta/2}^{\delta/2} [2\lambda_0(y)\, T_0((\delta/2-y)\,(1+i\,\omega)) + 4\lambda_1(y)\, T_1((\delta/2-y)\,(1+i\,\omega))] dy$$

$$+\sum_{R=1}^{\infty} \alpha^{2R-1}\,(1-\alpha)\, T_k((x+\delta/2)\,(1+i\,\omega))\,[2T_1(\delta\,(1+i\,\omega))]^{2R-2}$$

$$\times \int_{-\delta/2}^{\delta/2} [2\lambda_0(y)\, T_0((3\delta/2-y)\,(1+i\,\omega)) + 4\lambda_1(y)\, T_1((3\delta/2-y)\,(1+i\,\omega))] dy$$

$$+\sum_{R=1}^{\infty} \alpha^{2R-1}\,(1-\alpha)\, T_k((x+3\delta/2)\,(1+i\,\omega))\,[2T_1(\delta\,(1+i\,\omega))]^{2R-2}$$

$$\times \int_{-\delta/2}^{\delta/2} [2\lambda_0(y)\, T_0((\delta/2-y)\,(1+i\,\omega)) + 4\lambda_1(y)\, T_1((\delta/2-y)\,(1+i\,\omega))] dy$$

$$= O((1-\alpha)^2) + T_k((x+\delta/2)\,(1+i\,\omega))\, \frac{\alpha^2\,[2T_1(\delta\,(1+i\,\omega))]}{1-\alpha^2\,[2T_1(\delta\,(1+i\,\omega))]^2}$$

$$\times \int_{-\delta/2}^{\delta/2} [2\lambda_0(y)\, T_0((\delta/2-y)\,(1+i\,\omega)) + 4\lambda_1(y)\, T_1((\delta/2-y)\,(1+i\,\omega))] dy$$

$$+(1-\alpha)\,\alpha^2\, T_k((x+\delta/2)\,(1+i\,\omega))\,[2T_1(2\delta\,(1+i\,\omega))]\, \frac{\alpha\,[2T_1(\delta\,(1+i\,\omega))]}{(1-\alpha^2\,[2T_1(\delta\,(1+i\,\omega))]^2)^2}$$

$$\times \int_{-\delta/2}^{\delta/2} [2\lambda_0(y)\, T_0((\delta/2-y)\,(1+i\,\omega)) + 4\lambda_1(y)\, T_1((\delta/2-y)\,(1+i\,\omega))] dy$$

$$+(1-\alpha)\,\alpha\, T_k((x+\delta/2)\,(1+i\,\omega))\, \frac{1}{1-\alpha^2\,[2T_1(\delta\,(1+i\,\omega))]^2}$$

$$\times \int_{-\delta/2}^{\delta/2} [2\lambda_0(y)\, T_0((3\delta/2-y)\,(1+i\,\omega)) + 4\lambda_1(y)\, T_1((3\delta/2-y)\,(1+i\,\omega))] dy$$

$$+(1-\alpha)\,\alpha\, T_k((x+3\delta/2)\,(1+i\,\omega))\, \frac{1}{1-\alpha^2\,[2T_1(\delta\,(1+i\,\omega))]^2}$$

$$\times \int_{-\delta/2}^{\delta/2} [2\lambda_0(y)\, T_0((\delta/2-y)\,(1+i\,\omega)) + 4\lambda_1(y)\, T_1((\delta/2-y)\,(1+i\,\omega))] dy;$$

$$f_3 = O((1-\alpha)^2) + (-1)^k \sum_{R=0}^{\infty} \alpha^{2R+1} T_k((x-\delta/2)\,(-1-i\,\omega))\,[2T_1(\delta\,(1+i\,\omega))]^{2R}$$

$$\times \int_{-\delta/2}^{\delta/2} [2\lambda_0(y)\, T_0((\delta/2-y)\,(1+i\,\omega)) + 4\lambda_1(y)\, T_1((\delta/2-y)\,(1+i\,\omega))] dy$$

