On the scattering by a cavity in an impedance plane in 3D : boundary integral equations and novel Green's function

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summary :
The problem of the field scattered by a cavity embedded in an impedance (or Robin) plane is considered for the 3D Helmholtz equation in acoustics. Its resolution is more complex than for a scatterer above the plane, in particular because the Green's function for the unperturbed plane has a singular part unsuitable for applications below the plane. It is why the free space Green's function is commonly used in boundary integral equations for the cavity, and three unknowns are necessary. We propose here to use a novel Green's function below the impedance plane, which has the advantage to reduce the number of unknowns, and to simplify the problem. This specific Green's function derives from our recent study for passive and active unperturbed impedance planes. The uniqueness property is studied in passive case. The application to small cavity leads us to new analytical results.

1) Introduction

This paper presents novel integral equations for the field scattered by a cavity embedded in an imperflectly reflective plane with impedance boundary conditions, for the three-dimensional Helmholtz equation.

The development of boundary integral equation methods, in 2D and 3D, for this scattering problem is rather recent [1], [2], seemingly because of specific difficulties due to the representation of the field in the cavity. Indeed, the Green's function G_a, defined as the field of a monopole in presence of the unperturbed impedance plane, that is well adapted for solving the case of a perturbation in relief, has a logarithmic singularity in lower half space, which prevents it from being applied below the plane. It is why another representation is necessary in the cavity, and, until now, the Green's function in free space was used, which implies three distinct integral equations for three surface field unknowns [1] : the field and its normal derivative on the aperture of the cavity, and the field on the surface of the cavity.

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To reduce the number of unknowns and simplify the system of integral equations, we here develop an original way, consisting in the definition of a new Green's function that we name the 'below' Green's function G_b. Both functions G_a and G_b satisfy the impedance condition on the unperturbed surface, but the scattered fields attached to them are respectively regular above and below the plane. They derive from the solution for an arbitrary constant impedance plane (passive or active) given in [3]-[4].
Morever, our system of two novel integral equations has the property of uniqueness of the solution. It is an important point, particularly if we notice that most of the boundary integral equation methods in the related problem of electromagnetism, which use the generalized network formulation [5], are not uniquely solvable at some discrete frequencies [6]. Otherwise, other methods verify uniqueness, in particular the one developed by Chandler-Wilde in acoustics, with a system of three integral equations [1], and the one used by Xu for perfectly conducting surface in electromagnetism, with a system of two vectorial integral equations [6]. It is worth noticing that, in [6], the generalized network formulation is corrected by the image theory, which is equivalent to using a specific Green's function in the cavity that takes account of the plane.
This scattering problem can be also analyzed in complex spectral domain in 2D, or by asymptotic methods in 2D and 3D. So, integral equations with smooth kernels, which permit various approximations for large or small polygonal cavity in 2D [7], or asymptotic expressions for a large cavity [8]-[9], have been developed.
The paper is organized as follows. In section 2, we define the properties of the acoustic field, and analyze the uniqueness of the boundary value problem. We present in section 3, the expressions of the Green's functions G_a and G_b, derived from the solution for an unperturbed impedance plane. In section 4, we use the second Green's theorem and give a representation of the field above the plane and in the cavity. We then deduce the system of integral equations in section 5 and show the property of uniqueness in section 6. In section 7, this new system is considered for small cavity and original analytical results are derived.

2) Formulation of the boundary value problem and uniqueness

2.1) Boundary value problem

We consider the pressure field p_s scattered by an imperfectly reflective plane that is perturbed by a cavity (figure 1), when it is illuminated by the incident pressure field p_{inc}, radiated by a bounded source W above the plane, and satisfying the Helmholtz equation,

$$(\Delta + k^2)p_{inc} = W \tag{1}$$

in R^3, with $|\arg(ik)| \leq \pi/2$.
The plane S_0 is defined by $z = 0$ in Cartesian coordinates (x, y, z). The domain of the cavity with $z < 0$, and the half-space above the plane with $z \geq 0$, are respectively denoted Ω_2 and Ω_1. The aperture and the surface of the cavity, respectively denoted S_1 and S_2, are assumed to be piecewise analytic (with no zero exterior angles, i.e. no points of Ω_2 inside a cusp), bounded by a Jordan curve C_1.

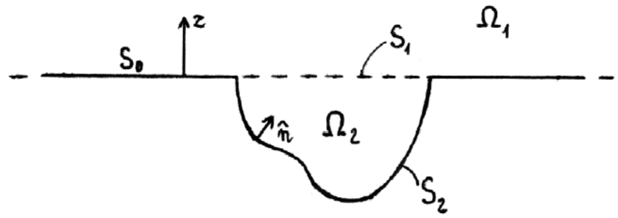

figure 1 : geometry and definition of the cavity

For any plane wave of incidence angle β composing p_{inc}, the infinite plane, when it is unperturbed, has a reflection coefficient $R(\beta)$ given by,

$$R(\beta) = \frac{\cos\beta - g}{\cos\beta + g}, \tag{2}$$

so that $p = p_s + p_{inc}$ verifies the impedance (or Robin) boundary condition,

$$(\frac{\partial}{\partial z} - ikg)p = 0 \tag{3}$$

on the plane S_0, except on the aperture S_1 of the cavity. The term $g = \sin\theta_1$ is denoted the impedance parameter. In (3), it is a constant, with $\mathrm{Re}(ik\cos\theta_1) \neq 0$ when $\mathrm{Re}\theta_1 \leq 0$. This condition on g is due to the presence of a cut in the solution for an unperturbed plane [3]-[4], along the path $\mathrm{Re}(ik\cos\theta_1) = 0$ as $\mathrm{Re}\theta_1 \leq 0$. Therefore, the surface waves, which radiate without exponential decay at infinity, can only be considered in the sense of the limit for $\mathrm{Re}(ik\cos\theta_1) = 0^+$or 0^- when $\mathrm{Re}\theta_1 \leq 0$.

Some general properties are considered for the scattered field in Ω_1 and Ω_2:
(a) $p_s(z)$, which satisfies the Helmholtz equation

$$(\Delta + k^2)p_s = 0 \ \text{ with } |\arg(ik)| \leq \pi/2, \tag{4}$$

is regular in $\Omega_1 \cup \Omega_2$, except at edges and corners of S_2 where

$$p_s = O(1) \text{ and grad}(p_s) = O(|r|^{\alpha}),\ -1 < \alpha \leq 0, \tag{5}$$

as the distance $|r|$ to the edge or corner vanishes [8], and p_s is continuous on the scatterer;

(b) p_s is constituted of outgoing waves, with guiding waves exponentially vanishing at infinity (Re($ik\cos\theta_1$) $\neq 0$ as Re$\theta_1 \leq 0$), and, the field at M, with $r = \overline{OM}$, verifies,

$$p_s = O(e^{-\delta|r|}), \tag{6}$$

$\delta > 0$, as z or $\rho = \sqrt{x^2+y^2} \rightarrow \infty$, $z > 0$, when $|\arg(ik)| < \pi/2$, and

$$\frac{\partial p_s}{\partial |r|} + ikp_s = o(|r|^{-1}),\ p_s = O(|r|^{-1}), \tag{7}$$

as $|r| = \sqrt{x^2+y^2+z^2} \rightarrow \infty$, $z \geq 0$, when $|\arg(ik)| = \pi/2$.

In addition, an impedance boundary condition is assumed on the surface of the cavity,

$$(\frac{\partial}{\partial n} - ikg_c)p|_{S_2} = 0, \tag{8}$$

where $\hat{n}$ is the normal to S_2 directed inside Ω_2, g_c is a function piecewise analytic on S_2.

Remark 1 :

Let us notice that the definitions of the 'acoustic impedance' ($\equiv A_0 p/\frac{\partial p}{\partial n}$, A_0 a constant) generally used in physics [20], and of our 'impedance parameter' ($\equiv \frac{\partial p}{\partial n}/(ikp)$), are different.

2.2) Uniqueness of the solution of the boundary value problem from [10, sect.7]

In [10], Levine develops an uniqueness theorem, i.e. a proof that $p_{inc} \equiv 0$ implies $p \equiv 0$, in the case of a scatterer with impedance boundary conditions. He considers piecewise $C^{(2+\lambda)}$ surface (with no zero exterior angle), $\lambda > 0$, without auxiliary 'edge conditions' at edges or corner points, except that p is continuous. He studies at first bounded scatterers, but he also gives, in section 7 of his paper, the elements to generalize his results to scatterers with infinite boundaries, in particular by the use of Jones' uniqueness theorem [11], that we follow.

We begin to notice first that the conditions given by Levine to apply the Green's first theorem are satisfied : the cavity is piecewise analytic (with no zero exterior angle), the field is countinuous on the scatterer, it satisfies impedance boundary conditions and the conditions (b) at infinity. So, we can write,

$$\int_{\Omega}(p^*(r)\Delta p(r) + \text{grad}p^*(r)\text{grad}p(r))dV = -\int_{S} p^*(r)(\widehat{n}\,\text{grad}p)dS + \\ + \lim_{a\to\infty}\int_{r=a,\, z\geq 0} p^*(r)(\frac{\partial p(r)}{\partial r})dS, \qquad (9)$$

where $\Omega \equiv \Omega_1 \cup \Omega_2,\ S \equiv S_2 \cup (S_0 \backslash S_1)$, $\widehat{n}$ is the inward normal to Ω, and from (3)-(8),

$$\text{Re}(\int_{\Omega} -ik|p(r)|^2 + \frac{|\text{grad}p(r)|^2}{-ik}dV) = \int_{S_2}\text{Re}(g_c)|p(r)|^2 dS + \\ + \int_{S_0\backslash S_1}\text{Re}(g)|p(r)|^2 dS + I_\infty\,, \qquad (10)$$

where

$$I_\infty = \text{Re}\lim_{a\to\infty} O(e^{-\delta a}) = 0 \text{ for } |\text{arg}(ik)| < \pi/2\,, \\ I_\infty = \lim_{a\to\infty}\int_{r=a,\, z\geq 0}|p(r)|^2 dS > 0 \text{ for } |\text{arg}(ik)| = \pi/2 \qquad (11)$$

For $\text{Re}(g) \geq 0,\ \text{Re}(g_c) \geq 0$ and $|\text{arg}(ik)| \leq \pi/2$, the left-hand term is negative since $\text{Re}(ik) \geq 0$, while the right-hand term is positive, and thus both terms vanish. Consequently, we have, when $|\text{arg}(ik)| < \pi/2$,

$$p(r) = 0 \text{ in } \Omega,\ \text{for Re}(g) \geq 0,\ \text{Re}(g_c) \geq 0, \qquad (12)$$

and, when $|\text{arg}(ik)| = \pi/2$,

$$p(r) = 0 \text{ on } S,\ \text{for Re}(g) > 0, \text{Re}(g_c) > 0, \\ \partial_n p(r) = 0 \text{ on } S,\ \text{for Re}(g) > 0, \text{Re}(g_c) > 0, \text{ or } g = g_c = 0. \qquad (13)$$

In the latter case, we can use, as suggested by Levine, the Jones' uniqueness theorem [11] for surfaces conical at infinity, when Neumann boundary condition ($\partial_n p(r)|_S = 0$) is satisfied, which implies $u \equiv 0$ in the entire domain Ω, and thus completes the proof of uniqueness. Let us notice, that another proof has been independently developed in [1] when S is smooth.

3) The 'above' Green's function G_a and the 'below' Green's function G_b

The integral representations of the field with single and double layers potentials generally derive from the use of free space Green's function [8], but more complex Green's

functions, verifying particular boundary conditions, can be used. In this latter case, a particular attention must be paid to the regularity of these functions.
So, when we consider a perturbation, due to a scatterer above an impedance plane, we can use the solution G_a for a monopole above this plane to express the field everywhere, while it is generally not possible when we have a cavity, because of the logarithmic singularity of G_a below the plane.
Therefore, we here develop an original way consisting in using another Green's function that we name the 'below' Green's function G_b. Both functions G_a and G_b derive from the solution for an unperturbed plane, respectively with the impedances g and $-g$.
In this section, the solution for active and passive plane [3]-[4] are briefly presented, then G_a and G_b are developed.

3.1) The solution for an unperturbed impedance plane with arbitrary impedance

3.1.1) Solution for a monopole

The incident field radiated at $M(x,y,z)$ by a monopole at $r'(x',y',z'=h)$ (figure 2) is given by $p_{inc}=e^{-ikR(z)}/kR(z)$, with $R(z)=\sqrt{(x-x')^2+(y-y')^2+(z-z')^2}$.

