# On the scattering by a cavity in an impedance plane in 3D : boundary integral equations and novel Green's function

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**Abstract**

The problem of the field scattered by a cavity embedded in an impedance (or Robin) plane is considered for the 3D Helmholtz equation in acoustics. Its resolution is more complex than for a scatterer above the plane, in particular because the Green's function for the unperturbed plane has a singular part unsuitable for applications below the plane. It is why the free space Green's function is commonly used in boundary integral equations for the cavity, and three unknowns are necessary. We propose here to use a novel Green's function below the impedance plane, which has the advantage to reduce the number of unknowns, and to simplify the problem. This specific Green's function derives from our recent study for passive and active unperturbed impedance planes. The uniqueness property is studied in passive case. The application to small cavity leads us to new analytical results.

## 1 Introduction

This paper presents novel integral equations for the field scattered by a cavity embedded in an imperflectly reflective plane with impedance boundary conditions, for the three-dimensional Helmholtz equation, and analytical approximations for small cavities.

The development of boundary integral equation methods, in 2D and 3D, for this scattering problem is rather recent [1],[2], seemingly because of specific difficulties due to the representation of the field in the cavity. Indeed, the Green's function $G_a$, defined as the field of a monopole in presence of the unperturbed impedance plane, which is perfectly adapted to reduce the radiation of an aperture in the plane to an expression depending only on one unknown, has a defect: it has a logarithmic singularity (a logarithmic branch cut) in lower half space that prevents it from being applied below the plane. It is why, until now, the Green's function in free space was preferred for the representation of the field in the cavity. That induces an additional unknown to characterize the radiation of the aperture below the plane, and finally implies three distinct integral equations for three surface field unknowns [1].

To reduce the number of unknowns and simplify the system of integral equations, we here develop an original way, consisting in the definition of a new Green's function that we name the 'below' Green's function $G_b$. Both functions $G_a$ and $G_b$ satisfy the

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impedance condition on the unperturbed surface, but the scattered fields attached to them are respectively regular above and below the plane. They derive from the solution for an arbitrary constant impedance plane (passive or active) given in [3]-[4].

Morever, our system of two novel integral equations has the property of uniqueness of the solution. It is an important point, particularly if we notice that most of the boundary integral equation methods in the related problem of electromagnetism, which use the generalized network formulation [5], are not uniquely solvable at some discrete frequencies [6]. Otherwise, other methods verify uniqueness, in particular the one developed by Chandler-Wilde in acoustics, with a system of three integral equations [1], and the ones used by Xu [6], Asvestas et al. [7], or Wood et al. [8], for perfectly conducting surface in electromagnetism, with a system of two vectorial integral equations. It is worth noticing that, in [6] (see also [9]-[10]), the generalized network formulation is corrected by the image theory, which is equivalent to using a specific Green's function in the cavity that takes account of the plane, while, in [7] and [8], the Green's functions for Dirichlet and Neumann plane are combined in an original way to derive novel boundary integral equations.

This scattering problem can be also analyzed in complex spectral domain in 2D, or by asymptotic methods in 2D and 3D. So, integral equations with smooth kernels in 2D [11], which permit various approximations for large or small polygonal cavity, or asymptotic expressions for a large cavity [12]-[13], have been developed.

The paper is organized as follows. In section 2, we define the properties of the acoustic field, and analyze the uniqueness of the boundary value problem. We present in section 3, the expressions of the Green's functions $G_a$ and $G_b$, derived from the solution for an unperturbed impedance plane. In section 4, we use the second Green's theorem and give a representation of the field above the plane and in the cavity. We then deduce the system of integral equations in section 5 and show the property of uniqueness in section 6. In section 7, this new system is considered for small cavity and original analytical results are derived. Some particular developments concerning applications to 2D cases, filled cavities, protuberances, and electromagnetism are also given in remarks and appendices.

# 2 Formulation of the boundary value problem and uniqueness

## 2.1 Boundary value problem

We consider the pressure field $p_s$ scattered by an imperfectly reflective plane that is perturbed by a cavity (figure 1), when it is illuminated by the incident pressure field $p_{inc}$, radiated by a bounded source $W$ above the plane and satisfying the Helmholtz equation,

$$(\Delta + k^2)p_{inc} = W \tag{2.1}$$

in $R^3$, with $|\arg(ik)| \leq \pi/2$.

The unperturbated plane $S_0$ is defined by $z = 0$ in Cartesian coordinates $(x, y, z)$. The domain of the cavity with $z < 0$, and the half-space above the plane with $z \geq 0$, are respectively denoted $\Omega_2$ and $\Omega_1$. The aperture and the surface of the cavity, respectively denoted $S_1$ and $S_2$, are assumed to be piecewise analytic (with no zero exterior angles, i.e. no points of $\Omega_2$ inside a cusp), bounded by a Jordan curve $C_1$.

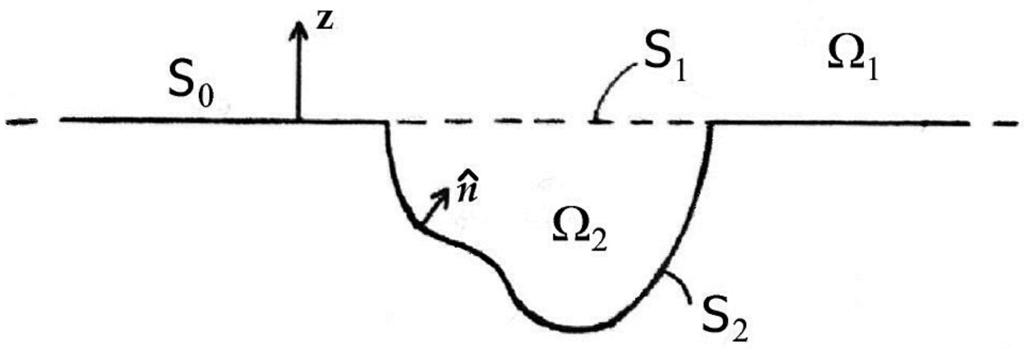


Figure 1: geometry and definition of the cavity

For any plane wave of incidence angle $\beta$ composing $p_{inc}$, the infinite plane, when it is unperturbed, has a reflection coefficient $R(\beta)$ given by,