$$+(-1)^k\sum_{R=1}^{\infty}(2R-1)\,\alpha^{2R}\,(1-\alpha)\,T_k((x-\delta/2)\,(-1-i\,\omega))\,[2T_1(2\delta\,(1+i\,\omega))]\,[2T_1(\delta\,(1+i\,\omega))]^{2R-2}$$
$$\times\int_{-\delta/2}^{\delta/2}[2\lambda_0(y)\,T_0((\delta/2-y)\,(1+i\,\omega))+4\lambda_1(y)\,T_1((\delta/2-y)\,(1+i\,\omega))]dy$$
$$+(-1)^k\sum_{R=1}^{\infty}\alpha^{2R}\,(1-\alpha)\,T_k((x-\delta/2)\,(-1-i\,\omega))\,[2T_1(\delta\,(1+i\,\omega))]^{2R-1}$$
$$\times\int_{-\delta/2}^{\delta/2}[2\lambda_0(y)\,T_0((3\delta/2-y)\,(1+i\,\omega))+4\lambda_1(y)\,T_1((3\delta/2-y)\,(1+i\,\omega))]dy$$
$$+(-1)^k\sum_{R=1}^{\infty}\alpha^{2R}\,(1-\alpha)\,T_k((x-3\delta/2)\,(-1-i\,\omega))\,[2T_1(\delta\,(1+i\,\omega))]^{2R-1}$$
$$\times\int_{-\delta/2}^{\delta/2}[2\lambda_0(y)\,T_0((\delta/2-y)\,(1+i\,\omega))+4\lambda_1(y)\,T_1((\delta/2-y)\,(1+i\,\omega))]dy$$
$$+(-1)^k(1-\alpha)\int_{-\delta/2}^{\delta/2}[\lambda_0(y)\,T_{k-1}((x+y-\delta)\,(-1-i\,\omega))+2\lambda_1(y)\,T_k((x+y-\delta)\,(-1-i\,\omega))]dy$$
$$=O((1-\alpha)^2)+(-1)^kT_k((x-\delta/2)\,(-1-i\,\omega))\,\frac{\alpha}{1-\alpha^2\,[2T_1(\delta\,(1+i\,\omega))]^2}$$
$$\times\int_{-\delta/2}^{\delta/2}[2\lambda_0(y)\,T_0((\delta/2-y)\,(1+i\,\omega))+4\lambda_1(y)\,T_1((\delta/2-y)\,(1+i\,\omega))]dy$$
$$+(-1)^k(1-\alpha)\,\alpha^2\,T_k((x-\delta/2)\,(-1-i\,\omega))\,[2T_1(2\delta\,(1+i\,\omega))]\,\frac{1+\alpha^2\,[2T_1(\delta\,(1+i\,\omega))]^2}{(1-\alpha^2\,[2T_1(\delta\,(1+i\,\omega))]^2)^2}$$
$$\times\int_{-\delta/2}^{\delta/2}[2\lambda_0(y)\,T_0((\delta/2-y)\,(1+i\,\omega))+4\lambda_1(y)\,T_1((\delta/2-y)\,(1+i\,\omega))]dy$$
$$+(-1)^k(1-\alpha)\,\alpha^2\,T_k((x-\delta/2)\,(-1-i\,\omega))\,\frac{2T_1(\delta\,(1+i\,\omega))}{1-\alpha^2\,[2T_1(\delta\,(1+i\,\omega))]^2}$$
$$\times\int_{-\delta/2}^{\delta/2}[2\lambda_0(y)\,T_0((3\delta/2-y)\,(1+i\,\omega))+4\lambda_1(y)\,T_1((3\delta/2-y)\,(1+i\,\omega))]dy$$
$$+(-1)^k(1-\alpha)\,\alpha^2\,T_k((x-3\delta/2)\,(-1-i\,\omega))\,\frac{2T_1(\delta\,(1+i\,\omega))}{1-\alpha^2\,[2T_1(\delta\,(1+i\,\omega))]^2}$$
$$\times\int_{-\delta/2}^{\delta/2}[2\lambda_0(y)\,T_0((\delta/2-y)\,(1+i\,\omega))+4\lambda_1(y)\,T_1((\delta/2-y)\,(1+i\,\omega))]dy$$
$$+(-1)^k(1-\alpha)\int_{-\delta/2}^{\delta/2}[\lambda_0(y)\,T_{k-1}((x+y-\delta)\,(-1-i\,\omega))+2\lambda_1(y)\,T_k((x+y-\delta)\,(-1-i\,\omega))]dy;$$