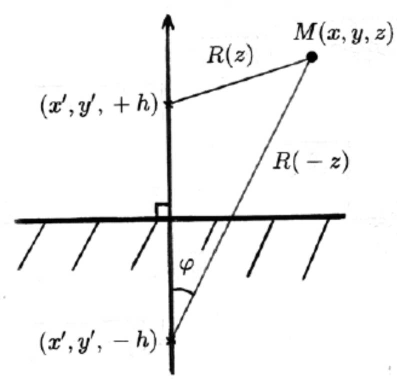

figure 2 : geometry and definition of φ for the radiation at M

From [3], the field p_s scattered by the impedance plane is given by

$$p_s=\frac{e^{-ikR(-z)}}{kR(-z)}+2ige^{ikg(z+h)}\mathcal{J}_g(\rho,-z-h)\,, \tag{14}$$

where $R(-z)=\sqrt{\rho^2+(z+h)^2}$, $z+h=R(-z)\cos\varphi$, $\rho=R(-z)\sin\varphi$, and,

$$\mathcal{J}_g(\rho,\,-z)=\frac{e^{-ikgz}}{2}\int_{\mathcal{D}}\frac{H_0^{(2)}(k\rho\sin\beta)e^{-ikz\cos\beta}}{\cos\beta+g}\sin\beta d\beta, \tag{15}$$

for $z>0$, $g=\sin\theta_1$, with $\mathrm{Re}(ik\sin\beta)=0$ on $\mathcal{D}$ from $-i\infty-\arg(ik)$ to $i\infty+\arg(ik)$. This function is a Fourier-Bessel integral commonly encountered in scattering theory [12, p.234], also called a Sommerfeld-type integral [13], which has a cut described by $\mathrm{Re}(ik\cos\theta_1)=0$ when $\mathrm{Re}(g)\leq 0$ and a singularity at $g=-1$.
A correct definition of $\mathcal{J}_g$ for arbitrary $g=\sin\theta_1$, active ($\mathrm{Re}g<0$) or passive ($\mathrm{Re}g>0$), except on the cut, is also given [3] by,

$$\mathcal{J}_g(\rho,\,-z-h)=-\int_{-ib}^{\infty}e^{-a\cosh t}dt=i\int_{b}^{i\infty}e^{-a\cos\alpha}d\alpha \tag{16}$$

where $a=\epsilon ikR(-z)\sin\varphi\cos\theta_1$, $\epsilon=\mathrm{sign}(\mathrm{Re}(ik\cos\theta_1))$ ($\mathrm{Re}(a)=0$ is on a cut of $\mathcal{J}_g$ and it can be only considered in the sense of the limit), and b satisfies

$$e^{\mp ib}=\frac{ikR(-z)}{a}(1\pm\sin\theta_1)(1\pm\cos\varphi), \tag{17}$$

with $|\mathrm{Re}b|<\pi$, $e^{-2ib}=\frac{(1+\sin\theta_1)(1+\cos\varphi)}{(1-\sin\theta_1)(1-\cos\varphi)}$, $|\mathrm{Re}(\theta_1)|\leq\pi/2$. As g varies, this expression has a correct cut as ϵ changes of sign for $\mathrm{Re}g<0$, and is regular elsewhere (note: for $\mathrm{Re}g>0$, the change of sign of ϵ does not induce a cut as g varies). The figure 3 shows the agreement of $\mathcal{J}_g$ given by (16) and by Fourier-Bessel expansion (15).

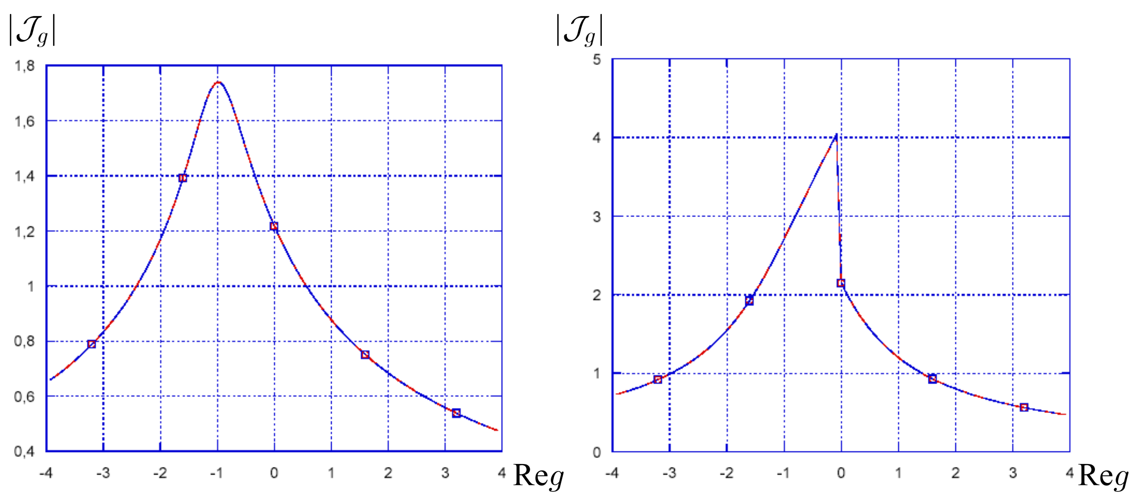

figure 3) Comparison of $\mathcal{J}_g$ given by (16) ($-\square-$) and by Fourier-Bessel expansion when (15) is used ($-\circ-$), when $\mathrm{Re}g$ varies; left : $|\mathcal{J}_g|$ when $\mathrm{Im}(g)=-0.4$, $z+h=.2$, $\rho=.3$, $ik=.01+i1.$; right : $|\mathcal{J}_g|$ when $\mathrm{Im}(g)=1.2$, $z+h=1.$, $\rho=1.$, $ik=.01+i1.$

3.1.2) Some properties of $\mathcal{J}_g$

Some general properties of $\mathcal{J}_g$, derived from (16), are worth noticing. Using the integral expression of the modified Bessel function K_0 [17], we can write,

$$\mathcal{J}_g(\rho,\,-z-h)+K_0(a)=\,-i\int_0^b e^{-a\cos\alpha}d\alpha=\,-i\int_{-b}^0 e^{-a\cos\alpha}d\alpha, \tag{18}$$

which implies, by definition of b and a, that

$$\mathcal{J}_g(\rho,\,-z-h)+K_0(a)=\,-\mathcal{J}_{-g}(\rho,z+h)-K_0(a) \tag{19}$$

where $a=\epsilon ik\rho\cos\theta_1,\ \epsilon=\mathrm{sign}(\mathrm{Re}(ik\cos\theta_1))$. From the regularity of $\mathcal{J}_{\pm g}(\rho,\,-z)$ for $z>0$ and the expression of b, we deduce that $\mathcal{J}_g(\rho,\,-z)$ has a logarithmic singularities when $z\le 0$ at $\rho=0$. So, when a, and thus, when ρ vanishes, we have [3]

$$\begin{aligned}&\mathcal{J}_g(\rho,\,-z)\sim\,-2K_0(a)\text{ when }z<0,g\neq\,-1,\\&\mathcal{J}_g(\rho,\,-z)\sim\,-K_0(a)\text{ when }z=0,\ g\neq\,-1.\\&\mathcal{J}_g(\rho,\,-z)\sim\,-E_1(\frac{ik(1+g)}{2}(|r|+z))\end{aligned} \tag{20}$$

Moreover, the reader can verify by inspection that,

$$\frac{\partial\mathcal{J}_g(\rho,\,-z-h)}{\partial z}=\frac{e^{-ik(R(-z)+gz)}}{R(\,-z)}, \tag{21}$$

and,

$$(\Delta+k^2)(e^{ikgz}\mathcal{J}_g(\rho,\,-z))=4\pi e^{ikgz}U(\,-z)\delta(x)\delta(y) \tag{22}$$

where U is the unit step function, δ is the Dirac function.

Remark 2 :
Let us notice [3] that, for Re$g>0$ and arg$(ik)=\pi/2$,

$$\mathcal{J}_g(\rho,\,-z-h)=\int_{-i\infty}^0 e^{-ikg(z_1+z+h)}\,\frac{e^{-ikR(-z_1-z)}}{kR(\,-z_1-z)}k\,dz_1, \tag{23}$$

where $R(\,-z)=\sqrt{\rho^2+(z+h)^2}$, and that, for $g=1$,

$$\mathcal{J}_{g=1}(\rho,\,-z-h)=\,-E_1(ik(R(\,-z)+(z+h))) \tag{24}$$

where E_1 is the exponential integral [17].

3.2) The functions G_a and G_b

3.2.1) The Green's functions G_a above the plane

The Green's function G_a is given by the solution for a monopole above the plane with impedance parameter g . From the previous section, it is given by

$$G_a(r,r') = G^0(x-x',y-y',z-z') + G_g^s(x-x',y-y',\,-z-z')\,, \qquad (25)$$

where G^0 is the free space Green's function,

$$G^0(r) = \frac{e^{-ik|r|}}{k|r|}, \qquad (26)$$

and G_g^s is the scattered Green's function,

$$G_g^s(r) = \frac{e^{-ik|r|}}{k|r|} + 2ige^{-ikgz}\mathcal{J}_g(\rho,z), \qquad (27)$$

with $|r| = \sqrt{\rho^2+z^2}$ and $\rho = \sqrt{x^2+y^2}$.
Because

$$(\Delta+k^2)G^0(r) = \frac{-4\pi}{k}\delta(x)\delta(y)\delta(z), \qquad (28)$$

and the equation (22) satisfied by $\mathcal{J}_g(\rho,\,-z)$, the function G_a verifies in R^3,

$$\begin{aligned}(\Delta+k^2)G_a(r,r') &= \frac{-4\pi}{k}(\delta(r-r') + \\ &+ \delta(r-r'_{im}) - 2ikge^{ikg(z+z')}U(\,-z-z')\delta(x-x')\delta(y-y'))\end{aligned} \qquad (29)$$

where $r'_{im} \equiv (x',y',\,-z'),\ \delta(r) \equiv \delta(x)\delta(y)\delta(z)$. It satisfies correct radiation conditions at infinity for $z \geq 0$ (condition (b)), and will be our choice for the Green's function above the plane for arbitrary $g = \sin\theta_1$, except for $\mathrm{Re}(ik\cos\theta_1) = 0$ when $\mathrm{Re}(g) \leq 0$ (i.e. except on the cut of $\mathcal{J}_g$).

3.2.2) The Green's functions G_b below the impedance plane

The function G_a cannot be used to describe the field in the cavity, when it is influenced by fictitious sources on the aperture, in particular because of the presence of a logarithmic singularity of $\mathcal{J}_g(\rho,\,-z)$ for negative z when $\rho = 0$.

However, we can consider $\mathcal{J}_{-g}(\rho, z)$ instead of $\mathcal{J}_g(\rho, -z)$, and obtain an original Green's function G_b, which is suitable for an integral representation of the field in the cavity, and continues to satisfy the impedance boundary condition (3). This choice will be corrected in the vicinity of $g = 1$ to take account of the singularity of $\mathcal{J}_{-g}$ at this point.

3.2.2.1) The function G_b for $g \neq 1$

We remark that, below the plane where $z + z' < 0$, the function

$$G_b(r, r') = G^0(x - x', y - y', z - z') + G^s_{-g}(x - x', y - y', z + z')\,, \qquad (30)$$

with

$$G^s_{-g}(r) = \frac{e^{-ik|r|}}{k|r|} - 2ige^{ikgz}\mathcal{J}_{-g}(\rho, z), \qquad (31)$$

continues to satisfy the impedance boundary condition (3) on the plane $z = 0$, is regular for $z + z' < 0$, except for the singularity of G^0 at $z = z'$, and verifies in R^3,

$$(\Delta + k^2)G_b(r, r') = \frac{-4\pi}{k}(\delta(r - r') + \\ + \delta(r - r'_{im}) + 2ikge^{ikg(z+z')}U(z + z')\delta(x - x')\delta(y - y')) \qquad (32)$$

where $r'_{im} \equiv (x', y', -z')$, $\delta(r) \equiv \delta(x)\delta(y)\delta(z)$.
This will be our choice for the Green's function below the plane, except in the vicinity of $g = 1$ (where $\mathcal{J}_{-g}$ is singular) and on the cut of $\mathcal{J}_{-g}$ (the case with $\mathrm{Re}(ik\cos\theta_1) = 0$ has to be taken in the sense of the limit). Let us notice that it satisfies the usual radiation conditions at infinity, similar to (b) but in lower space instead of upper space.

Remark 3 :
In the case of a cavity Ω_2 filled with a material, we can consider the wave number k_2 instead of k, and $g_2 = kg/k_2$ in place of g, so that G_b continues to satisfies the impedance boundary condition (3) on the plane $z = 0$.

3.2.2.2) A suitable choice for G_b when $g \simeq 1$, regular on the cut of $\mathcal{J}_{-g}$

The function $\mathcal{J}_{-g}(\rho, z)$ is singular at $g = 1$. However, considering the equations (19) and the domain of regularity of $\mathcal{J}_g$ [3], the function $\mathcal{J}_{-g}(\rho, z) + 2K_0(a)$ is regular for $\rho \neq 0$ in vicinity of $g = 1$, as $g = \sin\theta_1$ varies, with $a = ik\epsilon\rho\cos\theta_1$, $\epsilon = \mathrm{sign}(\mathrm{Re}(ik\cos\theta_1))$. We can then use that

$$K_0(a) + \ln(a) I_0(a) \tag{33}$$

is an entire function of a [17], and

$$(\Delta + k^2)(e^{ik\sin\theta_1 z} I_0(ik\epsilon\rho\cos\theta_1) = 0, \tag{34}$$

and choose to add the term

$$D_b(r, r') = 4ig \ln(ikd\cos\theta_1) I_0(ik\cos\theta_1 \sqrt{(x-x')^2 + (y-y')^2}) e^{ikg(z+z')}, \tag{35}$$

to G_b for $g \simeq 1$, where d is an arbitrary constant. So defined,

$$G_b(r, r') = G^0(x-x', y-y', z-z') + G^s_{-g}(x-x', y-y', z+z') + D_b(r, r'), \tag{36}$$

becomes regular for Re$g \geq 0$, and presents, as g varies, the same cut and singularities as G_a for Re$g \leq 0$.

This function continues to satisfy the impedance boundary condition (3) on the plane $z = 0$, is regular for $z + z' < 0$ except for the singularity of G^0, and verifies (32). The corrective term $D_b(r, r')$ does not satisfy the usual radiation conditions at infinity but it will be of no consequence for our demonstration in further sections, and this function can be used when $|ik\epsilon\rho\cos(\theta_1)| \ll 1$ is verified in the whole cavity.

Remark 4 :

For $g \rightarrow 1,$ we notice [3] that

$$\begin{aligned} \mathcal{J}_{-g}(\rho, z) = {} & E_1(\frac{ik(1+g)(|r|+z)}{2}) - 2K_0(a) + \\ & + O(ik(1-g)(|r|-z)E_2(\frac{ik(1+g)(|r|+z)}{2})) \end{aligned} \tag{37}$$

and thus

$$G^s_{-g}(r) + D_b(r) \rightarrow \frac{e^{-ik|r|}}{k|r|} - 2ie^{ikz}(E_1(ik(|r|+z)) + 2\ln(\rho/d)), \tag{38}$$

which is regular for $\rho = 0$, $z < 0$, since, $|r| + z = \frac{\rho^2}{|r|-z}$ and $E_1(v) = -\ln(v) + O(1)$.