$$R(\beta) = \frac{\cos\beta - g}{\cos\beta + g}, \tag{2.2}$$

so that $p = p_s + p_{inc}$ verifies the impedance (or Robin) boundary condition,

$$(\frac{\partial}{\partial z} - ikg)p = 0, \tag{2.3}$$

on the plane $S_0$, except on the aperture $S_1$ of the cavity. The term $g = \sin\theta_1$ is denoted the impedance parameter. In (2.3), it is a constant, with Re$(ik\cos\theta_1)\neq 0$ when Re$\theta_1 \leq 0$. This condition on $g$ is due to the presence of a cut in the solution for an unperturbed plane [3]-[4], along the path Re$(ik\cos\theta_1)= 0$ as Re$\theta_1 \leq 0$. Therefore, the surface waves, which radiate without exponential decay at infinity, can only be considered in the sense of the limit for Re$(ik\cos\theta_1)= 0^+$or $0^-$ when Re$\theta_1 \leq 0$.

Some general properties are considered for the scattered field in $\Omega_1$ and $\Omega_2$:

(a) $p_s$, which satisfies the Helmholtz equation

$$(\Delta + k^2)p_s = 0 \text{ with } |\arg(ik)| \leq \pi/2, \tag{2.4}$$

in $\Omega_1 \cup \Omega_2$, is regular in this domain, except at edges and corners of $S_2$ where

$$p_s = O(1) \text{ and } \mathrm{grad}(p_s) = O(|r|^\alpha), -1 < \alpha \leq 0, \tag{2.5}$$

as the distance $|r|$ to the edge or corner vanishes [12], and $p_s$ is continuous on the scatterer;

(b) $p_s$ is constituted of outgoing waves, with guiding waves exponentially vanishing at infinity (Re$(ik\cos\theta_1)\neq 0$ as Re$\theta_1 \leq 0$), and, the field at $M$, with $r = \overline{OM}$, verifies,

$$p_s = O(e^{-\delta|r|}), \tag{2.6}$$

$\delta > 0$, as $z$ or $\rho = \sqrt{x^2+y^2} \to \infty$, $z > 0$, when $|\arg(ik)| < \pi/2$, and

$$\frac{\partial p_s}{\partial |r|} + ikp_s = o(|r|^{-1}),\; p_s = O(|r|^{-1}), \tag{2.7}$$

as $|r| = \sqrt{x^2+y^2+z^2} \to \infty$, $z \geq 0$, when $|\arg(ik)| = \pi/2$.

In addition, an impedance boundary condition is assumed on the surface of the cavity,

$$(\frac{\partial}{\partial n} - ikg_c)p|_{S_2} = 0, \tag{2.8}$$

where $\widehat{n}$ is the normal to $S_2$ directed inside $\Omega_2$, $g_c$ is a function piecewise analytic on $S_2$.

**Remark 1.** *Let us notice that the definitions of the 'acoustic impedance' ($\equiv A_0 p/\frac{\partial p}{\partial n}$, $A_0$ a constant) generally used in physics [14], and of our 'impedance parameter' ($\equiv \frac{\partial p}{\partial n}/(ikp)$), are different.*

**Remark 2.** *The demonstrations concerning the Hölder regularity of the field near surfaces with general boundary conditions like (2.8) are lengthy, and some authors often assume that $ikg_c$ is a positive real number to simplify the development (see the remark of Levine in [15] after lemma 5.2).*

## 2.2 Uniqueness of the solution of the boundary value problem from [15, sect.7]

In [15], Levine develops an uniqueness theorem, i.e. a proof that $p_{inc} \equiv 0$ implies $p \equiv 0$, in the case of a scatterer with impedance boundary conditions. He considers piecewise $C^{(2+\lambda)}$ surface (with no zero exterior angle), $\lambda > 0$, without auxiliary 'edge conditions' at edges or corner points, except that $p$ is continuous. He studies at first bounded scatterers, but he also gives, in section 7 of his paper, the elements to generalize his results to scatterers with infinite boundaries, in particular by the use of Jones' uniqueness theorem [16], that we follow.

We begin to notice first that the conditions given by Levine to apply the Green's first theorem are satisfied: the cavity is piecewise analytic (with no zero exterior angle), the field is countinuous on the scatterer, it satisfies impedance boundary conditions and the conditions (b) at infinity. So, we can write,

$$\int_\Omega (p^*(r)\Delta p(r) + \mathrm{grad}p^*(r)\mathrm{grad}p(r))dV = -\int_S p^*(r)(\widehat{n}\,\mathrm{grad}(p(r)))dS + \\ + \lim_{a\to\infty}\int_{r=a,\, z\geq 0} p^*(r)(\frac{\partial p(r)}{\partial r})dS, \tag{2.9}$$

where $\Omega \equiv \Omega_1 \cup \Omega_2$, $S \equiv S_2 \cup (S_0 \backslash S_1)$, $\widehat{n}$ is the inward normal to $\Omega$, and from (2.3)-(2.8),

$$\mathrm{Re}(\int_\Omega -ik|p(r)|^2 + \frac{|\mathrm{grad}p(r)|^2}{-ik}dV) = \int_{S_2} \mathrm{Re}(g_c)|p(r)|^2 dS + \\ + \int_{S_0\backslash S_1} \mathrm{Re}(g)|p(r)|^2 dS + \; I_\infty, \tag{2.10}$$

where

$$I_\infty = \lim_{a\to\infty} e^{-\delta a} = 0 \text{ for } |\arg(ik)| < \pi/2,$$

$$I_\infty = \lim_{a\to\infty} \int_{r=a,\, z\geq 0} |p(r)|^2 dS > 0 \text{ for } |\arg(ik)| = \pi/2. \tag{2.11}$$