$$f_4=O((1-\alpha)^2)+(-1)^k\sum_{R=0}^{\infty}\alpha^{2R}T_k((x-\delta/2)\,(-1-i\,\omega))\,[2T_1(\delta\,(1+i\,\omega))]^{2R-1}$$

$$\times \int_{-\delta/2}^{\delta/2} [2\lambda_0(y)\, T_0((\delta/2+y)\,(1+i\,\omega)) - 4\lambda_1(y)\, T_1((\delta/2+y)\,(1+i\,\omega))]dy$$

$$+(-1)^k \sum_{R=2}^{\infty} (2R-2)\,\alpha^{2R-1}\,(1-\alpha)\, T_k((x-\delta/2)\,(-1-i\,\omega))\,[2T_1(2\delta\,(1+i\,\omega))]\,[2T_1(\delta\,(1+i\,\omega))]^{2R-3}$$

$$\times \int_{-\delta/2}^{\delta/2} [2\lambda_0(y)\, T_0((\delta/2+y)\,(1+i\,\omega)) - 4\lambda_1(y)\, T_1((\delta/2+y)\,(1+i\,\omega))]dy$$

$$+(-1)^k \sum_{R=1}^{\infty} \alpha^{2R-1}\,(1-\alpha)\, T_k((x-\delta/2)\,(-1-i\,\omega))\,[2T_1(\delta\,(1+i\,\omega))]^{2R-2}$$

$$\times \int_{-\delta/2}^{\delta/2} [2\lambda_0(y)\, T_0((3\delta/2+y)\,(1+i\,\omega)) - 4\lambda_1(y)\, T_1((3\delta/2+y)\,(1+i\,\omega))]dy$$

$$+(-1)^k \sum_{R=1}^{\infty} \alpha^{2R-1}\,(1-\alpha)\, T_k((x-3\delta/2)\,(-1-i\,\omega))\,[2T_1(\delta\,(1+i\,\omega))]^{2R-2}$$

$$\times \int_{-\delta/2}^{\delta/2} [2\lambda_0(y)\, T_0((\delta/2+y)\,(1+i\,\omega)) - 4\lambda_1(y)\, T_1((\delta/2+y)\,(1+i\,\omega))]dy$$

$$= O((1-\alpha)^2) + (-1)^k T_k((x-\delta/2)\,(-1-i\,\omega))\,\frac{\alpha^2\,[2T_1(\delta\,(1+i\,\omega))]}{1-\alpha^2\,[2T_1(\delta\,(1+i\,\omega))]^2}$$

$$\times \int_{-\delta/2}^{\delta/2} [2\lambda_0(y)\, T_0((\delta/2+y)\,(1+i\,\omega)) - 4\lambda_1(y)\, T_1((\delta/2+y)\,(1+i\,\omega))]dy$$

$$+(-1)^k (1-\alpha)\,\alpha^2\, T_k((x-\delta/2)\,(-1-i\,\omega))\,[2T_1(2\delta\,(1+i\,\omega))]\,\frac{2\alpha\,[2T_1(\delta\,(1+i\,\omega))]}{(1-\alpha^2\,[2T_1(\delta\,(1+i\,\omega))]^2)^2}$$