3.2.3) Some additional properties of $G_{a,b}(r, r')$

From the derivative of $\mathcal{J}_g$ given in (21), we have

$$(\frac{\partial.}{\partial z'}-ikg)G_{a,b}(r,r')$$
$$=(\frac{\partial.}{\partial z'})(G^0(r-r')+G^0(r-r'_{im}))-ikg(G^0(r-r')-G^0(r-r'_{im})), \quad (39)$$

where $r'_{im}\equiv(x',y',-z')$. This leads us to write, when $z=0$,

$$(\frac{\partial.}{\partial z'}-ikg)G_{a,b}(r,r')|_{z=0}=(\frac{\partial.}{\partial z'})(2G^0(r-r'))|_{z=0}\,, \quad (40)$$

and, when $z'\to 0$, $z\neq 0$,

$$(\frac{\partial.}{\partial z'}-ikg)G_{a,b}(r,r')\to 0. \quad (41)$$

These properties will be particularly useful to prove the continuity of the normal derivative of the field, deduced from our solution, through the aperture of the cavity. Moreover, for our choice of G_b in section 3.2.2.1 (for $g\neq 1$), we have

$$G_b(r,r')=G_a(r,r')+4ige^{ikg(z+z')}K_0(a)$$
$$G_b(r,r')|_{g=v}=G_a(r_{im},r'_{im})|_{g=-v}$$
$$(G_a(r,r')+G_b(r,r'))|_{z=z'=0}=4(\frac{e^{-ik\rho}}{k\rho}+ig(\mathcal{J}_g(\rho,0)+K_0(a))), \quad (42)$$

while, for our choice of G_b in section 3.2.2.2 (for $g\simeq 1$),

$$G_b(r,r')=G_a(r,r')+4ige^{ikg(z+z')}(K_0(a)+\ \ln(ikd\cos(\theta_1))I_0(ik\rho\cos(\theta_1)))$$
$$(G_a(r,r')+G_b(r,r'))|_{z=z'=0}=4(\frac{e^{-ik\rho}}{k\rho}+ig(\mathcal{J}_g(\rho,0)+K_0(a)+$$
$$+\ln(ikd\cos(\theta_1))I_0(ik\rho\cos(\theta_1)))), \quad (43)$$

where

$$\mathcal{J}_g(\rho,0)+K_0(a)=\ -i\int_0^b e^{-a\cos\alpha}d\alpha,\ b=\ \mp i\ln(\,\epsilon\frac{(1\mp\sin\theta_1)}{\cos\theta_1}), \quad (44)$$

with $g=\sin\theta_1,\ a=\epsilon ik\rho\cos\theta_1,\ \epsilon=\mathrm{sign}(\mathrm{Re}(ik\rho\cos\theta_1))$. Let us also notice that, in agreement with the reciprocity principle [8], we have $G_{a,b}(r,r')=G_{a,b}(r',r)$.

Remark 5 :

We can use (24) in (43) for $g\to 1$, and notice that,

$$(G_a(r,r')+G_b(r,r'))|_{z=z'=0}\to 4(\frac{e^{-ik\rho}}{k\rho}-i(E_1(ik\rho)+\ln(\rho/d))), \quad (45)$$

which is regular for $\rho = 0$ since $E_1(v) = -\ln(v) + O(1)$.

Remark 6 :
For $r' \to \pm\infty$, $r - r_{im} = 2\widehat{z}(\widehat{z}.r)$, we have

$$G_a(r,r') = \frac{e^{-ik|r'|}}{k|r'|}([e^{ik(r.r')/|r'|}(1 + e^{-2ik(\widehat{z}\frac{r'}{|r'|})\widehat{z}.r}(\frac{\widehat{z}\frac{r'}{|r'|} - g}{\widehat{z}\frac{r'}{|r'|} + g}))] + o(1)) \tag{46}$$

4) Integral representation of the field with G_a and G_b

4.1) The representation of the field from the second Green's theorem

Let us consider the pressure fields p and G, satisfying the Helmholtz equation

$$\begin{aligned}(\Delta + k^2)p &= W, \\ (\Delta + k^2)G &= W_G\,,\end{aligned} \tag{47}$$

in the domain Ω, bounded by the surface $\partial\Omega$, piecewise analytic. If the functions p and G have the regularity which permit the application of the second Green's theorem, we can write

$$\int_\Omega W(r)\,G(r)\,dV - \int_\Omega W_G(r)\,p(r)\,dV = \int_{\partial\Omega^+} \widehat{n}.(\text{grad}(G)p - \text{grad}(p)G)dS, \tag{48}$$

where $\partial\Omega^+$ denotes the internal surface to Ω, $\widehat{n}$ is the unit normal, piecewise defined, directed inside Ω, and the surface integral is taken in the sense of principal value of Cauchy. Thereafter, we omit the sign for $\partial\Omega^+$, and we write $\partial\Omega$ instead of $\partial\Omega^+$.

4.2) The case $W_G(r) = -w\delta(r - r')$

Let us consider $W_G(r)$ as a generalized function in (48), with $W_G(r) = -w\delta(r - r')$, w being a constant. In this case, we have

$$1_\Omega(r')p(r') - p_i(r') = \frac{1}{w}\int_{\partial\Omega} \widehat{n}.(\text{grad}(G(r,r'))p - \text{grad}(p)G(r,r'))dS, \tag{49}$$

for $r' \in \overline{\Omega}$, where

$$p_i \;=\; -\frac{1}{w}\int_\Omega W(r)\,G(r,r')\,dV,$$
$$1_\Omega(r') = \int_\Omega \delta(r-r')dr = \frac{1}{4\pi}\int_{\partial\Omega} \widehat{n}\;\text{grad}(\frac{1}{|r'-r|})dS = \frac{1}{4\pi}\int_{\partial\Omega} \frac{(r'-r)}{|r'-r|^3}\widehat{n}dS, \quad (50)$$

and the integrals are considered in the sense of the principal value of Cauchy. The reader can easily recover 1_Ω, by letting $k=0$, $G(r,r') = \frac{w}{4\pi|r'-r|}$ and $p \equiv 1$.

remark 7 :
$1_\Omega = 1$ in Ω, $=0$ in $R^3\backslash\overline{\Omega}$, is fractional on $\partial\Omega$ ($=\frac{1}{2}$ when $\partial\Omega$ is smooth), $=0$ if $\Omega \equiv 0$. For an external problem in $R^3\backslash\Omega'$, the surface can be considered to be closed at infinity so that $1_{\mathrm{R}^3\backslash\Omega'}(r') = 1 - 1_{\Omega'}(r')$, where $\widehat{n}$ is the outward normal to Ω'.

remark 8 :
Considering the continuity of the single-layer potential in (50), we notice that

$$\begin{aligned}
&\frac{1}{w}\int_{\partial\Omega}(\widehat{n}.\text{grad}(G(r,r'))p_e(r) - q_e(r)G(r,r'))dS|_{r'\in\Omega\to r_0\in\partial\Omega}\\
&\to (1-1_\Omega(r_0))p_e(r_0) + \\
&+\frac{1}{w}\text{p.v.}\int_{\partial\Omega}(\widehat{n}.\text{grad}(G(r,r_0))p_e(r) - q_e(r)G(r,r_0))dS
\end{aligned} \quad (51)$$

when p_e is continuous on $\partial\Omega$, $q_e(r)G(r,r')$ is summable and its integral is continuous.

4.4) Integral representation of the field above the plane and in the cavity

4.4.1) Integral representation of the field above the plane

From the definitions of G_a and $p = p_s + p_{inc}$, we can use the second Green's theorem for Ω tending to the infinite half-space Ω_1 above the plane. Indeed, considering the condition (b), and the impedance boundary condition (3), satisfied by p and G_a on the plane $z=0$, the surface integral at infinity and on $S_0\backslash S_1$ vanishes, so that we obtain,

$$(1_{\Omega_1}(r') + 1_{\Omega_1}(r'_{im}))p(r') - p_i\,(r') = \frac{-\,k}{4\pi}\int_{S_1} G_a(r,r')(\partial_z p(r) - ikgp(r))dS \quad (52)$$

for $z \geq 0$, where $(1_{\Omega_1}(r') + 1_{\Omega_1}(r'_{im})) = 1$, and $p_i = \frac{-k}{4\pi}\int_{\Omega_1} W(r)\,G_a(r,r')\,dV$ is the field in presence of the plane without cavity.

4.4.2) Integral representation of the field in the cavity

From the definitions of G_b and $p = p_s + p_{inc}$, we can use the second Green's theorem in the domain Ω_2 of the cavity, which gives us,

$$(1_{\Omega_2}(r') + 1_{\Omega_2}(r'_{im}))p(r') + \frac{k}{4\pi}\int_{\Omega_2} W(r)\, G_b(r, r')\, dV$$
$$= \frac{k}{4\pi}\int_{\partial\Omega_2} \widehat{n}.(\text{grad}(G_b)p - \text{grad}(p)G_b)dS \quad (53)$$

where $1_{\Omega_2}(r') = \int_{\Omega_2}\delta(r - r')dr = \frac{1}{4\pi}\int_{\partial\Omega_2}\frac{(r'-r)}{|r'-r|^3}\widehat{n}dS$, and $\widehat{n}$ is the unit normal to S_2 directed inside Ω_2. Considering that the source W is above the plane, and that G_b (resp. p) satisfies the impedance boundary condition (3) (resp. (8)), the equation (53) becomes

$$(1_{\Omega_2}(r') + 1_{\Omega_2}(r'_{im}))p(r') = \frac{k}{4\pi}\int_{S_1} G_b(r, r')(\partial_z(p(r)) - ikgp(r))dS$$
$$+ \frac{k}{4\pi}\int_{S_2} p(r)(\partial_n G_b(r, r') - ikg_c G_b(r, r'))dS \quad (54)$$

where, we notice that,

$$(1_{\Omega_2}(r') + 1_{\Omega_2}(r'_{im})) = 1 \text{ in } \overline{\Omega}_2\backslash\overline{S}_2$$
$$1_{\Omega_2}(r'_{im}) = 0 \text{ when } r' \in \overline{\Omega}_2\backslash\overline{S}_1 \quad (55)$$

Remark 9 :
even if $\partial_n(G_b(r, r')|_{r\in S_2}$ diverges when $r' \notin \partial\Omega_2 \to r$, it is continuous when r' belongs to smooth parts of S_2.

5) The integral equations on the aperture S_1 and on the surface of the cavity S_2

On the aperture S_1, we can substract the equation (52) from (54), and obtain

$$(1_{\Omega_2}(r') + 1_{\Omega_2}(r'_{im}) - 1)p(r') + p_i\,(r')|_{r'\in S_1} =$$
$$= \frac{k}{4\pi}\int_{S_1} (G_a(r, r') + G_b(r, r'))(\partial_z(p(r)) - ikgp(r))dS$$
$$+ \frac{k}{4\pi}\int_{S_2} p(r)(\partial_n(G_b(r, r')) - ikg_c G_b(r, r'))dS, \quad (56)$$

where we notice that $(1_{\Omega_2}(r') + 1_{\Omega_2}(r'_{im})) = 1$ on S_1, except possibly on $S_1 \cap S_2$, while, on the surface $\overline{S}_2$ of the cavity, we can write, from (54),

$$(1_{\Omega_2}(r') + 1_{\Omega_2}(r'_{im}))p(r')|_{r' \in \overline{S}_2} = \frac{k}{4\pi}\int_{S_1} G_b(r,r')(\partial_z(p(r)) - ikgp(r))dS$$
$$+ \frac{k}{4\pi}\int_{S_2} p(r)(\partial_n(G_b(r,r')) - ikg_c G_b(r,r'))dS, \tag{57}$$

where $1_{\Omega_2}(r') = \int_{\Omega_2}\delta(r - r')dr = \frac{1}{4\pi}\int_{\partial\Omega_2}\widehat{n}.\text{grad}(\frac{1}{|r'-r|})dS$ ($= \frac{1}{2}$ on smooth parts), and the surface integrals are taken in the sense of principal value of Cauchy.
The integral equations (56)-(57) represent a system for two unknowns,

$$\begin{aligned} &q_1(r) = (\partial_z(p(r)) - ikgp(r))|_{r \in S_1}, \\ &p_2(r) = p(r)|_{r \in S_2}, \end{aligned} \tag{58}$$

respectively on the aperture and on the surface of cavity, whose solution permits to express the field everywhere.

6) Uniqueness property of the integral equations

We consider the solutions of our integral equations, $q_1(r)$ on the aperture, and $p_2(r)$ on the surface of the cavity, which satisfy the conditions (a), so that $q_1 = O(r^{\alpha})$, $-1 < \alpha \leq 0$, as the distance to edges or corners vanishes, and p_2 is continuous. We then study the uniqueness of q_1 and p_2 when Re$g > 0$ and Re$g_c > 0$, or $g = g_c = 0$, and verify that q_1 and p_2 vanish when $p_i \equiv 0$.
For this, we show that we can define a field $p_e(r')$, derived from q_1, p_2 and p_i, which verifies $p_2(r') = p_e(r')$ on S_2 and $q_1(r') = \partial_z p_e(r') - ikgp_e(r')$ on S_1, and satisfies the boundary value problem with the conditions of uniqueness given in section 2.

6.1) A field $\boldsymbol{p_e(r')}$ derived from $\boldsymbol{p_2}$ and $\boldsymbol{q_1}$

We consider the field p_e derived, from q_1 and p_2, following

$$p_e(r') = \frac{-k}{4\pi}\int_{S_1} G_a(r,r')q_1(r)dS + p_i(r'), \tag{59}$$

in the domain Ω_1 above the plane, and,

$$p_e(r') = (1 - (1_{\Omega_2}(r') + 1_{\Omega_2}(r'_{im})))\, p_2(r') + \frac{k}{4\pi}\int_{S_1} G_b(r,r')q_1(r)dS +$$
$$+ \frac{k}{4\pi}\int_{S_2} p_2(r)(\partial_n G_b(r,r') - ikg_c G_b(r,r'))dS, \tag{60}$$

in the domain Ω_2 of the cavity, where the surface integrals are taken in the sense of principal value of Cauchy.