For $\mathrm{Re}(g) \geq 0$, $\mathrm{Re}(g_c) \geq 0$ and $|\arg(ik)| \leq \pi/2$, the left-hand term is negative since $\mathrm{Re}(ik) \geq 0$, while the right-hand term is positive, and thus both terms vanish. Consequently, we have, when $|\arg(ik)| < \pi/2$,

$$p(r) = 0 \text{ in } \Omega, \text{ for } \mathrm{Re}(g) \geq 0,\ \mathrm{Re}(g_c) \geq 0, \tag{2.12}$$

and, when $|\arg(ik)| = \pi/2$,

$$\begin{aligned} &p(r) = 0 \text{ on } S, \text{ for } \mathrm{Re}(g) > 0, \mathrm{Re}(g_c) > 0, \\ &\partial_n p(r) = 0 \text{ on } S, \text{ for } \mathrm{Re}(g) > 0, \mathrm{Re}(g_c) > 0, \text{ or for } g = g_c = 0. \end{aligned} \tag{2.13}$$

In the latter case, we can use, as suggested by Levine, the Jones' uniqueness theorem [16] for surfaces conical at infinity, when Neumann boundary condition ($\partial_n p(r)|_S = 0$) is satisfied, which implies $p \equiv 0$ in the entire domain $\Omega$, and thus completes the proof of uniqueness. Let us notice, that another proof has been independently developed in [1] when $S$ is smooth.

# 3 The 'above' Green's function $G_a$ and the 'below' Green's function $G_b$

The integral representations of the field with single and double layers potentials generally derive from the use of free space Green's function [12], but more complex Green's functions, verifying particular boundary conditions, can be used. In this latter case, a particular attention must be paid to the regularity of these functions.

So, when we consider a perturbation, due to a scatterer above an impedance plane, we can use the solution $G_a$ for unperturbed case to express the field everywhere, while it is generally not possible for a cavity, because of the logarithmic singularity of $G_a$ below the plane.

Therefore, we here develop an original way consisting in using another Green's function in the cavity that we name the 'below' Green's function $G_b$. Both functions $G_a$ and $G_b$ satisfy the impedance boundary condition (2.3) at $z = 0$, and derive from the solution for an unperturbed plane, respectively with the impedances $g$ and $-g$.

In this section, the solution for active and passive plane [3]-[4] are briefly presented, then $G_a$ and $G_b$ are developed.

## 3.1 The solution for an unperturbed impedance plane with arbitrary impedance

### 3.1.1 Solution for a monopole

We consider the incident field, radiated by a monopole at $r'(x', y', z' = h)$ (figure 2), $p_{inc} = e^{-ikR(z)}/kR(z)$ at $M(x, y, z)$, with $R(z) = \sqrt{(x - x')^2 + (y - y')^2 + (z - z')^2}$.

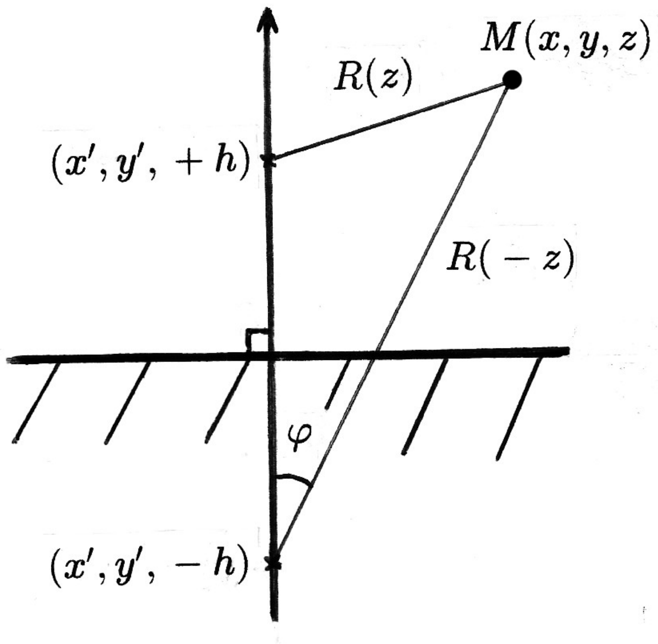


Figure 2: geometry and definition of $\varphi$ for the radiation at $M$

From [3], the field $p_s$ scattered by the impedance plane is then given by

$$p_s = \frac{e^{-ikR(-z)}}{kR(-z)} + 2ige^{ikg(z+h)}\mathcal{J}_g(\rho, -z - h), \tag{3.1}$$

where $R(-z) = \sqrt{\rho^2 + (z + h)^2}$, $z + h = R(-z)\cos\varphi$, $\rho = R(-z)\sin\varphi$, and,

$$\mathcal{J}_g(\rho, -z) = \frac{e^{-ikgz}}{2}\int_{\mathcal{D}} \frac{H_0^{(2)}(k\rho\sin\beta)e^{-ikz\cos\beta}}{\cos\beta + g}\sin\beta d\beta, \tag{3.2}$$

for $z > 0$, $g = \sin\theta_1$, with $\mathrm{Re}(ik\sin\beta) = 0$ on $\mathcal{D}$ from $-i\infty - \arg(ik)$ to $i\infty + \arg(ik)$. This function is a Fourier-Bessel integral commonly encountered in scattering theory [17, p.234], also called a Sommerfeld-type integral [18], which has a cut described by $\mathrm{Re}(ik\cos\theta_1) = 0$ when $\mathrm{Re}(g) \leq 0$ and a singularity at $g = -1$.

A correct definition of $\mathcal{J}_g$ for arbitrary $g = \sin\theta_1$, active ($\mathrm{Re}g < 0$) or passive ($\mathrm{Re}g > 0$), except on the cut, is also given [3] by,

$$\mathcal{J}_g(\rho, -z - h) = -\int_{-ib}^{\infty} e^{-a\cosh t}dt = i\int_{b}^{i\infty} e^{-a\cos\alpha}d\alpha, \tag{3.3}$$

where $a = \epsilon ikR(-z)\sin\varphi\cos\theta_1$, $\epsilon$ =sign(Re$(ik\cos\theta_1)$) (Re$(a) = 0$ is on a cut of $\mathcal{J}_g$ and it can be only considered in the sense of the limit), and $b$ satisfies

$$e^{\mp ib} = \frac{ikR(-z)}{a}(1 \pm \sin\theta_1)(1 \pm \cos\varphi), \tag{3.4}$$

with $|\text{Re}b| < \pi$, $e^{-2ib} = \frac{(1+\sin\theta_1)(1+\cos\varphi)}{(1-\sin\theta_1)(1-\cos\varphi)}$, $|\text{Re}(\theta_1)| \leq \pi/2$. As $g$ varies, this expression has a correct cut as $\epsilon$ changes of sign for Re$g < 0$, and is regular elsewhere (note: for Re$g > 0$, the change of sign of $\epsilon$ does not induce a cut as $g$ varies). The figure 3 shows the perfect agreement of $\mathcal{J}_g$ given respectively by (3.3) and by Fourier-Bessel expansion (3.2).