$$\times \int_{-\delta/2}^{\delta/2} [2\lambda_0(y)\, T_0((\delta/2+y)\,(1+i\,\omega)) - 4\lambda_1(y)\, T_1((\delta/2+y)\,(1+i\,\omega))]dy$$

$$+(-1)^k (1-\alpha)\,\alpha\, T_k((x-\delta/2)\,(-1-i\,\omega))\,\frac{1}{1-\alpha^2\,[2T_1(\delta\,(1+i\,\omega))]^2}$$

$$\times \int_{-\delta/2}^{\delta/2} [2\lambda_0(y)\, T_0((3\delta/2+y)\,(1+i\,\omega)) - 4\lambda_1(y)\, T_1((3\delta/2+y)\,(1+i\,\omega))]dy$$

$$+(-1)^k (1-\alpha)\,\alpha\, T_k((x-3\delta/2)\,(-1-i\,\omega))\,\frac{1}{1-\alpha^2\,[2T_1(\delta\,(1+i\,\omega))]^2}$$

$$\times \int_{-\delta/2}^{\delta/2} [2\lambda_0(y)\, T_0((\delta/2+y)\,(1+i\,\omega)) - 4\lambda_1(y)\, T_1((\delta/2+y)\,(1+i\,\omega))]dy;$$

$$c = \sum_{R=1}^{\infty} \alpha^{2R+1} T_k((x+\delta/2)\,(1+i\,\omega))\,[2T_1(\delta\,(1+i\,\omega))]^{2R-1}\,[4T_2(\delta\,(1+i\,\omega)) + 2\sqrt{\pi}\, T_1(\delta\,(1+i\,\omega))]$$

$$+ \sum_{R=1}^{\infty} (1-\alpha)\,\alpha^{2R} T_k((x+\delta/2)\,(1+i\,\omega))\,[2T_1(\delta\,(1+i\,\omega))]^{2R-1}\,[8T_2(\delta\,(1+i\,\omega))] + O((1-\alpha)^2)$$

$$
\begin{aligned}
&+\sum_{R=2}^{\infty}(2R-2)\,(1-\alpha)\,\alpha^{2R}T_k((x+\delta/2)\,(1+i\,\omega))\,[2T_1(2\delta\,(1+i\,\omega))]\,[2T_1(\delta\,(1+i\,\omega))]^{2R-3}\\
&\qquad\qquad\times[4T_2(\delta\,(1+i\,\omega))+2\sqrt{\pi}T_1(\delta\,(1+i\,\omega))]\\
&+\sum_{R=1}^{\infty}(1-\alpha)\,\alpha^{2R}T_k((x+\delta/2)\,(1+i\,\omega))\,[2T_1(\delta\,(1+i\,\omega))]^{2R-2}\,[4T_2(2\delta\,(1+i\,\omega))+2\sqrt{\pi}T_1(2\delta\,(1+i\,\omega))]\\
&+\sum_{R=1}^{\infty}(1-\alpha)\,\alpha^{2R-1}T_k((x+3\delta/2)\,(1+i\,\omega))\,[2T_1(\delta\,(1+i\,\omega))]^{2R-2}\,[4T_2(\delta\,(1+i\,\omega))+2\sqrt{\pi}T_1(\delta\,(1+i\,\omega))]\\
&+4\,(1-\alpha)\,T_{k+1}((x+\delta/2)\,(1+i\,\omega))+\alpha\,[2T_{k+1}((x+\delta/2)\,(1+i\,\omega))+\sqrt{\pi}\,T_k((x+\delta/2)\,(1+i\,\omega))]\\
&=O((1-\alpha)^2)+T_k((x+\delta/2)\,(1+i\,\omega))\,[4T_2(\delta\,(1+i\,\omega))+2\sqrt{\pi}T_1(\delta\,(1+i\,\omega))]\,\frac{\alpha^3[2T_1(\delta\,(1+i\,\omega))]}{1-\alpha^2\,[2T_1(\delta\,(1+i\,\omega))]^2}\\
&\qquad+(1-\alpha)\,T_k((x+\delta/2)\,(1+i\,\omega))\,[8T_2(\delta\,(1+i\,\omega))]\,\frac{\alpha^2[2T_1(\delta\,(1+i\,\omega))]}{1-\alpha^2\,[2T_1(\delta\,(1+i\,\omega))]^2}\\
&+(1-\alpha)\,T_k((x+\delta/2)\,(1+i\,\omega))\,[2T_1(2\delta\,(1+i\,\omega))]\,[4T_2(\delta\,(1+i\,\omega))+2\sqrt{\pi}T_1(\delta\,(1+i\,\omega))]\\
&\qquad\qquad\times\frac{\alpha^4[2T_1(\delta\,(1+i\,\omega))]}{(1-\alpha^2\,[2T_1(\delta\,(1+i\,\omega))]^2)^2}\\
&+(1-\alpha)\,T_k((x+\delta/2)\,(1+i\,\omega))\,[4T_2(2\delta\,(1+i\,\omega))+2\sqrt{\pi}T_1(2\delta\,(1+i\,\omega))]\,\frac{\alpha^2}{1-\alpha^2\,[2T_1(\delta\,(1+i\,\omega))]^2}\\
&+(1-\alpha)\,T_k((x+3\delta/2)\,(1+i\,\omega))\,[4T_2(\delta\,(1+i\,\omega))+2\sqrt{\pi}T_1(\delta\,(1+i\,\omega))]\,\frac{\alpha}{1-\alpha^2\,[2T_1(\delta\,(1+i\,\omega))]^2}\\
&+4(1-\alpha)\,T_{k+1}((x+\delta/2)\,(1+i\,\omega))+\alpha\,[2T_{k+1}((x+\delta/2)\,(1+i\,\omega))+\sqrt{\pi}\,T_k((x+\delta/2)\,(1+i\,\omega))];
\end{aligned}
$$