The expression (60) verifies, like G_b, the Helmholtz equation in Ω_2, while (59) satisfies, like G_a, the Helmholtz equation in Ω_1 with the radiation conditions at infinity given in (b), and the impedance conditions on $S_0 \backslash S_1$. Moreover, from the equation of continuity (51), the function $p_e(r')$ is continuous up to S_2.

It then remains to analyze the continuity through the aperture S_1 of the cavity, the impedance boundary condition on S_2, and the expressions of q_1 and p_2 with p_e. Therefore, we show that we have,

- $p_e(r') = p_2(r')$ on the surface of the cavity S_2;
- the continuity of $p_e(r')$ through the aperture S_1;
- the continuity of $\partial_z p_e(r') - ikgp_e(r')$ through S_1;
- $\partial_z p_e(r') - ikgp_e(r') = q_1(r')$ on S_1;
- $\partial_n p_e = ikg_c p_2$ on S_2,

in the case $p_i \equiv 0$, considered for the uniqueness.

6.2) $p_e(r') = p_2(r')$ on S_2

Substracting the integral equation (57) from (60) for $r' \in S_2$, we obtain

$$p_e(r') + (1_{\Omega_2}(r') + 1_{\Omega_2}(r'_{im}) - 1)\, p_2(r') = (1_{\Omega_2}(r') + 1_{\Omega_2}(r'_{im}))p_2(r'), \tag{61}$$

on S_2, and thus,

$$p_e(r')|_{r' \in S_2} = p_2(r'). \tag{62}$$

6.3) Continuity of $p_e(r')$ through S_1

The integrals in the expressions (59) and (60) of $p_e(r')$ remain convergent when the point of observation approaches the aperture respectively above and below S_1. Morever, $(1_{\Omega_2}(r') + 1_{\Omega_2}(r'_{im})) = 1$ in $\overline{\Omega}_2 \backslash S_2$, and, from the integral equation (56) with $p_i \equiv 0$, the expressions (59) and (60) tend to the same limit, which proves the continuity of $p_e(r')$ through the aperture S_1.

6.4) Continuity of $\partial_{z'} p_e(r') - ikgp_e(r')$, equal to $q_1(r')$ on S_1

Using (40) in the expressions (59) and (60) of $p_e(r')$, we can write

$$\begin{aligned}
\partial_{z'} p_e(r') - ikgp_e(r')|_{z'=h>0} \;&=\; \frac{-\,k}{4\pi}(\frac{\partial.}{\partial z'})\int_{S_1} 2G^0(r,r')q_1(r)dS|_{z'=h},\\
\partial_{z'} p_e(r') - ikgp_e(r')|_{z'=-h<0} \;&=\; \frac{k}{4\pi}(\frac{\partial.}{\partial z'})\int_{S_1} 2G^0(r,r')q_1(r)dS|_{z'=-h} + \\
+\,\frac{k}{4\pi}(\frac{\partial.}{\partial z'} - ikg)\int_{S_2} & p_2(r)(\partial_n(G_b(r,r')) - ikg_c G_b(r,r'))dS|_{z'=-h} \qquad (63)
\end{aligned}$$

We then apply that,

$$\begin{aligned}
&(\frac{\partial.}{\partial z'} - ikg)G_b(r,r') \to 0 \text{ when } z' \to 0,\, z \neq 0\\
&(\frac{\partial.}{\partial z'})G^0(r(x,y,0),r')|_{z'=h} \;=\; -\,(\frac{\partial.}{\partial z'})G^0(r(x,y,0),r')|_{z'=-h} \qquad (64)
\end{aligned}$$

This implies that the contribution of the integral term along S_2 vanishes when $h \to 0$, and that we have the continuity of $\partial_z p_e(r') - ikgp_e(r')$ through the aperture S_1. Indeed, we then have

$$\pm\,(\partial_{z'} p_e(r') - ikgp_e(r'))|_{z'=h>0} \;\to\; \frac{-\,k}{4\pi}(\frac{\partial.}{\partial z'})\int_{S_1} 2G^0(r,r')q_1(r)dS|_{z'=\pm h}, \qquad (65)$$

when $h \to 0$, while, by application of the discontinuity property of the normal derivative of the single-layer potential [14], substracting the relations in (65) for plus and minus signs, we can write

$$\partial_z(p_e(r')) - ikgp_e(r') = q_1(r') \text{ on } S_1. \qquad (66)$$

6.5) $\partial_n p_e(r') = ikg_c p_e$ on S_2

The field $p_e(r')$, defined by (60), satisfies the Helmholtz equation in Ω_2, and we can write in this domain, from the second Green's theorem,

$$\begin{aligned}
(1_{\Omega_2}(r') + 1_{\Omega_2}(r'_{im}))p_e(r') \;&=\; \frac{k}{4\pi}\int_{S_1} G_b(r,r')(\partial_z(p_e(r)) - ikgp_e(r))dS + \\
+\,\frac{k}{4\pi}\int_{S_2} & (p_e(r)\partial_n G_b(r,r') - G_b(r,r')\partial_{n_2} p_e(r))dS. \qquad (67)
\end{aligned}$$

We have proved that $\partial_z(p_e(r)) - ikgp_e(r) = q_1(r)$ on S_1 and $p_e(r) = p_2(r)$ on S_2, and substracting (67) from (60), we obtain

$$\frac{k}{4\pi}\int_{S_2} G_b(r,r')\mu(r)dS = 0, \tag{68}$$

for $r' \in \Omega_2$, with $\mu(r) \equiv \partial_n p_e(r) - ikg_c p_e(r)$.

The surface S_2, bounded by the curve C_1, is open, and, considering the domain of analyticity of $G_b(r,r')$, we can use the analytic continuation principle through S_1. So, the potential

$$\mathcal{P}(r') = \frac{k}{4\pi}\int_{S_2} G_b(r,r')\mu(r)dS, \tag{69}$$

vanishes in the domain $\Omega \equiv \Omega_2 \cup \Omega_2^i$, where Ω_2^i (resp. S_2^i) is the symmetric of Ω_2 (resp. S_2) relative to the plane $z = 0$. From the properties of G_b, $\mathcal{P}$ is also regular in $R^3 \backslash (\Omega \cup \Omega_c)$, where Ω_c is the upper part of the cylinder along z-axis bounded by S_2^i.

It is then possible to prove that $\mu \equiv 0$. For this, two distinct proofs are detailed in appendix A, successively for $g = 0$ or $g \to \infty$, and, for $g \neq 0,\ |g| < \infty$.

7) Some simplifications of the integral equations for a shallow cavity

The integrals with $\partial_n(\frac{e^{-ik|r-r'_{im}|}}{k|r-r'_{im}|})$ terms, in the equations (56)-(57), becomes difficult to calculate when $|r - r'_{im}| \to 0$ and the depth vanishes. Therefore, we develop our integral equations in a new form, and analytical expressions are derived.

7.1) A new form of the integral terms for shallow cavity

For a shallow cavity, we let

$$\begin{aligned} &G_{bs}(r,r') = G_b(r,r') - G_{st}(r,r') \\ &G_{st}(r,r') = \frac{1}{k|r-r'|} + \frac{1}{k|r-r'_{im}|} \\ &r'_2(r_1) = r_1 + \alpha(r_1)\widehat{z} \in S_2, r_1 \in S_1, \alpha(r_1) \in R \end{aligned} \tag{70}$$

where $r'_2(r_1)$ is the projection of $r_1 \in S_1$ on S_2 along z. We then consider the domain Ω

defined so that $1_\Omega(r') = 1_{\Omega_2}(r') + 1_{\Omega_2}(r'_{im})$, and notice that

$$\begin{aligned}
&1_\Omega(r')p_2(r') \\
&= \frac{p_2(r')}{4\pi}\int_{\partial\Omega} \widehat{n}\,\mathrm{grad}(\frac{1}{|r-r'|})dS \\
&= \frac{k}{4\pi}\int_{S_2} \widehat{n}\,\mathrm{grad}(G_{st}(r,r'))p_2(r')dS \qquad (71)
\end{aligned}$$

We can use this equation, and derive a new form of integrals along S_2 in our system of equations. So, we obtain, for $r' \in S_1$,

$$\begin{aligned}
p_i\,(r') - p_2(r'_2(r')) &= \frac{k}{4\pi}\int_{S_1} (G_a(r,r') + G_b(r,r'))q_1(r)dS \\
&+ \frac{k}{4\pi}\int_{S_2} p_2(r)(\partial_n(G_{bs}(r,r')) - ikg_cG_b(r,r'))dS \\
&+ \frac{k}{4\pi}\int_{S_2} (p_2(r) - p_2(r'_2(r')))\partial_n(G_{st}(r,r'))dS \qquad (72)
\end{aligned}$$

while, for $r' \in \overline{S}_2$,

$$\begin{aligned}
&-k\int_{S_2} (p_2(r) - p_2(r'))\partial_n(G_{st}(r,r'))dS = k\int_{S_1} G_b(r,r')q_1(r)dS \\
&+ k\int_{S_2} p_2(r)(\partial_{n_2}(G_{bs}(r,r')) - ikg_cG_b(r,r'))dS \qquad (73)
\end{aligned}$$

Comparing with previous integral equations system, we notice that the term $\partial_n(\frac{1}{k|r-r'_{im}|})$ is multiplied by terms that vanish as $|r-r'_{im}| \to 0$, so that the difficulty of calculus for a small cavity depth has disappeared. Let us remark that this modification can be applied whenever a part of S_2 is close to S_1.

7.2) The limit case of an impedance patch

In the limit case where $S_2 \equiv S_1$, the integral with $\partial_n G_{st}(r,r')$ vanishes, and $\partial_{n_2}(G_{bs}(r,r')) = ikg\,G_b(r,r')$, so that we obtain, for $r' \in S_1$,

$$\begin{aligned}
&k\int_{S_1} G_b(r,r')(q_1(r) + ik(g-g_c)p_2(r))dS|_{z'=0^-} = 0 \\
&p_i\,(r') - p_2(r') = \frac{k}{4\pi}\int_{S_1} G_a(r,r')q_1(r)dS|_{z'=0^+} \qquad (74)
\end{aligned}$$

where $q_1(r)$ and $p_2(r)$ are $O(1)$. The first equation implies $q_1(r) = ik(g_c - g)p_2(r)$ (see appendix C), which leads us to recover the well-known integral equation [18] for an impedance patch,

$$p_2(r') - p_i\,(r') = \frac{k}{4\pi}\int_{S_1} G_a(r,r') ik(g-g_c) p_2(r) dS \tag{75}$$

Remark 10 :
Let us notice that

$$k\int_{S_1} G_a(r,r')\mu(r) dS|_{z'=0^+} = 0 \tag{76}$$

when $\mu(r) = q_1(r) + ik(g-g_c)p_2(r) = O(1)$, implies $\mu(r) \equiv 0$ (see appendix C).

7.3) On some Approximations for a small cavity, and validation.

7.3.1) Approximate expressions for a small cavity

For small dimensions of the cavity, we assume that

$$\begin{aligned}
&p_2(r) - p_c = o(1),\ p_c = \frac{\int_{S_2} p_2 dS}{\int_{S_2} dS},\\
&\int_{S_2} (p_2(r) - p_c)\int_{S_2} \partial_n (G_{bs(,st)}(r,r')) dS dS' = o(1),\\
&\int_{S_1} (q_1(r) - q_c)\int_{S_{2(,1)}} G_{b(,a)}(r,r') dS' dS = o(1),\ q_c = \frac{\int_{S_1} q_1(r) dS}{\int_{S_1} dS}.
\end{aligned} \tag{77}$$

In this case, we can calculate q_c and p_c, by integration of our integral equations over S_2 and S_1, and define an approximate expression of the field radiated above the plane. For this, we use that,

$$\begin{aligned}
&(\Delta + k^2) G_{bs}(r,r') = -k^2 G_{st}(r,r'),\ r' \in S_2,\ r \in S_1,\\
&\int_{\partial\Omega_2} \partial_{n_2} G_{bs}(r,r') dS = k^2 \int_{\Omega_2} G_{st}(r,r') dV,\ r' \in S_2,\\
&\partial_z G_{bs}(r,r') = ikg G_b(r,r'),\ r' \in S_2,\ r \in S_1,
\end{aligned} \tag{78}$$

and,

$$\int_{S_2} \partial_n G_{bs}(r,r') dS = \int_{S_1} ikg\, G_b(r,r') dS + k^2 \int_{\Omega_2} G_{st}(r,r') dV,\ r' \in S_2. \tag{79}$$

So, summing the integral equation (73) over S_2, we obtain

$$q_c\int_{S_1}\int_{S_2}G_b(r,r')dS'dS = ikp_c(\int_{S_2}g_c-\int_{S_1}g)\int_{S_2}G_b(r,r')dS'dS$$
$$-k^2p_c\int_{\Omega_2}\int_{S_2}G_{st}(r,r')dV+o(1) \tag{80}$$

and deduce that

$$q_c = ikp_c[(r_c(g_c)-g)+ikl_c]+o(1) \tag{81}$$

where

$$r_c(g_c)=\frac{\int_{S_2}g_c\int_{S_2}G_b(r,r')dS'dS}{\int_{S_1}\int_{S_2}G_b(r,r')dS'dS}\sim\frac{\int_{S_2}g_c dS}{\int_{S_1}dS},$$
$$l_c=\frac{\int_{\Omega_2}\int_{S_2}G_{st}(r,r')dV}{\int_{S_1}\int_{S_2}G_b(r,r')dS'dS}\sim\frac{\int_{\Omega_2}dV}{\int_{S_1}dS}. \tag{82}$$

We then use the integral equation (72) for $r'\in S_1$, and sum it over S_1. This gives us,

$$\int_{S_1}p_i\,(r)dS-p_c\int_{S_1}dS=\frac{kq_c}{4\pi}\int_{S_1}\int_{S_1}G_a(r,r')dS'dS\,(1+o(k)) \tag{83}$$

so that we can write, for the approximate expressions of p_c and q_c,

$$p_c=\frac{\int_{S_1}p_i\,(r)dS/\int_{S_1}dS}{1+\frac{ik}{4\pi}[(r_c(g_c)-g)+ikl_c]\frac{k\int_{S_1}\int_{S_1}G_a(r,r')dS'dS}{\int_{S_1}dS}},$$
$$q_c=ikp_c[(r_c(g_c)-g)+ikl_c]\,, \tag{84}$$

The expression of q_c can be used, for the field radiated by the cavity above the plane,

$$p(r')-p_i\,(r')=\frac{-\,k}{4\pi}\,q_c\int_{S_1}G_a(r,r')dS\;+o(1). \tag{85}$$

when $r'\notin S_1$ and $k\int_{S_1}(q_1(r)-q_c)G_a(r,r')dS=o(1)$, in particular for the far field.