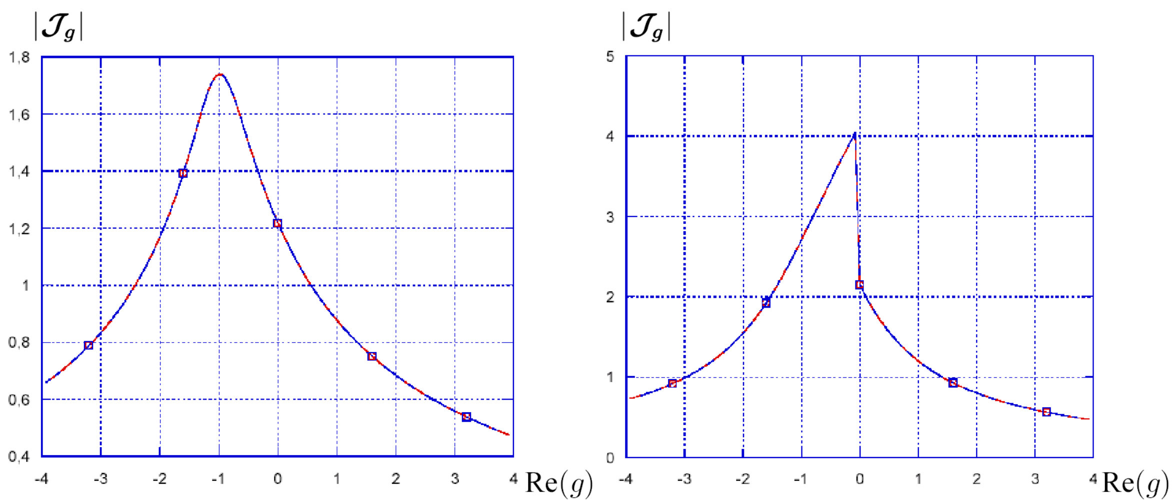


Figure 3: Comparison of $\mathcal{J}_g$ given by (3.3) $(-\square-)$ and by Fourier-Bessel expansion when (3.2) is used $(-\circ-)$, when Re$g$ varies; left: $|\mathcal{J}_g|$ when Im$(g) = -0.4$, $z + h = .2$, $\rho = .3$, $ik = .01 + i1.$; right: $|\mathcal{J}_g|$ when Im$(g) = 1.2$, $z + h = 1.$, $\rho = 1.$, $ik = .01 + i1.$

### 3.1.2 Some properties of $\mathcal{J}_g$

Some general properties of $\mathcal{J}_g$, derived from (3.3), are worth noticing. Using the integral expression of the modified Bessel function $K_0$ [19], we can write,

$$\mathcal{J}_g(\rho, -z-h) + K_0(a) = -i\int_0^b e^{-a\cos\alpha}d\alpha = -i\int_{-b}^0 e^{-a\cos\alpha}d\alpha, \tag{3.5}$$

which implies, by definition of $b$ and $a$, that

$$\mathcal{J}_g(\rho, -z-h) + K_0(a) = -\mathcal{J}_{-g}(\rho, z+h) - K_0(a), \tag{3.6}$$

where $a = \epsilon ik\rho\cos\theta_1$, $\epsilon$ =sign(Re$(ik\cos\theta_1)$). From the regularity of $\mathcal{J}_{\pm g}(\rho, -z)$ for $z > 0$ and the expression of $b$, we deduce that $\mathcal{J}_g(\rho, -z)$ has a logarithmic singularities when

$z \leq 0$ at $\rho = 0$. So, when $a$, and thus, when $\rho$ vanishes, we have [3]

$$\begin{aligned} \mathcal{J}_g(\rho, -z) &\sim -2K_0(a) \text{ when } z < 0, g \neq -1, \\ \mathcal{J}_g(\rho, -z) &\sim -K_0(a) \text{ when } z = 0,\ g \neq -1, \\ \mathcal{J}_g(\rho, -z) &\sim -E_1(\frac{ik(1+g)}{2}(|r| + z)). \end{aligned} \tag{3.7}$$

where $E_1$ is the exponential integral [19]. Moreover, the reader can verify by inspection that,

$$\frac{\partial \mathcal{J}_g(\rho, -z-h)}{\partial z} = \frac{e^{-ik(R(-z)+g(z+h))}}{R(-z)}, \tag{3.8}$$

and,

$$(\Delta + k^2)(e^{ikgz}\mathcal{J}_g(\rho, -z)) = 4\pi e^{ikgz} U(-z)\delta(x)\delta(y), \tag{3.9}$$

where $U$ is the unit step function, $\delta$ is the Dirac function.