$$
\begin{aligned}
d&=(-1)^k\sum_{R=0}^{\infty}\alpha^{2R+1}T_k((\delta/2-x)\,(1+i\,\omega))\,[2T_1(\delta\,(1+i\,\omega))]^{2R}\\
&\quad\times[(1-\alpha)\,(8T_2(\delta\,(1+i\,\omega)))+\alpha\,(2\sqrt{\pi}T_1(\delta\,(1+i\,\omega))+4T_2(\delta\,(1+i\,\omega)))]\\
&+(-1)^k\sum_{R=1}^{\infty}(2R-1)\,(1-\alpha)\,\alpha^{2R}T_k((\delta/2-x)\,(1+i\,\omega))\,[2T_1(2\delta\,(1+i\,\omega))]\,[2T_1(\delta\,(1+i\,\omega))]^{2R-2}\\
&\qquad\times[\alpha\,(2\sqrt{\pi}T_1(\delta\,(1+i\,\omega))+4T_2(\delta\,(1+i\,\omega)))]\\
&\quad+(-1)^k\sum_{R=1}^{\infty}(1-\alpha)\,\alpha^{2R}T_k((\delta/2-x)\,(1+i\,\omega))\,[2T_1(\delta\,(1+i\,\omega))]^{2R-1}\\
&\qquad\times[\alpha\,(2\sqrt{\pi}T_1(2\delta\,(1+i\,\omega))+4T_2(2\delta\,(1+i\,\omega)))]\\
&\quad+(-1)^k\sum_{R=1}^{\infty}(1-\alpha)\,\alpha^{2R}T_k((3\delta/2-x)\,(1+i\,\omega))\,[2T_1(\delta\,(1+i\,\omega))]^{2R-1}\\
&\qquad\times[\alpha\,(2\sqrt{\pi}T_1(\delta\,(1+i\,\omega))+4T_2(\delta\,(1+i\,\omega)))]\\
&\quad+(-1)^k(1-\alpha)\,[\alpha\,[2T_{k+1}(3\delta/2-x)\,(1+i\,\omega))+\sqrt{\pi}\,T_k((3\delta/2-x)\,(1+i\,\omega))]]
\end{aligned}
$$