Remark 11 :
In the case of a cavity Ω_2 filled with a homogenous material, we can consider G_b with k_2 instead of k, and $g_2=kg/k_2$ in place of g, and write

$$q_c=ik_2p_c[(r_c(g_c)-g_2)+ik_2l_c]+o(1) \tag{86}$$

Remark 12 :
To our knowledge, our approximate expressions are original, but a similar low frequency analysis could also be done with the integral equations given in [1].

7.3.2) Validation in the case of a small cylindrical cavity with impedance wall

For the validation, we choose to compare the impedance on the aperture, expressed, from (81)-(82), by

$$\eta_a = \frac{\int_{S_1} \frac{\partial p}{\partial z} dS / \int_{S_1} dS}{ikp_c} = \frac{q_c}{ikp_c} + g = r_c(g_c) + ikl_c \sim \frac{\int_{S_2} g_c dS}{\int_{S_1} dS} + ik\frac{\int_{\Omega_2} dV}{\int_{S_1} dS}, \quad (87)$$

with the impedance given for a cavity with well-tabulated results.
For this, we consider the particular case of a cylindrical cavity of radius a and depth d with an imperfectly reflective surface, characterized by impedances g_{cw} on the wall and g_{ce} on the bottom, with $ka = o(1)$ and $d/a = O(1)$. So, we have, from (87),

$$\eta_a \sim \frac{g_{ce}\pi a^2 + g_{cw} 2\pi a d}{\pi a^2} + ik\frac{\pi a^2 d}{\pi a^2} = g_{ce} + \frac{2 g_{cw} d}{a} + ikd \quad (88)$$

while, from the modal expansion of the field [19],

$$\eta_m = \frac{\frac{\partial p}{\partial z}}{ikp}|_{S_1} \simeq \frac{\alpha_1}{k}\frac{(1 + \frac{g_{ce} - \frac{\alpha_1}{k}}{g_{ce} + \frac{\alpha_1}{k}} e^{-2i\alpha_1 d})}{(1 - \frac{g_{ce} - \frac{\alpha_1}{k}}{g_{ce} + \frac{\alpha_1}{k}} e^{-2i\alpha_1 d})} \sim g_{ce} + i\alpha_1^2 \frac{d}{k} \simeq g_{ce} + \frac{2 g_{cw} d}{a} + ikd,$$
$$-ikag_{cw} J_0(\xi_1) + \xi_1 J_1(\xi_1) = 0,\ \alpha_1^2 = k^2 - (\frac{\xi_1}{a})^2 \simeq k^2 - \frac{2ikg_{cw}}{a}, \quad (89)$$

As expected for a small cavity, η_m perfectly recovers η_a, and the expression (87) is validated.

Remark 13 :
For a perfectly rigid small cavity, we have $g_c = 0$ and thus $\eta_a = ikl_c$, and we recover the result given in [20, equ. (3)-(6)].

8) Conclusion

We have developed novel integral equations which permit to simplify the calculus of the field scattered by a cavity in an impedance plane. For this, a new Green's function is used for the expression of the field in the cavity which leads to reduce the number of unknowns. Moreover, a particular attention is paid to the uniqueness of the solution. In the case of a small cavity, our equations are detailed and developed in a new form. In this case, analytical results are derived and our expression for approximate aperture impedance is validated.

Appendix A : $\int_{S_2(open)} G_b(r,r')\mu(r)dS = 0$ *in* $\Omega \equiv \Omega_2 \cup \Omega_2^i$ *implies* $\mu(r) \equiv 0$

This appendix concerns the study of the solution $\mu(r)$ of

$$\mathcal{P}(r') = 0 \text{ where } \mathcal{P}(r') = \frac{k}{4\pi}\int_{S_2} G_b(r,r')\mu(r)dS, \tag{90}$$

when S_2 is open, and the proof that $\mu(r)$ (in some function class) vanishes.

A.1) $\mu \equiv 0$ *in the cases* $g = 0$ *(Neumann) or* $g \to \infty$ *(Dirichlet)*

In the respective cases $g = 0$ (Neumann boundary condition) and $g \to \infty$ (Dirichlet boundary condition), we have

$$\begin{aligned} G_b(r,r')|_{g=0} &= [G^0(r-r') + G^0(r-r'_{im})] \\ G_b(r,r')|_{g\to\infty} &= [G^0(r-r') - G^0(r-r'_{im})] \end{aligned} \tag{91}$$

and thus,

$$\begin{aligned} \mathcal{P}(r')|_{g=0} &= \frac{k}{4\pi}\int_{\partial\Omega} G^0(r-r')\Xi_0(r)dS \\ \mathcal{P}(r')|_{g\to\infty} &= \frac{k}{4\pi}\int_{\partial\Omega} G^0(r-r')\Xi_\infty(r)dS \end{aligned} \tag{92}$$

where $\Xi_0(r_{im}) = \Xi_0(r) = \mu(r)$ and $\Xi_\infty(r_{im}) = -\Xi_\infty(r) = -\mu(r)$. We assume that μ is a function, piecewise continuous (except possibly for singularities of μ at the edge of $\partial\Omega$), so that $\mathcal{P}$ is continuous on $\partial\Omega$. We then use a proof similar to the ones given by Colton and Kress in [14] to prove that $\mu(r) \equiv 0$.

The potential $\mathcal{P}$ vanishes in Ω, and thus, by continuity, on $\partial\Omega$. Moreover, $\mathcal{P}$ satisfies the Helmholtz equation and the Sommerfeld radiation condition at infinity in R^3. Hence by Rellich's uniqueness theorem generalized by Levine for non smooth domain [10], $\mathcal{P}(r')$ also vanishes outside Ω. We can then conclude, from the discontinuity property of the normal derivative of the single layer potential [14],

$$\frac{\partial\mathcal{P}(r')}{\partial n}|_+ - \frac{\partial\mathcal{P}(r')}{\partial n}|_- = -\Xi(r')\,, \tag{93}$$

at any non singular points of S_2, that $\Xi \equiv 0$ and thus $\mu \equiv 0$.

A.2) A proof that $\mu \equiv 0$ for $g \neq 0$, $|g| < \infty$

When $g \neq 1$, we notice that $G_b(r, r')|_{g=v} = G_a(r_{im}, r'_{im})|_{g=-v}$, and the problem is then equivalent to a boundary value problem in the upper half-space, concerning a perturbation in relief on a plane of impedance $-g$, with a field vanishing inside and on the surface of the perturbation. For Re$(-g) > 0$, the uniqueness theorem of Levine [10, sect.7] applies, and we can deduce that $\mu \equiv 0$.
For Re$g > 0$, this demonstration is no more valid, and we develop here a more general proof which uses that S_2 is an open surface.

A.2.1) Definition of the function $\mathcal{P}_1$

For this, we begin to define new functions $\mathcal{P}_0$ and $\mathcal{P}_1$, and we write,

$$\begin{aligned}
\mathcal{P}(r') &= (\mathcal{P}_0 + 2ig\mathcal{P}_1) \\
\mathcal{P}_0(r') &= \frac{k}{4\pi}\int_{S_2}(G^0(r-r') + G^0(r-r'_{im}))\mu(r)dS \\
\mathcal{P}_1(r') &= \frac{k}{4\pi}\int_{S_2}\mathcal{V}_b(r-r'_{im})\mu(r)dS
\end{aligned} \tag{94}$$

where, from (21), the function $\mathcal{V}_b(r) = -e^{ikgz}\mathcal{J}_{-g}(\rho, z)$ satisfies

$$\begin{aligned}
\frac{\partial \mathcal{V}_b(r-r'_{im})}{\partial z} &= \frac{e^{-ik|r-r'_{im}|}}{|r-r'_{im}|} + ikg\mathcal{V}_b(r-r'_{im}) \\
&= kG^0(r-r'_{im}) + ikg\mathcal{V}_b(r-r'_{im}).
\end{aligned} \tag{95}$$

We notice that $\mathcal{V}_b(r)$ is regular for $z < 0$, and has a weak singularity, like lnρ, at $\rho = 0$ for $z \geq 0$. Thus, the potential $\mathcal{P}_1(r')$ is an analytic function in $R^3 \backslash \Omega_c$, i.e. everywhere except at points above the surface S_2^i, which is the image of S_2.

A.2.2) A problem for $\mathcal{P}_1$ equivalent to the problem $\mathcal{P} \equiv 0$

Since $\mathcal{P}$ vanishes in Ω, and $\mathcal{P}_0(r'_{im}) = \mathcal{P}_0(r')$, we can write that $\mathcal{P}_1(r'_{im}) = \mathcal{P}_1(r')$ in this domain. So, we have

$$
\begin{aligned}
&\mathcal{P}_1(r') = \frac{k}{4\pi}\int_{S_2} \mathcal{V}_b(r - r'_{im})\mu(r)dS \\
&\mathcal{P}_1(r'_{im}) = \mathcal{P}_1(r'),\ r' \in \Omega \equiv \Omega_2 \cup \Omega_2^i \\
&(\Delta + k^2)\mathcal{P}_1 = 0 \text{ in } R^3\backslash\Omega_c,
\end{aligned} \tag{96}
$$

where S_2 is an open surface. This implies reciprocally that $\mathcal{P} \equiv 0$. Indeed, we have

$$
\partial_z \mathcal{P}_1(r') - ikg\mathcal{P}_1(r') = \frac{k}{4\pi}\int_{S_2} kG^0(r - r'_{im})\mu(r)dS. \tag{97}
$$

Therefore, using the parity of $\mathcal{P}_1$ in $\Omega \equiv \Omega_2 \cup \Omega_2^i$, we have $\partial_z(\mathcal{P}_1(r') + \mathcal{P}_1(r'_{im})) = 0$, and we can write, as $r' \in \Omega$,

$$
\frac{1}{k}(\partial_z \mathcal{P}_1(r') + \partial_z \mathcal{P}_1(r'_{im})) = \frac{k}{4\pi}\int_{S_2} G_b(r, r')\mu(r)dS = 0 \tag{98}
$$

where we have used that $\mathcal{P}_1(r'_{im}) = \mathcal{P}_1(r')$ and $\mathcal{P}_1(r'_{im}) + \mathcal{P}_1(r') = 2\mathcal{P}_1(r')$. This implies, by definition of $\mathcal{P}$, that $\mathcal{P} = 0$ in Ω, which shows the equivalence.

A.2.3) a proof that $\mu \equiv 0$, by the analysis of the singularities at the ends of S_2

The singularities of the field at the ends of S_2, i.e. in vicinity of the curve C_1, depends on the geometry. For this, we denote $\widehat{n}_0$, the unit vector, normal to C_1 at r_0 and orthogonal to the normal $\widehat{n}$ to S_2, and $\widehat{c}$ the unit vector tangent to C_1, so that $(\widehat{c}, \widehat{n}, \widehat{n}_0)$ is an orthonormal basis (figure 4), and (ρ, φ) the cylindrical coordinates associated to $(\widehat{n}, \widehat{n}_0)$, with $\rho\cos\varphi = \widehat{n}_0.(r - r_0)$, $\rho\sin\varphi = -\widehat{n}.(r - r_0)$. We also denote $\widehat{y}$ the unit vector perpendicular to $\widehat{z}$ and to $\widehat{c}$ so that $(\widehat{c}, \widehat{y}, \widehat{z})$ is an orthonormal basis.

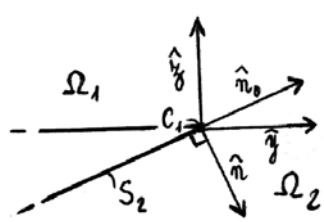

figure 4 : definitions of unit vectors on the curve C_1 limiting the aperture

Let us consider S'_2, a part of S_2 bounded by an analytic arc C'_1 of C_1, and consider to simplify, without losing generality, that the function $\mu(r)$ satisfies

$$\begin{aligned}
\mu(r) &= \mu_f(r) + \mu_a(r) \\
\mu_f(r) &= \sum_{p\geq 1} a_p J_{\alpha_p}(k\rho), \\
\mu_a(r) &= \sum_{m\geq 0} b_m \rho^m
\end{aligned} \tag{99}$$

on S_2' where the α_p are not entire numbers, $\alpha_p < \alpha_{p+1}$, $\alpha_1 > -1$, $a_1 \neq 0$ except if $\mu_f \equiv 0$, and $J_\nu(z) = (\frac{z}{2})^\nu \sum_{k\geq 0} \frac{(-z/4)^k}{k!\Gamma(\nu+k+1)}$ is the bessel function of order ν [17]. The terms with powers of lnρ could be considered in the method but are omitted for simplification. Thereafter, we prove that the conditions on $\mathcal{P}_1$ implies the vanishing of μ_f and μ_a on S_2', and that, by the continuation principle through a hole and the nullity of $\mathcal{P}$ in Ω_2, $\mu \equiv 0$ on S_2. To simplify the analysis, we will only detail the demonstration in the case where $(\widehat{y}.\widehat{n}_0) = \cos\Phi' \neq 0$.