**Remark 3.** *Let us notice [3] that, for $Reg > 0$ and $\arg(ik) = \pi/2$,*

$$\mathcal{J}_g(\rho, -z-h) = \int_{-i\infty}^{0} e^{-ikg(z_1+z+h)} \frac{e^{-ikR(-z_1-z)}}{kR(-z_1-z)} k\, dz_1, \tag{3.10}$$

*where $R(-z) = \sqrt{\rho^2 + (z+h)^2}$, and that, for $g = 1$,*

$$\mathcal{J}_{g=1}(\rho, -z-h) = -E_1(ik(R(-z) + (z+h))). \tag{3.11}$$

## 3.2 The functions $G_a$ and $G_b$

### 3.2.1 The Green's functions $G_a$ above the plane

The Green's function $G_a$ is given by the solution for a monopole above the plane with impedance parameter $g$ . From the previous section, it is given by

$$G_a(r, r') = G^0(x - x', y - y', z - z') + G^s_g(x - x', y - y', -z - z')\,, \tag{3.12}$$

where $G^0$ is the free space Green's function,

$$G^0(r) = \frac{e^{-ik|r|}}{k|r|}, \tag{3.13}$$

and $G^s_g$ is the scattered Green's function,

$$G^s_g(r) = \frac{e^{-ik|r|}}{k|r|} + 2ige^{-ikgz}\mathcal{J}_g(\rho, z), \tag{3.14}$$

with $|r| = \sqrt{\rho^2 + z^2}$ and $\rho = \sqrt{x^2 + y^2}$.

Because

$$(\Delta + k^2)G^0(r) = \frac{-4\pi}{k}\delta(x)\delta(y)\delta(z), \tag{3.15}$$

and the equation (3.9) satisfied by $\mathcal{J}_g(\rho, -z)$, the function $G_a$ verifies in $R^3$,

$$\begin{aligned}(\Delta + k^2)G_a(r, r') &= \frac{-4\pi}{k}(\delta(r - r')+ \\ &+ \delta(r - r'_{im}) - 2ikge^{ikg(z+z')}U(-z - z')\delta(x - x')\delta(y - y')),\end{aligned} \tag{3.16}$$

where $r'_{im} \equiv (x', y', -z')$, $\delta(r) \equiv \delta(x)\delta(y)\delta(z)$. It satisfies correct radiation conditions at infinity for $z \geq 0$ (equ. (2.6)-(2.7) in condition (b)), and will be our choice for the Green's function above the plane for arbitrary $g = \sin\theta_1$, except for $\mathrm{Re}(ik\cos\theta_1) = 0$ when $\mathrm{Re}(g) \leq 0$ (i.e. except on the cut of $\mathcal{J}_g$).

### 3.2.2 The Green's functions $G_b$ below the impedance plane

The function $G_a$ cannot be used to describe the field in the cavity, when it is influenced by fictitious sources on the aperture, in particular because of the presence of a logarithmic singularity of $\mathcal{J}_g(\rho, -z)$ for negative $z$ when $\rho = 0$.

However, we can consider $\mathcal{J}_{-g}(\rho, z)$ instead of $\mathcal{J}_g(\rho, -z)$, and obtain an original Green's function $G_b$, which is suitable for an integral representation of the field in the cavity, and continues to satisfy the impedance boundary condition (2.3). This choice will be corrected in the vicinity of $g = 1$ to take account of the singularity of $\mathcal{J}_{-g}$ at this point.

#### The function $G_b$ for $g \neq 1$

We remark that, below the plane where $z + z' < 0$, the function

$$G_b(r, r') = G^0(x - x', y - y', z - z') + G^s_{-g}(x - x', y - y', z + z'), \tag{3.17}$$

with

$$G^s_{-g}(r) = \frac{e^{-ik|r|}}{k|r|} - 2ige^{ikgz}\mathcal{J}_{-g}(\rho, z), \tag{3.18}$$

continues to satisfy the impedance boundary condition (2.3) on the plane $z = 0$, is regular for $z + z' < 0$, except for the singularity of $G^0$ at $z = z'$, and verifies in $R^3$,

$$\begin{aligned}(\Delta + k^2)G_b(r, r') &= \frac{-4\pi}{k}(\delta(r - r')+ \\ &+ \delta(r - r'_{im}) + 2ikge^{ikg(z+z')}U(z + z')\delta(x - x')\delta(y - y'))\end{aligned} \tag{3.19}$$

where $r'_{im} \equiv (x', y', -z')$, $\delta(r) \equiv \delta(x)\delta(y)\delta(z)$.

This will be our choice for the Green's function below the plane, except in the vicinity of $g = 1$ (where $\mathcal{J}_{-g}$ is singular) and on the cut of $\mathcal{J}_{-g}$ (the case with $\mathrm{Re}(ik\cos\theta_1) = 0$ has to be taken in the sense of the limit). Let us notice that it satisfies the usual radiation conditions at infinity, similar to (b) but in lower space instead of upper space.

**Remark 4.** *In the case of a cavity $\Omega_2$ filled with a material of wave number $k_2$ instead of $k$, and the conditions of continuity $p|_{z=0^-} = p|_{z=0^+}$ and $\partial_z p|_{z=0^-} = a_2 \partial_z p|_{z=0^+}$, we consider $G_b$ with the parameter $g_2$ instead of $g$ satisfying $k_2 g_2 = a_2 k g$ so that $(\partial_z p(r') - ik_2 g_2 p(r'))|_{z=0^-} = a_2(\partial_z p(r') - ikgp(r'))|_{z=0^+}$.*

### A suitable choice for $G_b$ when $g \simeq 1$, regular on the cut of $\mathcal{J}_{-g}$

The function $\mathcal{J}_{-g}(\rho, z)$ is singular at $g = 1$. However, considering the equations (3.6) and the domain of regularity of $\mathcal{J}_g$ [3], the function $\mathcal{J}_{-g}(\rho, z) + 2K_0(a)$ is regular for $\rho \neq 0$ in vicinity of $g = 1$, as $g = \sin\theta_1$ varies, with $a = ik\epsilon\rho\cos\theta_1$, $\epsilon =$sign(Re$(ik\cos\theta_1)$). We can then use that

$$K_0(a) + \ln(a) I_0(a) \tag{3.20}$$

is an entire function of $a$ [19], and

$$(\Delta + k^2)(e^{ik\sin\theta_1 z} I_0(ik\epsilon\rho\cos\theta_1)) = 0, \tag{3.21}$$

and choose to add the term

$$D_b(r, r') = 4ig\ln(ikd\cos\theta_1) I_0(ik\cos\theta_1\sqrt{(x-x')^2 + (y-y')^2}) e^{ikg(z+z')}, \tag{3.22}$$

to $G_b$ for $g \simeq 1$, where $d$ is an arbitrary constant. So defined,

$$G_b(r, r') = G^0(x - x', y - y', z - z') + G^s_{-g}(x - x', y - y', z + z') + D_b(r, r'), \tag{3.23}$$

becomes regular for Re$g \geq 0$, and presents, as $g$ varies, the same cut and singularities as $G_a$ for Re$g \leq 0$.