$$
\begin{aligned}
&= O((1-\alpha)^2) + (-1)^k T_k((\delta/2-x)(1+i\,\omega))\,\frac{\alpha}{1-\alpha^2\,[2T_1(\delta\,(1+i\,\omega))]^2}\\
&\times[8(1-\alpha)\,T_2(\delta\,(1+i\,\omega)) + \alpha\,(2\sqrt{\pi}T_1(\delta\,(1+i\,\omega)) + 4T_2(\delta\,(1+i\,\omega)))]\\
&+(-1)^k(1-\alpha)\,T_k((\delta/2-x)(1+i\,\omega))\,[2T_1(2\delta\,(1+i\,\omega))]\,\frac{\alpha^2[1+\alpha^2\,[2T_1(\delta\,(1+i\,\omega))]^2]}{(1-\alpha^2\,[2T_1(\delta\,(1+i\,\omega))]^2)^2}\\
&\times[\alpha\,(2\sqrt{\pi}T_1(\delta\,(1+i\,\omega)) + 4T_2(\delta\,(1+i\,\omega)))]\\
&+(-1)^k(1-\alpha)\,T_k((\delta/2-x)(1+i\,\omega))\,\frac{\alpha^2(2T_1(\delta\,(1+i\,\omega)))}{1-\alpha^2\,[2T_1(\delta\,(1+i\,\omega))]^2}\\
&\times[\alpha\,(\sqrt{\pi}T_1(2\delta\,(1+i\,\omega)) + 2T_2(2\delta\,(1+i\,\omega)))]\\
&+(-1)^k(1-\alpha)\,T_k((3\delta/2-x)(1+i\,\omega))\,\frac{\alpha^2(2T_1(\delta\,(1+i\,\omega)))}{1-\alpha^2\,[2T_1(\delta\,(1+i\,\omega))]^2}\\
&\times[\alpha\,(2\sqrt{\pi}T_1(\delta\,(1+i\,\omega)) + 4T_2(\delta\,(1+i\,\omega)))]\\
&+(-1)^k(1-\alpha)\,[\alpha\,[2T_{k+1}(3\delta/2-x)(1+i\,\omega)) + \sqrt{\pi}\,T_k((3\delta/2-x)(1+i\,\omega))]].
\end{aligned}
$$

We then briefly present some comparison of different numerical methods for eq. (6) – (10).

NUMERICAL SIMULATIONS

Finite differences-1

Numerical simulations were performed in [5] by using a standard finite difference scheme for eq. (6) – (10). More precisely, we used an implicit first order upwind scheme which also was used to establish the validity of this model (that is, the scheme can be used for more refined models enabling to take into account the temperature, to use the ES model instead of the BGK one, or even to introduce 2D effects, cf. [5]). The scheme is described in detail for example in [9].

Finite differences-2

Formulas (11), (12) can be understood (when $k = 0, 1$) as as a system of integrodifferential equations which can be numerically solved thanks to a finite difference scheme. We discretized the integrals w.r.t. t and v with a second-order (trapeze) rule and truncated the series in such a way that the remainder term is negligible.

Since except in the case $\alpha = 0$ (pure specular reflexion), the Rth-term of the series involves an integration on a $2R+2$ or $2R+3$-dimensional space, the computation can be performed only if all terms but the first in the series are discarded. This means that (when $\alpha \neq 0$), only a case with δ sufficiently large (i.-e. only slightly rarefied) can be computed.

Unfortunately, such cases are known to be rather poorly modeled by eq. (6) – (10), because the temperature plays a non negligible role in this situation, cf. [5]. As a consequence, the discretization of (11), (12) can be used efficiently only when $\alpha = 0$ (pure specular reflexion case). In practice, even in this case, the numerical code leads to slightly longer computations than the standard finite difference scheme (defined in the previous subsection) for a given required precision. As a consequence, this approach is merely useful for cross checking the results of other codes, but cannot be considered as an efficient method for solving eq. (6) – (10).