A.2.3.1) $\boldsymbol{\mu_a = 0}$ *on* $\boldsymbol{S_2'}$ *when* $\boldsymbol{\widehat{y}.\widehat{n}_0 = \cos\Phi' \neq 0}$ *on* $\boldsymbol{C_1'}$

Let us consider the analytic part of μ in vicinity of C_1' and the singularities of $\mathcal{P}_1$ induced by it. Since we have

$$(\partial_z(\partial_y \mathcal{P}_1(r')) - ikg(\partial_y \mathcal{P}_1(r'))) = \partial_y \frac{k}{4\pi} \int_{S_2} kG^0(r - r'_{im})\mu(r)dS \tag{100}$$

a singularity appears (see [15] or appendix B), following

$$\partial_y \frac{k}{4\pi} \int_{S_2} kG^0(r - r'_{im})\mu(r)dS = -\frac{k\cos\Phi'(r_0)}{2\pi}\mu(r_0)\ln|r' - r_0| + O(1) \tag{101}$$

as r' tends normally to $q_0 \in C_1'$. This implies, from $\partial_y \mathcal{P}_1(r') = O(1)$, that

$$\partial_z(\partial_y \mathcal{P}_1(r')) = -\frac{k\cos\Phi'(r_0)}{2\pi}\mu(r_0)\ln|r' - r_0| + O(1) \tag{102}$$

Considering the parity of $\mathcal{P}_1(r')$, this equality is impossible except if $\mu(q_0) = 0$. In the same manner, the case of higher order terms of μ_a, $b_1\rho^1$, $b_2\rho^2$, ... can be considered successively with higher order y-derivatives of $\mathcal{P}_1(r')$, so that $b_m = 0$, $m \geq 0$.

A.2.3.2) $\boldsymbol{\mu_f = 0}$ *on* $\boldsymbol{S_2'}$ *for arbitrary* $\boldsymbol{\cos\Phi'}$ *on* $\boldsymbol{C_1'}$

Let us consider the fractional part μ_f of μ and the single layer potential induced by it, in the expression (97) of $\partial_z \mathcal{P}_1(r') - ikg\mathcal{P}_1(r')$, when S_2' is assumed to simplify with null curvature. From the appendix B, the potential has a fractional part of order $1+\alpha_1$ ($\sim\rho^{1+\alpha_1}$ as $\rho\to 0$), which is thus the fractional order of $\partial_z\mathcal{P}_1(r')$. We then deduce that $\mathcal{P}_1$ has a fractional order $2+\alpha_1$. Since $J_{\alpha_1}(k\rho)$ radiates, for its fractional part, like $\frac{4\pi}{k\text{sin}(\nu\pi)}J_{1+\alpha_1}(k\rho)\text{cos}((1+\alpha_1)\varphi')+O(J_{3+\alpha_1}(k\rho))$, which does not contains $\rho^{2+\alpha_1}$ terms in its expansion, the order $2+\alpha_1$ of $\mathcal{P}_1$ comes from the next term $a_2J_{\alpha_2}(k\rho)$. This implies $\alpha_2=\alpha_1+1$, and $a_2\neq 0$ if $a_1\neq 0$. Consequently, when $a_1\neq 0$, we can write,

$$
\begin{aligned}
&\frac{k}{4\pi}\int_{S_2}kG^0(r-r'_{im})\mu(r)dS=\\
&=\frac{1}{\text{sin}(\alpha_1\pi)}(a_1J_{1+\alpha_1}(k\rho')\text{cos}((1+\alpha_1)\varphi')-a_2J_{2+\alpha_1}(k\rho')\text{cos}((2+\alpha_1)\varphi'))\\
&+O(J_{3+\alpha_1}(k\rho'))+O(J_{1+\alpha_3}(k\rho'))+\text{entire function of }\rho'
\end{aligned}
\tag{103}
$$

as $\rho'\to 0$, $\rho'\text{cos}\varphi'=\widehat{n}_0.(r'-r_0)$, $\rho'\text{sin}\varphi'=-\widehat{n}.(r'-r_0)$, $\alpha_3>\alpha_2=\alpha_1+1$. Then, from (97), we have

$$
\begin{aligned}
&(\partial_z\mathcal{P}_1(r'))-ikg\mathcal{P}_1(r'))\\
&=\frac{1}{\text{sin}(\alpha_1\pi)}(a_1J_{1+\alpha_1}(k\rho')\text{cos}((1+\alpha_1)\varphi')-a_2J_{2+\alpha_1}(k\rho')\text{cos}((2+\alpha_1)\varphi'))\\
&+O(J_{3+\alpha_1}(k\rho'))+O(J_{1+\alpha_3}(k\rho'))+\text{entire function of }\rho'
\end{aligned}
\tag{104}
$$

From the parity of $\mathcal{P}_1$ and $\partial_z\mathcal{P}_1$, we then derive

$$
\begin{aligned}
&a_1\text{cos}((1+\alpha_1)\Phi'+\varphi)=-a_1\text{cos}((1+\alpha_1)\Phi'-\varphi),\\
&a_2\text{cos}((2+\alpha_1)\Phi'+\varphi)=a_2\text{cos}((2+\alpha_1)\Phi'-\varphi).
\end{aligned}
\tag{105}
$$

Thus, when $a_1\neq 0$, we can write,

$$
\begin{aligned}
&\text{cos}((1+\alpha_1)\Phi')=0\\
&\text{sin}((2+\alpha_1)\Phi')=0
\end{aligned}
\tag{106}
$$

This implies $\text{cos}\Phi'=0$, and α_1 is entire, which is impossible. We then deduce that the first order coefficient a_1 of μ_f is null, which implies, by definition, that $\mu_f\equiv 0$.

A.2.3.3) μ vanishes on S'_2 implies $\mu\equiv 0$

From the previous results, it exists a subdomain S'_2 of S_2 where $\mu=0$, that we can substract of the support of μ, assuming without losing generality, that $|\text{cos}\Phi'|\neq 1$ along C'_1. In this case, we can use the continuation principle through the hole S'_2, and the field $\mathcal{P}(r')$, null in $\overline{\Omega}_2$, also vanishes outside the cavity below the plane $z=0$.

Noticing the regularity of $\mathcal{P}_1(r')$ for $z' < 0$, and thus the continuity of the normal derivative of $\mathcal{P}_1(r')$ through S_2, we can apply the discontinuity property of the normal derivative of single-layer potentials with free space Green's function [14],

$$\frac{\partial \mathcal{P}(r')}{\partial n}|_{+} - \frac{\partial \mathcal{P}(r')}{\partial n}|_{-} = -\mu(r) \tag{107}$$

at any non singular points of S_2, then deduce, from the vanishing of the left side, that $\mu \equiv 0$.

A.2.4) elements of proof for the particular case $\widehat{\boldsymbol{y}}.\widehat{\boldsymbol{n}}_0 = \cos\boldsymbol{\Phi}' = 0$ on C_1

From the previous analysis, the fractional part μ_f vanishes, and we can assume that μ is analytic. In this case, we can choose to study the function,

$$\mathcal{P}'(r') = (\partial_{z'} - ikg)\mathcal{P}(r') = \frac{k}{4\pi}\int_{S_2} G_b'(r, r')\mu(r)dS \tag{108}$$

where, from (39),

$$\begin{aligned} G_b'(r,r') &= \partial_{z'}(G^0(r-r') + G^0(r-r'_{im})) - ikg(G^0(r-r') - G^0(r-r'_{im})) \\ &= (-\partial_z - ikg)(G^0(r-r') - G^0(r-r'_{im})) \end{aligned} \tag{109}$$

The function $\mathcal{P}'$ and its derivatives vanish in Ω, since $\mathcal{P} = 0$ in Ω. Let us show that it is also the case for $\mathcal{P}'$ outside Ω, then for $\mathcal{P}$, and thus for μ.
Since $\mathcal{P}'$ vanishes along the plane $z = 0$, $(\widehat{z}.\text{grad})^{2n}\mathcal{P}' = 0$ along C_1. Using integration by parts and continuity, we have $(\widehat{z}.\text{grad})^{2n+1}\mathcal{P}' = 0$ along C_1. Considering to simplify that the cylinder S_c along z-axis defined with a section C_1 does not have common points with S_2, except on C_1, we then deduce that $\mathcal{P}'$ vanishes on S_c and thus, by uniqueness principle, everywhere. Since $\mathcal{P}$ and all z-derivatives of $\mathcal{P}'$ vanishes on S_2, $\mathcal{P}$ vanishes below S_2, and, by continuation principle, everywhere below the plane $z = 0$.
We can then use the discontnuity property of the normal derivative of single layer potential (107), and deduce that $\mu \equiv 0$.

Appendix B :
behaviour of single layer potentials on open surfaces

Let S be an open analytic, orientable surface in three-dimensional space bounded by a Jordan curve C, and C' an arc belonging to it. Let r' and r be two points, and $\mu(r)$ an analytical function defined for all $r \in S$ except possibly for a singularity on the edge C'.

We study the behaviour of single potentials

$$U_0(r') = \int_S \frac{\mu(r)}{|r-r'|} dS_q,\ U_k(r') = \int_S \frac{\mu(r) e^{-ik|r-r'|}}{|r-r'|} dS_q \tag{110}$$

B.1) Principal part of grad$(U_0(r'))$ when $\mu(r) = O(1)$ on C'

If $\mu(r)$ is finite on C', we can write, from Rolf Leis [15], in vicinity of C'

$$\mathrm{grad}(U_0(r')) = -\int_S \mu(r)\widehat{n}\frac{\partial}{\partial n}\frac{1}{|r-r'|}dS_q + \int_S \frac{\mathrm{grad}_S(\mu(r))}{|r-r'|}dS_q + 2\int_S \frac{\widehat{n}H\mu(r)}{|r-r'|}dS_q$$
$$-\int_C \frac{\widehat{n}_0\mu(r)}{|r-r'|}dc \tag{111}$$

where $\widehat{n}_0$ is a unit vector, normal to C and orthogonal to the normal $\widehat{n}$, grad_S is the surface gradient, H is a function depending on the characteristics of the surface. The line integral becomes logarithmically singular, while the other surface integrals are regular. The singularity, as $r' \notin C \to r_0$, r_0 being the projection of r' on C', can be described by

$$\int_C \frac{\widehat{n}_0\mu(r)}{|r-r'|}dc = -2\widehat{n}_0(r_0)\mu(r_0)\ln|r'-r_0| + O(1) \tag{112}$$

where $\widehat{c}$ is the unit vector, tangent to C' at r_0, and $(\widehat{c}, \widehat{n}, \widehat{n}_0)$ is an orthonormal basis.

B.2) Principal fractional part of $U_k(r')$ when $\mu(r)$ is of fractional order

In the case of $\mu(r)$ of fractional order (with fractional power of $|r-r_0|$ near $r_0 \in C'$), it is possible to analyze the fractional part of the field, letting the curvature of the edge C' tending to 0. In this case, we can write,

$$U_k(r') \sim -i\pi\int_L \mu_f(\rho)H_0^{(2)}(k|\overline{\rho}-\overline{\rho}'|)d\rho$$
$$\sim -i\int_{-i\infty}^{+i\infty}\int_0^{\infty}\mu_f(\rho)e^{-ik\rho\cos\alpha}d\rho e^{-ik\rho'\cos(\alpha-\varphi')}d\alpha \tag{113}$$

when $\rho' \to 0$, ρ' denoting the radial distance to the edge of the point r', with $\rho'\cos(\varphi') = \widehat{n}_0(r'-r_0)$, $\rho'\sin(\varphi') = -\widehat{n}(r'-r_0)$, $r_0 \in C'$, and $\mu_f(\rho) = \mu(r)$.

So, for $\mu_f(\rho) = J_\nu(k\rho) \sim (\frac{\beta}{k})^{-\nu}\lim_{\beta\to 0} J_\nu(\beta\rho)$, $\nu > -1$, $\nu \neq 0, 1, 2, ...$, we obtain $U_k(r')$, from [16, eq. 6.611.1], following

$$
\begin{aligned}
U_k(r') &\sim -\frac{ie^{-i(1+\nu)\pi/2}}{k\,2^\nu}\int_{-i\infty}^{+i\infty}\frac{1}{(\cos\alpha)^{1+\nu}}e^{-ik\rho'\cos(\alpha-\varphi')}d\alpha \\
&\sim -\frac{ie^{-i(1+\nu)\pi/2}}{k\,2^\nu}\int_{0}^{+i\infty}\left(\frac{1}{(\cos(\alpha+\varphi'))^{1+\nu}}+\frac{1}{(\cos(\alpha-\varphi'))^{1+\nu}}\right)e^{-ik\rho'\cos\alpha}d\alpha \\
&\sim -\frac{i\,4e^{-i(1+\nu)\pi/2}}{k}\cos((1+\nu)\varphi')\int_{0}^{+i\infty}e^{i(1+\nu)\alpha}e^{-ik\rho'\cos\alpha}d\alpha \\
&\sim \frac{4\pi}{k\sin(\nu\pi)}\cos((1+\nu)\varphi')(J_{1+\nu}(k\rho') + \text{an entire function of } \rho')
\end{aligned}
\tag{114}
$$

Then, we can rewrite (114), using the discontinuity property of the normal derivative of U_k through S [14], and $2(1+\nu)J_{1+\nu}(k\rho')/k\rho' = J_\nu(k\rho') + J_{2+\nu}(k\rho')$ [17], following

$$
U_k(r') = \frac{4\pi}{k\sin(\nu\pi)}J_{1+\alpha_1}(k\rho')\cos((1+\alpha_1)\varphi') + O(J_{3+\alpha_1}(k\rho')) + U_a(p) \tag{115}
$$

where $U_a(r')$ is an entire function of ρ'.