This function continues to satisfy the impedance boundary condition (2.3) on the plane $z = 0$, is regular for $z + z' < 0$ except for the singularity of $G^0$, and verifies (3.19). The corrective term $D_b(r, r')$ does not satisfy the usual radiation conditions at infinity but it will be of no consequence for our demonstration in further sections, and this function can be used when $|ik\epsilon\rho\cos(\theta_1)| \ll 1$ is verified in the whole cavity.

**Remark 5.** *For $g \to 1$, we notice [3] that*

$$\begin{aligned}\mathcal{J}_{-g}(\rho, z) = E_1\Big(\frac{ik(1+g)(|r|+z)}{2}\Big) - 2K_0(a) + \\ + O\Big(ik(1-g)(|r|-z) E_2\Big(\frac{ik(1+g)(|r|+z)}{2}\Big)\Big)\end{aligned} \tag{3.24}$$

*and thus*

$$G^s_{-g}(r) + D_b(r) \to \frac{e^{-ik|r|}}{k|r|} - 2ie^{ikz}(E_1(ik(|r|+z)) + 2\ln(\rho/d)), \tag{3.25}$$

*which is regular for $z < 0$, $\rho \to 0$, since $|r| + z = \frac{\rho^2}{|r|-z}$ and $E_1(v) = -\ln(v) + O(1)$.*

### 3.2.3 Some additional properties of $G_{a(,b)}(r,r')$

From the derivative of $\mathcal{J}_g$ given in (3.8), we have

$$
\begin{aligned}
&(\frac{\partial .}{\partial z'} - ikg)G_{a(,b)}(r,r') \\
&= (\frac{\partial .}{\partial z'})(G^0(r-r') + G^0(r-r'_{im})) - ikg(G^0(r-r') - G^0(r-r'_{im})), \qquad (3.26)
\end{aligned}
$$

where $r'_{im} \equiv (x', y', -z')$. This leads us to write, when $z = 0$,

$$(\frac{\partial .}{\partial z'} - ikg)G_{a(,b)}(r,r')|_{z=0} = (\frac{\partial .}{\partial z'})(2G^0(r-r'))|_{z=0}, \tag{3.27}$$

and, when $z' \to 0$, $z \neq 0$,

$$(\frac{\partial .}{\partial z'} - ikg)G_{a(,b)}(r,r') \to 0. \tag{3.28}$$

These properties will be particularly useful to prove the continuity of the normal derivative of the field, deduced from our solution, through the aperture of the cavity.

Moreover, for our choice of $G_b$ for $g \neq 1$ (in section 3.2.2), we have

$$
\begin{aligned}
&G_b(r,r') = G_a(r,r') + 4ige^{ikg(z+z')}K_0(a) \\
&G_b(r,r')|_{g=v} = G_a(r_{im}, r'_{im})|_{g=-v} \\
&(G_a(r,r') + G_b(r,r'))|_{z=z'=0} = 4(\frac{e^{-ik\rho}}{k\rho} + ig(\mathcal{J}_g(\rho,0) + K_0(a))), \qquad (3.29)
\end{aligned}
$$

while, for our choice of $G_b$ for $g \simeq 1$ (in section 3.2.2),

$$
\begin{aligned}
&G_b(r,r') = G_a(r,r') + 4ige^{ikg(z+z')}(K_0(a) + \ln(ikd\cos(\theta_1))I_0(ik\rho\cos(\theta_1))) \\
&(G_a(r,r') + G_b(r,r'))|_{z=z'=0} = 4(\frac{e^{-ik\rho}}{k\rho} + ig(\mathcal{J}_g(\rho,0) + K_0(a) + \\
&+ \ln(ikd\cos(\theta_1))I_0(ik\rho\cos(\theta_1)))), \qquad (3.30)
\end{aligned}
$$

where

$$\mathcal{J}_g(\rho,0) + K_0(a) = -i\int_0^b e^{-a\cos\alpha}d\alpha,\ b = \mp i\ln(\epsilon\frac{(1\mp\sin\theta_1)}{\cos\theta_1}), \tag{3.31}$$

with $g = \sin\theta_1$, $a = \epsilon ik\rho\cos\theta_1$, $\epsilon$ =sign(Re($ik\rho\cos\theta_1$)). Let us also notice that, in agreement with the reciprocity principle [12], we have $G_{a(,b)}(r,r') = G_{a(,b)}(r',r)$.

**Remark 6.** *We can use (3.11) in (3.30) for $g \to 1$, and notice that, in this case,*

$$(G_a(r,r') + G_b(r,r'))|_{z=z'=0} \to 4(\frac{e^{-ik\rho}}{k\rho} - i(E_1(ik\rho) + \ln(\rho/d))). \tag{3.32}$$

**Remark 7.** *For $|r'| \to \infty$, $r - r_{im} = 2\widehat{z}(\widehat{z}.r)$, we have*

$$G_a(r,r') = \frac{e^{-ik|r'|}}{k|r'|}([e^{ik(r.r')/|r'|}(1 + e^{-2ik(\widehat{z}\frac{r'}{|r'|})\widehat{z}.r}(\frac{\widehat{z}\frac{r'}{|r'|} - g}{\widehat{z}\frac{r'}{|r'|} + g}))] + o(1)). \tag{3.33}$$

# 4 Integral representation of the field with $G_a$ and $G_b$

## 4.1 The representation of the field from the second Green's theorem

Let us consider the pressure fields $p$ and $G$, satisfying the Helmholtz equation

$$\begin{aligned}(\Delta + k^2)p &= W,\\ (\Delta + k^2)G &= W_G,\end{aligned} \tag{4.1}$$

in the domain $\Omega$, bounded by the surface $\partial\Omega$, piecewise analytic. If the functions $p$ and $G$ have the regularity which permits the application of the second Green's theorem, we can write

$$\int_\Omega W(r)\,G(r)\,dV - \int_\Omega W_G(r)\,p(r)\,dV = \int_{\partial\Omega^+} \widehat{n}.(\mathrm{grad}(G)p - \mathrm{grad}(p)G)dS, \tag{4.2}$$

where $\partial\Omega^+$ denotes the internal surface to $\Omega$, $\widehat{n}$ is the unit normal, piecewise defined, directed inside $\Omega$, and the surface integral is taken in the sense of principal value of Cauchy. Thereafter, we omit the sign for $\partial\Omega^+$, and we write $\partial\Omega$ instead of $\partial\Omega^+$.