DSMC

It is also possible to perform the computations of the solutions of eq. (6) – (10) thanks to a particle method. Since h is a fluctuation, it is not nonnegative, and one is led to use a semi-discretization (that is, still continuous w.r.t. time) of the form

$$h(t,x,v) \sim \sum_{i=1}^{N} r_i(t)\, \delta_{x_i(t)}(x)\, \delta_{v_i(t)}(v),$$

where $x_i(t)$, $v_i(t)$ and $r_i(t)$ are the respective positions, velocities and numerical weights of the particles, and where the weights can be either bigger or smaller than 0.

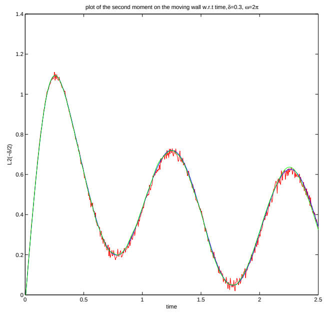
The method belongs to the class of DSMC methods in the sense that (diffuse) boundary conditions as well as collisions (source term) are treated by a Monte Carlo approach. We refer e.g. to [1] for the use of Monte Carlo methods in microfluidics.

As can be expected, since we are solving a time-dependent problem, the DSMC method needs a large number of particles and many averages in order to give acceptable results, which lead to a performance very far from that of the finite difference method. We think nonetheless that it was interesting to test this method since for future computations involving more complex models (Boltzmann equation, 2D or 3D extensions) it may be the only one available.

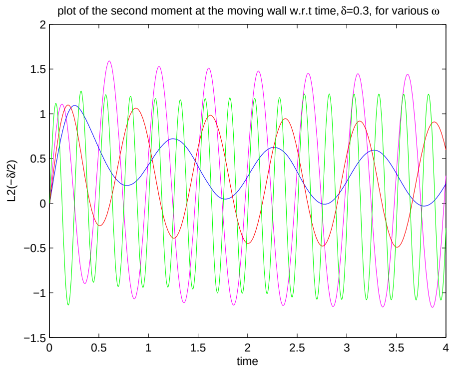
Comparisons of the different approaches

In order to show the coherence of the various numerical methods used to solve eq. (6) – (10), we show the results obtained in a typical simulation.

The first figure shows the evolution of the second moment (L_2) at point $-\delta/2$ (i.-e. a quantity which is directly related to the pressure at the moving wall), with respect to time. The length of the channel in this experiment is $\delta = 0.3$ and the frequency is $\omega = 2\pi$. In the DSMC computation, about 10^6 particles are used, and the result is averaged over 100 independent experiments. In the figure, the red curve corresponds to a DSMC computation (the random fluctuations can be observed) while the blue and green curves correspond to the two finite differences methods (and are almost indistinguishable). As can be seen, the convergence towards the permanent (periodic) regime is already visible on a few periods of the oscillating wall.



We recall that our main interest in [5] was to find the resonances/antiresonances in the channel of the device, that is (for different values of the channel dimension δ) the values of ω corresponding to extrema of the amplitude of the oscillation (in the permanent regime) of $L_2(t,-\delta/2)$. We show in figure 2 an illustration of the computations done in [5] with the finite difference scheme, where for $\delta = 0.3$ (quite rarefied gas), and $\alpha = 0$, the evolution of $L_2(t,-\delta/2)$ is shown at various values of ω (that is, $\omega = 2\pi$, $\omega = \frac{8}{3}\pi$, $\omega = 4\pi$, and $\omega = 8\pi$).



The interested reader will find in [5] the analytical and numerical study (in the case $\alpha = 1$) of the position of resonances/antiresonances when the length δ of the channel varies.

We hope that the computations presented in this paper will help to extend the results of [5] in more complex cases (when $\alpha \neq 1$, or when the modeling is more detailed).

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