Remark 14 :
In the case of logarithmic behaviour, we can let $\mu(\rho) = \frac{1}{2}\ln(\frac{\rho}{2}) = \lim_{\nu\to 0^+}\partial_\nu(K_\nu(\rho)/\Gamma(\nu))$, and derive, from [16, eq. 6.611.3],

$$
\begin{aligned}
U_k(p) &\sim -i\int_{-i\infty}^{+i\infty}\partial_\nu\left(\frac{\Gamma(1-\nu)\sin\nu\alpha}{\sin\alpha}\right)\Big|_{\nu=0}e^{-ik\rho'\cos(\alpha-\varphi')}d\alpha \\
&\sim -i\int_{-i\infty}^{+i\infty}\left(\gamma\alpha+\frac{\alpha}{\sin\alpha}\right)e^{-ik\rho'\cos(\alpha-\varphi')}d\alpha,\ \gamma = .577... \\
&\sim -i\gamma\varphi'\int_{-i\infty}^{+i\infty}e^{-ik\rho'\cos\alpha}d\alpha + o(\ln\rho) = 2\gamma\varphi' K_0(ik\rho) + o(\ln\rho)
\end{aligned}
\tag{116}
$$

Remark 15 :
Let $t_0(r_0) = a\widehat{n}_0(r_0) + b\widehat{n}(r_0)$ when $t_0\widehat{n}_0 \neq 0$. Considering higher derivatives of U_0, we can write

$$
\begin{aligned}
(t_0.\text{grad})^n(U_0(r')) = & -\int_S (t_0.\text{grad}_S)^{n-1}(\mu(r))(t_0.\widehat{n})\frac{\partial}{\partial n}(\frac{1}{|r-r'|})dS \\
& + 2\int_S \frac{(t_0.\widehat{n})H(t_0.\text{grad}_S)^{n-1}(\mu(r))}{|r-r'|}dS + \int_S \frac{(t_0.\text{grad}_S)^n(\mu(r))}{|r-r'|}dS \\
& - t_0\int_C \frac{(t_0.\widehat{n}_0)(t_0.\text{grad}_S)^{n-1}(\mu(r))}{|r-r'|}dc + \mathcal{R}_n
\end{aligned} \tag{117}
$$

when $(\widehat{n}_0\text{grad}_S)^j(\mu(r_0)) = 0$ (or $(t_0(r_0)\text{grad}_S)^j(\mu(r_0)) = 0$ when $t_0\widehat{n}_0 \neq 0$) on C', for $j < n-1$, $(\widehat{n}_0\text{grad}_S)^{n-1}(\mu(r_0)) = O(1)$, and, in this case, $\mathcal{R}_n$ is continuous on C'. This result also applies if we replace U_0 by U_k since the behaviour of highest rank is the same for U_k and U_0.

Appendix C :
$\int_{S_1} G_{b(a)}(r,r')\mu(r)dS|_{z=0^-} = 0$ *on* S_1 *implies* $\mu(r) \equiv 0$ *when* $\mu(r) = A_0 + o(1)$

Let us show that

$$
\mathcal{U}(r') = \int_{S_1} G_{b(a)}(r,r')\mu(r)dS|_{z=0^-} = 0 \text{ on } S_1 \tag{118}
$$

implies $\mu(r) \equiv 0$, when $\mu(r) = A_0 + o(1)$ as $r \to r_c \in \partial S_1 \equiv C_1$, A_0 is a constant.

C.1) the case with G_b

From the analysis of Rolf Leis (see [15] or appendix B), $\mu(r) = A_0 + o(1)$ as $r \to r_c \in \partial S_1 = C_1$ induces a singularity of tangential derivative in vicinity of C_1 of the form $A_0 \ln|r-r_c|$. This implies, from $\mathcal{U}(r')|_{S_1} \equiv 0$, that $A_0 = 0$.
We then choose to define the following functions u and w,

$$
\begin{aligned}
u(r') &= \frac{-k}{4\pi}\int_{S_1} G_b(r,r')\mu(r)dS \text{ with } u(r') = 0 \text{ on } S_1, \\
w(r') &= (\frac{\partial .}{\partial z'} - ikg)u(r') = \frac{-2k}{4\pi}\int_{S_1}(\frac{\partial .}{\partial z'})G^0(r,r')\mu(r)dS
\end{aligned} \tag{119}
$$

where we have used that

$$
\begin{aligned}
& (\frac{\partial .}{\partial z'} - ikg)G_b(r,r') \\
& = (\frac{\partial .}{\partial z'})(G^0(r-r') + G^0(r-r'_{im})) - ikg(G^0(r-r') - G^0(r-r'_{im}))
\end{aligned} \tag{120}
$$

Considering the property of the double layer potential with free space Green's function G^0, and F the radiation pattern (or scattering diagram) of w, we can write

$$w(r') = -\mu(r') = O(1) \text{ on } S_1,\ w(r') = 0 \text{ on } S_0\backslash\overline{S}_1$$
$$w(r') = \frac{e^{-ik|r'|}}{|r'|}(F(\frac{r'}{|r'|}) + o(1)) \text{ when } r' \to \infty. \tag{121}$$

From Leis's second theorem [15],

$$\text{grad}(w(r)) = o(1/|r - r_c|) \tag{122}$$

when $r \to r_c \in C$, and, from $u(r) = 0$ on S_1, we have

$$\lim_{z' \to 0^-} \frac{\partial w(r)}{\partial z} = (k^2 - \frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2})u(r) - ikgw(r) \to -ikg\,w(r) \text{ on } S_1 \tag{123}$$

Thus, we can apply the Green's first theorem on the domain $z < 0$, and we obtain

$$\text{Re}\,(\int_{z \leq 0^-} -ik|w(r)|^2 + \frac{|\text{grad}w(r)|^2}{-ik}dV) = (\int_{z=0^-} \text{Re}(g)|w(r)|^2 dS +$$
$$+\,\text{Re}\,(\int_0^{2\pi}\int_{\pi/2}^{\pi} |F(\Theta,\phi)|^2 \sin\Theta d\Theta d\phi) \tag{124}$$

For $\text{Re}(g) \geq 0$ and $|\arg(ik)| \leq \pi/2$, the left-hand term is ≤ 0, while the right-hand term is ≥ 0, and thus both terms vanish. So, we have,

$$w(r) = 0 \text{ as } z < 0, \text{ when } |\arg(ik)| < \pi/2, \text{Re}(g) \geq 0$$
$$w(r) = 0 \text{ as } z = 0^-, \text{ when } |\arg(ik)| = \pi/2, \text{Re}(g) > 0 \tag{125}$$

which implies in these cases, from $w(r') = -\mu(r')$ on S_1, that μ vanishes.
In the case $g = 0$, G_b can be replaced by $2G_0$ in the definition of u, and the demonstration of Colton and Kress [14, sect. 2], can be directly used to conclude that $\mu \equiv 0$.

Remark 16 :
the same property can be deduced for $\text{Re}g < 0$, except along the branch-cut of G_b with $\text{Re}(ik\cos\theta_1) = 0$, $g = \sin\theta_1$. For this, we can directly use the first Green's theorem with u instead of w, and deduce that $\mu \equiv 0$.

C.2) the case with G_a

If we consider in the definitions of u, G_a instead of G_b, and the domain $z > 0$ instead of the domain $z < 0$, we can directly use the first Green's theorem with u instead of w, and deduce that $\mu(r') = 0$ when $\mathrm{Re}(g) > 0$ or $g = 0$.

Appendix D :
The Green's tensors for an impedance plane in electromagnetism

In our method, a key point is the use of the 'below' Green's functions in the cavity which derives from our solution for an arbitrary impedance plane (passive or active). In a similar manner, an extension of our work to electromagnetism is based primarily on the Green's tensors for an arbitrary impedance plane, which are now developed from [3]-[4]. We consider, the electromagnetic field (E, H) that satisfies the Maxwell equation,

$$\mathrm{curl}(E) = -ik(Z_0H) - M,\ \mathrm{curl}(Z_0H) = ikE + Z_0J \tag{126}$$

above the plane, and the impedance boundary conditions,

$$\widehat{z} \wedge E|_{z=0} = g^e\ (\widehat{z} \wedge \widehat{z} \wedge (Z_0H))|_{z=0}\,, \tag{127}$$

or

$$(\partial_z - ikg^e)E_z|_{z=0} = 0,\ (\partial_z - ik/g^e)H_z|_{z=0} = 0, \tag{128}$$

The incident field, radiated by the sources J and M in free space, is given by

$$\begin{aligned} E_{inc} &= \mathrm{curl}(G*M) + \frac{i}{k}(\mathrm{grad}(\mathrm{div}(.)) + k^2)(G*Z_0J) \\ &= \frac{1}{8\pi k^2}(-M*[\underline{\mathcal{D}}_{e,i}(r',r)] + Z_0J*[\underline{\mathcal{F}}_{h,i}(r',r)]) \\ Z_0H_{inc} &= -\mathrm{curl}(G*Z_0J) + \frac{i}{k}(\mathrm{grad}(\mathrm{div}(.)) + k^2)(G*M) \\ &= \frac{1}{8\pi k^2}(Z_0J*[\underline{\mathcal{D}}_{h,i}(r',r)] + M*[\underline{\mathcal{F}}_{e,i}(r',r)]) \end{aligned} \tag{129}$$

where $G = -\frac{e^{-ik|r|}}{4\pi|r|}$, $|r| = \sqrt{x^2+y^2+z^2}$, and $*$ is the convolution product.

Developing the expressions of potentials given in [3]-[4] for the scattered field (E_s, H_s), we can write, when $M = M_{r'}\delta(r - r')$ and $J = J_{r'}\delta(r - r')$,

$$\begin{aligned}
E_s(r) &= -ik\mathrm{curl}(\mathcal{H}_s\widehat{z}) + (\mathrm{grad}(\mathrm{div}(.)) + k^2)(\mathcal{E}_s\widehat{z}) \\
&= \frac{1}{8\pi k^2}([\underline{\mathcal{F}}_{he}(r,r')].Z_0J_{r'} - [\underline{\mathcal{D}}_{he}(r,r')].M_{r'}) \\
&= \frac{1}{8\pi k^2}(Z_0J_{r'}.[\underline{\mathcal{F}}_{he}(r',r)] - M_{r'}.[\underline{\mathcal{D}}_{eh}(r',r)]) \\
&= \frac{1}{8\pi k^2}(Z_0J_{r'}.[\frac{1}{ik}\mathrm{curl}_{r'}([\underline{\mathcal{D}}_{eh}(r',r)])] - M_{r'}.[\underline{\mathcal{D}}_{eh}(r',r)])
\end{aligned} \tag{130}$$

and

$$\begin{aligned}
Z_0H_s(r) &= ik\mathrm{curl}(\mathcal{E}_s\widehat{z}) + (\mathrm{grad}(\mathrm{div}(.)) + k^2)(\mathcal{H}_s\widehat{z}) \\
&= \frac{1}{8\pi k^2}([\underline{\mathcal{D}}_{eh}(r,r')].Z_0J_{r'} + [\underline{\mathcal{F}}_{eh}(r,r')].M_{r'}) \\
&= \frac{1}{8\pi k^2}(Z_0J_{r'}.[\underline{\mathcal{D}}_{he}(r',r)] + M_{r'}.[\underline{\mathcal{F}}_{eh}(r',r)]) \\
&= \frac{1}{8\pi k^2}(Z_0J_{r'}.[\underline{\mathcal{D}}_{he}(r',r)] + M_{r'}.[\frac{1}{ik}\mathrm{curl}_{r'}([\underline{\mathcal{D}}_{he}(r',r)])]
\end{aligned} \tag{131}$$

where $\underline{\mathcal{F}}_{he(,eh)}(r',r)$ and $\underline{\mathcal{D}}_{eh(,he)}(r',r)$ are dyadic tensors. In these notations, we have $D.[\widehat{a}\,\widehat{b}] = (D.\widehat{a})\,\widehat{b}$, $[\widehat{a}\,\widehat{b}].D = \widehat{a}\,(\widehat{b}\,.D)$ and

$$[\underline{\mathcal{G}}(r,r')] \;\rightarrow\; [\underline{\mathcal{G}}(r',r)] \;\text{ if }\; (x,y,z) \leftrightarrow (x',y',z') \text{ and } (\widehat{x},\widehat{y},\widehat{z}) \leftrightarrow (\widehat{x}',\widehat{y}',\widehat{z}'). \tag{132}$$