## 4.2 The case $W_G(r) = -w\delta(r - r')$

Let us consider $W_G(r)$ as a generalized function in (4.2), with $W_G(r) = -w\delta(r - r')$, $w$ being a constant. In this case, we have

$$1_\Omega(r')p(r') - p_i(r') = \frac{1}{w}\int_{\partial\Omega} \widehat{n}.(\mathrm{grad}(G(r,r'))p - \mathrm{grad}(p)G(r,r'))dS, \tag{4.3}$$

for $r' \in \overline{\Omega}$, where

$$\begin{aligned} p_i &= -\frac{1}{w}\int_\Omega W(r)\,G(r,r')dV,\\ 1_\Omega(r') &= \int_\Omega \delta(r - r')dr = \frac{1}{4\pi}\int_{\partial\Omega} \widehat{n}\mathrm{grad}(\frac{1}{|r'-r|})dS = \frac{1}{4\pi}\int_{\partial\Omega} \frac{(r'-r)}{|r'-r|^3}\widehat{n}dS, \end{aligned} \tag{4.4}$$

and the integrals are considered in the sense of the principal value of Cauchy. The reader can easily recover $1_\Omega$, by letting $k = 0$, $G(r,r') = \frac{w}{4\pi|r'-r|}$ and $p \equiv 1$.

**Remark 8.** *$1_\Omega = 1$ in $\Omega$, $1_\Omega = 0$ in $R^3\backslash\overline{\Omega}$, and $1_\Omega$ is fractional on $\partial\Omega$ ($= \frac{1}{2}$ when $\partial\Omega$ is smooth). For an external problem in $R^3\backslash\Omega'$, the surface can be considered to be closed at infinity when the Sommerfeld conditions at infinity are satisfied (for example with $G(r,r') = \frac{we^{-ik|r'-r|}}{4\pi|r'-r|}$) so that $1_{R^3\backslash\Omega'}(r') = 1 - 1_{\Omega'}(r')$. Let us notice that, when $\widehat{n}grad(p)$ and $p$ vanishes on a continuous part of $\partial\Omega$, and $W \equiv 0$, we can use that $1_\Omega = 0$ in $R^3\backslash\overline{\Omega}$ and the analytical continuation through an hole, and conclude that $p = 0$ in $\Omega$.*

**Remark 9.** *Considering the continuity of the single-layer potential in (50), we notice that*

$$\frac{1}{w}\int_{\partial\Omega}(\widehat{n}.grad(G(r,r'))p_e(r) - q_e(r)G(r,r'))dS|_{r'\in\Omega\to r_0\in\partial\Omega}$$
$$\to (1 - 1_\Omega(r_0))p_e(r_0)+$$
$$+\frac{1}{w}p.v.\int_{\partial\Omega}(\widehat{n}.grad(G(r,r_0))p_e(r) - q_e(r)G(r,r_0))dS, \tag{4.5}$$

*when $p_e$ is continuous on $\partial\Omega$, $q_e(r)G(r,r')$ is summable and its integral is continuous.*

## 4.3 Integral representation of the field above the plane and in the cavity

### 4.3.1 Integral representation of the field above the plane

From the definitions of $G_a$ and $p = p_s + p_{inc}$, we can use the second Green's theorem for $\Omega$ tending to the infinite half-space $\Omega_1$ above the plane. Indeed, considering the condition (b), and the impedance boundary condition (2.3), satisfied by $p$ and $G_a$ on the plane $z = 0$, the surface integral at infinity and on $S_0 \backslash S_1$ vanishes, so that we obtain,

$$(1_{\Omega_1}(r') + 1_{\Omega_1}(r'_{im}))p(r') - p_i(r') = \frac{-k}{4\pi}\int_{S_1} G_a(r,r')(\partial_z p(r) - ikgp(r))dS, \tag{4.6}$$

for $z \geq 0$, where $(1_{\Omega_1}(r') + 1_{\Omega_1}(r'_{im})) = 1$ since $r'_{im} \equiv (x', y', -z')$, and

$$p_i(r') = \frac{-k}{4\pi}\int_{\Omega_1} W(r)\, G_a(r,r')dV \tag{4.7}$$

is the field in presence of the plane without cavity.

### 4.3.2 Integral representation of the field in the cavity

From the definitions of $G_b$ and $p = p_s + p_{inc}$, we can use the second Green's theorem in the domain $\Omega_2$ of the cavity, which gives us,

$$(1_{\Omega_2}(r') + 1_{\Omega_2}(r'_{im}))p(r') + \frac{k}{4\pi}\int_{\Omega_2} W(r)\, G_b(r,r')\, dV =$$
$$= \frac{k}{4\pi}\int_{\partial\Omega_2} \widehat{n}.(\mathrm{grad}(G_b(r,r'))p - \mathrm{grad}(p)G_b(r,r'))dS, \tag{4.8}$$

where $1_{\Omega_2}(r') = \int_{\Omega_2}\delta(r - r')dr = \frac{1}{4\pi}\int_{\partial\Omega_2}\frac{(r'-r)}{|r'-r|^3}\widehat{n}dS$, $\widehat{n}$ is the unit normal to $S_2$ directed inside $\Omega_2$, and $r'_{im} \equiv (x', y', -z')$.