The tensors verify the impedance boundary conditions,

$$\begin{aligned}
&\widehat{z} \wedge [(\underline{\mathcal{D}}_{he} + \underline{\mathcal{D}}_{h,i})(r,r')(r,r')]|_{z=0} = -g^e(\widehat{z} \wedge \widehat{z} \wedge [(\underline{\mathcal{F}}_{eh} + \underline{\mathcal{F}}_{e,i})(r,r')]|_{z=0}, \\
&\widehat{z} \wedge [(\underline{\mathcal{F}}_{he} + \underline{\mathcal{F}}_{h,i})(r,r')]|_{z=0} = g^e(\widehat{z} \wedge \widehat{z} \wedge [(\underline{\mathcal{D}}_{eh} + \underline{\mathcal{D}}_{e,i})(r,r')])|_{z=0}.
\end{aligned} \tag{133}$$

and can be written,

$$\begin{aligned}
\underline{\mathcal{F}}_{he(,eh)} &\equiv -\mathcal{B}(\underline{B}_{h(,e)}) + \mathcal{A}(\underline{A}_{e(,h)}) \\
\underline{\mathcal{D}}_{he(,eh)} &\equiv \mathcal{B}(\underline{A}_{h(,e)}) + \mathcal{A}(\underline{B}_{e(,h)})
\end{aligned} \tag{134}$$

where

$$\begin{aligned}
&[\mathcal{A}(\underline{B}_{e(,h)})(r,r')] = \\
&= [ik(\widehat{x}\partial_x + \widehat{y}\partial_y + \widehat{z}\partial_z)(\widehat{y}'\partial_x - \widehat{x}'\partial_y)(\partial_z\mathcal{S}_{e(,h)}(r,r')) + \\
&+ ik^3\widehat{z}(\widehat{y}'\partial_x - \widehat{x}'\partial_y)(\mathcal{S}_{e(,h)}(r,r'))]
\end{aligned} \tag{135}$$

$$[\mathcal{B}(\underline{A}_{e(,h)})(r,r')] =$$
$$= [ik(\widehat{x}\partial_y - \widehat{y}\partial_x)(\widehat{x}'\partial_x + \widehat{y}'\partial_y + \widehat{z}'\epsilon\partial_z)(\epsilon\partial_z\mathcal{S}_{e(,h)}(r,r')) +$$
$$+ ik^3(\widehat{x}\partial_y - \widehat{y}\partial_x)\widehat{z}'(\mathcal{S}_{e(,h)}(r,r'))] \qquad (136)$$

$$[\mathcal{A}(\underline{A}_{e(,h)})(r,r')] =$$
$$= [(\widehat{x}\partial_x + \widehat{y}\partial_y + \widehat{z}\partial_z)(\widehat{x}'\partial_x + \widehat{y}'\partial_y + \widehat{z}'\epsilon\partial_z)(\epsilon\partial_{z^2}\mathcal{S}_{e(,h)}(r,r')) +$$
$$+ k^2\widehat{z}(\widehat{x}'\partial_x + \widehat{y}'\partial_y + \widehat{z}'\epsilon\partial_z)(\epsilon\partial_z\mathcal{S}_{e(,h)}(r,r')) +$$
$$+ k^2(\widehat{x}\partial_x + \widehat{y}\partial_y + \widehat{z}\partial_z)(\widehat{z}')(\partial_z\mathcal{S}_{e(,h)}(r,r')) +$$
$$+ \widehat{z}\widehat{z}'k^4(\mathcal{S}_{e(,h)}(r,r'))] \qquad (137)$$

$$[\mathcal{B}(\underline{B}_{e(,h)})(r,r')] =$$
$$= [-k^2(-\widehat{x}\partial_y + \widehat{y}\partial_x)(\widehat{x}'\partial_y - \widehat{y}'\partial_x)(\mathcal{S}_{e(,h)}(r,r'))], \qquad (138)$$

with $\epsilon = -1$, $\widehat{x}' \equiv \widehat{x}, \widehat{y}' \equiv \widehat{y}, \widehat{z}' \equiv \widehat{z}$. The functions $\mathcal{S}_{e(,h)}$ verify the conditions [3],

$$(\partial_z - ikg^{e(,h)})\mathcal{S}_{e(,h)}(r,r') = (\partial_z + ikg^{e(,h)})\mathcal{S}_i(r_{im},r'))|_{z=0}, \qquad (139)$$

where $g^h = 1/g^e$, $r_{im} - r = 2\widehat{z}.r$, $\mathcal{S}_i(r,r'_{im}) = \mathcal{S}_i(r_{im},r')$, and

$$\mathcal{S}_i(r,r') = (e^{ik|\widehat{z}.(r-r')|}E_1(ik(|(r-r')| + |\widehat{z}.(r-r')|) +$$
$$+ e^{-ik|\widehat{z}.(r-r')|}(E_1(ik(|(r-r')| - |\widehat{z}.(r-r')|)) + 2\ln|\widehat{z}\wedge(r-r')|)), \qquad (140)$$

Their expressions are given by [3], [4],

$$\mathcal{S}_e(r,r') = (\mathcal{S}_i(r_{im},r') + \sum_{\epsilon'=-1,1}\frac{-2g^e}{(g^e-\epsilon')}(\mathcal{V}_{\epsilon'} + \epsilon'\mathcal{K}_{g^e}))(x-x', y-y', -z-z'),$$
$$\mathcal{S}_h(r,r') = (-\mathcal{S}_i(r_{im},r') + \sum_{\epsilon'=-1,1}\frac{2g^e}{(g^e-\epsilon')}(\mathcal{V}_{\epsilon'} + \epsilon'\mathcal{K}_{g^h}))(x-x', y-y', -z-z'), \qquad (141)$$

for $z \geq 0$, $z' \geq 0$. The functions $\mathcal{V}_{\epsilon'}$ and $\mathcal{K}_g$, which satisfy the Helmholtz equation above the plane, are given by

$$\mathcal{V}_{\epsilon'}(x,y,-z) = e^{\epsilon' ikz}(E_1(ik(|r| + \epsilon' z)) + (1-\epsilon')\ln\rho),$$
$$\mathcal{K}_g(x,y,-z) = e^{ikgz}\mathcal{J}_g(\rho,-z), \qquad (142)$$

for $z \geq 0$, $\rho = \sqrt{x^2+y^2}$, $g = g^e$ or $g = g^h$, $g^h = 1/g^e$. Let us notice that we have

$$\frac{\partial}{\partial z}\mathcal{S}_i(-z) = ik(e^{ikz}E_1(ik(|r|+z)) - e^{-ikz}(E_1(ik(|r|-z)) + 2\ln\rho)),$$
$$\frac{\partial^2}{\partial z^2}\mathcal{S}_i(-z) = -2ik\frac{e^{-ik|r|}}{|r|} - k^2\mathcal{S}_i(-z), \qquad (143)$$

and

$$\frac{\partial}{\partial z}(\mathcal{V}_{\epsilon'}+\epsilon'\mathcal{K}_g)(x,y,-z)=ik\epsilon'(\mathcal{V}_{\epsilon'}+g\,\mathcal{K}_g)(x,y,-z),$$
$$\frac{\partial^2}{\partial z^2}(\mathcal{V}_{\epsilon'}+\epsilon'\mathcal{K}_g)=-ik\epsilon'((\epsilon'-g)\frac{e^{-ik|r|}}{|r|}-ik(\epsilon'\mathcal{V}_{\epsilon'}+g^2\,\mathcal{K}_g)),$$
$$\sum_{\epsilon'=-1,1}\frac{-2g^e}{(g^e-\epsilon')}\frac{\partial^2}{\partial z^2}(\mathcal{V}_{\epsilon'}+\epsilon'\mathcal{K}_{g^e})=2k^2\sum_{\epsilon'=-1,1}\frac{g^e(\mathcal{V}_{\epsilon'}+\epsilon'(g^e)^2\,\mathcal{K}_{g^e})}{(g^e-\epsilon')},$$
$$\sum_{\epsilon'=-1,1}\frac{2g^e}{(g^e-\epsilon')}\frac{\partial^2}{\partial z^2}(\mathcal{V}_{\epsilon'}+\epsilon'\mathcal{K}_{g^h})=-4ik\frac{e^{-ik|r|}}{|r|}-2k^2\sum_{\epsilon'=-1,1}\frac{(g^e\mathcal{V}_{\epsilon'}+\epsilon'g^h\mathcal{K}_{g^h})}{(g^e-\epsilon')}, \tag{144}$$

for $z\geq 0$. The term lnρ does not contribute to the field, except to suppress a singularity due to $E_1(ik(|r|-|z|))$ at $\rho=0$ [3]. From the behaviour of $\mathcal{J}_g$, $\mathcal{S}_{e(,h)}(r',r)$ remains definite for $g^e=1$ because $\mathcal{V}_{\epsilon'=1}+\mathcal{K}_{g^e}\to 0$ when $g^e\to 1$, while it is singular for $g^e=-1$. Moreover, when $g^h=(g^e)^{-1}\to\infty$, we have $g^h\mathcal{K}_{g^h}\to-\frac{e^{-ik|r|}}{ik|r|}$. In a similar manner, the functions $\underline{\mathcal{F}}_{h,i(e,i)}$ and $\underline{\mathcal{D}}_{h,i(e,i)}$ can be also expressed like $\underline{\mathcal{F}}_{he(,eh)}$ and $\underline{\mathcal{D}}_{he(,eh)}$, if we take $\mathcal{S}_i(r,r')$ in place of $\mathcal{S}_{e(,h)}(r,r')$, and $\epsilon=1$.

Remark 17 :
In a free domain Ω bounded by S, the field is the radiation of the surface sources [8],

$$M=-n\wedge E\,\delta_S,\,J=n\wedge H\,\delta_S, \tag{145}$$

with

$$Z_0\mathrm{div}(J)=Z_0\mathrm{div}(n\wedge H\delta_S)=-ikn.E\,\delta_S-Z_0(n\wedge H).v\delta_{\partial S},$$
$$\mathrm{div}(M)=-\mathrm{div}_S(n\wedge E\,\delta_S)=-ikZ_0n.H\,\delta_S+(n\wedge E).v\delta_{\partial S}, \tag{146}$$

where n is the normal to S directed inside Ω, v is the geodesic normal to ∂S directed outside S, and δ_S is the Dirac surface function.

Remark 18 :
We notice that

$$\mathrm{curl}_r([\underline{\mathcal{D}}_{he(,eh)}(r,r')].C_{r'})=ik([\underline{\mathcal{F}}_{eh(,he)}(r,r')].C_{r'}),$$
$$\mathrm{curl}_r([\underline{\mathcal{F}}_{he(,eh)}(r,r')].C_{r'})=-ik([\underline{\mathcal{D}}_{eh(,he)}(r,r')].C_{r'}), \tag{147}$$

and

$$
\begin{aligned}
&D_r.[\underline{\mathcal{F}}_{he(,eh)}(r,r')].C_{r'} = C_{r'}.[\underline{\mathcal{F}}_{he(,eh)}(r',r)].D_r,\\
&D_r.[\underline{\mathcal{D}}_{he(,eh)}(r,r')].C_{r'} = C_{r'}.[\underline{\mathcal{D}}_{eh(,he)}(r',r)].D_r,
\end{aligned}
\tag{148}
$$

with $C_{r'} = c_x\widehat{x}' + c_y\widehat{y}' + c_z\widehat{z}'$, $D_r = d_x\widehat{x} + d_y\widehat{y} + d_z\widehat{z}$ being two constant vectors.

Remark 19 :

The tensors also satisfy,

$$
\begin{aligned}
&[\mathcal{A}(\underline{B}_{e,h})(r,r')].C_{r'} =\\
&= ik(\text{grad}(\text{div}(\widehat{z}.)) + k^2\widehat{z}.)((C^t_{r'} \wedge \widehat{z})\text{grad}(\mathcal{S}_{e,h}(r,r'))\\
&= [ik(\widehat{x}\partial_x + \widehat{y}\partial_y)(\widehat{y}'\partial_x - \widehat{x}'\partial_y)(\partial_z\mathcal{S}_{e,h}(r,r')) +\\
&+ ik\,\widehat{z}(\widehat{y}'\partial_x - \widehat{x}'\partial_y)(\partial_{z^2} + k^2)\mathcal{S}_{e,h}(r,r')].C_{r'},
\end{aligned}
\tag{149}
$$

$$
\begin{aligned}
&[\mathcal{B}(\underline{A}_{e,h})(r,r')].C_{r'} =\\
&= ik\text{curl}(\widehat{z}(\epsilon\partial_z(C^t_{r'}\text{grad}(\mathcal{S}_{e,h}(r,r'))) +\\
&+ c_z((\partial_{z^2} + k^2)\mathcal{S}_{e,h}(r',r)))\\
&= [ik((\widehat{x}\partial_y - \widehat{y}\partial_x)(\widehat{x}'\partial_x + \widehat{y}'\partial_y)\epsilon\partial_z\mathcal{S}_{e,h}(r,r') +\\
&+ (\widehat{x}\partial_y - \widehat{y}\partial_x)\widehat{z}'(\partial_{z^2} + k^2)\mathcal{S}_{e,h}(r,r'))].C_{r'},
\end{aligned}
\tag{150}
$$

$$
\begin{aligned}
&[\mathcal{A}(\underline{A}_{e,h})(r,r')].C_{r'} =\\
&= (\text{grad}(\text{div}(\widehat{z}.)) + k^2\widehat{z}.)(C^t_{r'}(\epsilon\partial_z\text{grad}(\mathcal{S}^\epsilon_{e,h}(r,r'))) +\\
&+ c_z((\partial_{z^2} + k^2)\mathcal{S}_{e,h}(r,r')))\\
&= [\epsilon\partial_{z^2}(\widehat{x}\partial_x + \widehat{y}\partial_y)(\widehat{x}'\partial_x + \widehat{y}'\partial_y)\,\mathcal{S}_{e,h}(r,r') +\\
&+ \partial_z(\widehat{x}\partial_x + \widehat{y}\partial_y)(\widehat{z}')(\partial_{z^2} + k^2)\mathcal{S}_{e,h}(r,r') +\\
&+ \widehat{z}(\widehat{x}'\partial_x + \widehat{y}'\partial_y)(\partial_{z^2} + k^2)(\epsilon\partial_z\mathcal{S}_{e,h}(r,r')) +\\
&+ \widehat{z}\widehat{z}'(\partial_{z^2} + k^2)(\partial_{z^2} + k^2)\mathcal{S}_{e,h}(r,r')].C_{r'},
\end{aligned}
\tag{151}
$$

$$
\begin{aligned}
&[\mathcal{B}(\underline{B}_{e,h})(r,r')].C_{r'} =\\
&= ik\text{curl}(\widehat{z}(ik(C^t_{r'} \wedge \widehat{z})\text{grad}(\mathcal{S}_{e,h}(r,r'))))\\
&= [-\,k^2(-\widehat{x}\partial_y + \widehat{y}\partial_x)(\widehat{x}'\partial_y - \widehat{y}'\partial_x)(\mathcal{S}_{e,h}(r,r'))].C_{r'},
\end{aligned}
\tag{152}
$$

where $C_{r'} = C^t_{r'} + c_z\widehat{z}'$, and, from the Helmholtz equation satisfied by $\mathcal{S}_{e,h}$,

$$
\begin{aligned}
&[\mathcal{B}(\underline{B}_{e,h})(r,r')].C_{r'} =\\
&= -\,k^2((c_x\partial_x + c_y\partial_y)(\widehat{x}\partial_x + \widehat{y}\partial_y) +\\
&+ (c_x\widehat{x} + c_y\,\widehat{y})(\partial_{z^2} + k^2))(\mathcal{S}_{e,h}(r,r')).
\end{aligned}
\tag{153}
$$

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