Considering that the source $W$ is above the plane, and that $G_b$ (resp. $p$) satisfies the impedance boundary condition (2.3) (resp. (2.8)), the equation (4.8) becomes

$$(1_{\Omega_2}(r') + 1_{\Omega_2}(r'_{im}))p(r') = \frac{k}{4\pi}\int_{S_1} G_b(r,r')(\partial_z(p(r)) - ikgp(r))dS+$$
$$+\frac{k}{4\pi}\int_{S_2} p(r)(\partial_n G_b(r,r') - ikg_c G_b(r,r'))dS, \tag{4.9}$$

for $z' \leq 0$, where $\partial_n(.) = \widehat{n}.\text{grad}(.)$, and we notice that,

$$
\begin{aligned}
&(1_{\Omega_2}(r') + 1_{\Omega_2}(r'_{im})) = 1 \text{ in } \overline{\Omega}_2 \backslash \overline{S}_2, \\
&1_{\Omega_2}(r'_{im}) = 0 \text{ in } \overline{\Omega}_2 \backslash \overline{S}_1, \\
&(1_{\Omega_2}(r') + 1_{\Omega_2}(r'_{im})) = 0 \text{ when } r' \notin \overline{\Omega}_2.
\end{aligned} \tag{4.10}
$$

**Remark 10.** *Even if $\partial_n G_b(r, r')|_{r \in S_2}$ diverges when $r' \notin S_2 \to r$, it is continuous when $r'$ belongs to smooth parts of $S_2$.*

# 5 The integral equations on the aperture $S_1$ and on the surface of the cavity $S_2$

On the aperture $S_1$, we can substract the equation (4.6) from (4.9), and obtain

$$
\begin{aligned}
&((1_{\Omega_2}(r') + 1_{\Omega_2}(r'_{im}) - 1)p(r') + p_i(r'))|_{r' \in S_1} = \\
&= \frac{k}{4\pi} \int_{S_1} (G_a(r, r') + G_b(r, r'))(\partial_z p(r) - ikgp(r))dS \\
&+ \frac{k}{4\pi} \int_{S_2} p(r)(\partial_n(G_b(r, r')) - ikg_c G_b(r, r'))dS,
\end{aligned} \tag{5.1}
$$

where we notice that $(1_{\Omega_2}(r') + 1_{\Omega_2}(r'_{im})) = 1$ on $S_1$, except possibly on $S_1 \cap S_2$, while, on the surface $\overline{S}_2$ of the cavity, we can write, from (4.9),

$$
\begin{aligned}
&(1_{\Omega_2}(r') + 1_{\Omega_2}(r'_{im}))p(r')|_{r' \in \overline{S}_2} = \frac{k}{4\pi} \int_{S_1} G_b(r, r')(\partial_z p(r) - ikgp(r))dS + \\
&+ \frac{k}{4\pi} \int_{S_2} p(r)(\partial_n(G_b(r, r')) - ikg_c G_b(r, r'))dS,
\end{aligned} \tag{5.2}
$$

where $1_{\Omega_2}(r') = \int_{\Omega_2} \delta(r - r')dr = \frac{1}{4\pi} \int_{\partial\Omega_2} \widehat{n}.\text{grad}(\frac{1}{|r'-r|})dS$ ($= \frac{1}{2}$ on smooth parts), the surface integrals are taken in the sense of principal value of Cauchy, and $\partial_n(.) = \widehat{n}.\text{grad}(.)$, $\widehat{n}$ is the unit normal to $S_2$ directed inside $\Omega_2$.

The integral equations (5.1)-(5.2) represent a system for two unknowns,

$$
\begin{aligned}
&q_1(r) = (\partial_z p(r) - ikgp(r))|_{r \in S_1}, \\
&p_2(r) = p(r)|_{r \in S_2},
\end{aligned} \tag{5.3}
$$

respectively on the aperture and on the surface of cavity, whose solution permits to express the field everywhere.

# 6 Uniqueness property of the integral equations

From the conditions of regularity given in (a), we consider the solutions of our integral equations (5.1)-(5.2), $q_1(r)$ (on the aperture) and $p_2(r)$ (on the surface of the cavity), such that $q_1 = O(r^\alpha)$, $-1 < \alpha \leq 0$, as the distance to edges or corners vanishes, and $p_2$ is continuous. We then study the uniqueness of $q_1$ and $p_2$ when $\mathrm{Re}g > 0$ and $\mathrm{Re}g_c > 0$, or $g = g_c = 0$, and verify that $q_1$ and $p_2$ vanish when $p_i \equiv 0$.

For this, we show that we can define a field $p_e(r')$, derived from $q_1$, $p_2$ and $p_i$, which verifies $p_2(r') = p_e(r')$ on $S_2$ and $q_1(r') = \partial_z p_e(r') - ikgp_e(r')$ on $S_1$, and satisfies the boundary value problem with the conditions of uniqueness given in section 2.

## 6.1 A field $p_e(r')$ derived from $p_2$ and $q_1$

We consider the field $p_e$ derived, from $q_1$ and $p_2$, following

$$p_e(r')|_{r'\in\Omega_1} = \frac{-k}{4\pi}\int_{S_1} G_a(r,r')q_1(r)dS + p_i(r'), \tag{6.1}$$

in the domain $\Omega_1$ above the plane, and,

$$p_e(r')|_{r'\in\Omega_2} = (1 - (1_{\Omega_2}(r') + 1_{\Omega_2}(r'_{im})))p_2(r') + \frac{k}{4\pi}\int_{S_1} G_b(r,r')q_1(r)dS + \\ + \frac{k}{4\pi}\int_{S_2} p_2(r)(\partial_n G_b(r,r') - ikg_c G_b(r,r'))dS, \tag{6.2}$$

in the domain $\Omega_2$ of the cavity, where the surface integrals are taken in the sense of principal value of Cauchy.

The expression (6.2) verifies, like $G_b$, the Helmholtz equation in $\Omega_2$, while (6.1) satisfies, like $G_a$, the Helmholtz equation in $\Omega_1$ with the radiation conditions at infinity given in (b), and the impedance conditions on $S_0\backslash S_1$. Moreover, from the equation of continuity (4.5), the function $p_e(r')$ is continuous up to $S_2$.

It then remains to verify the continuity through the aperture $S_1$ of the cavity, the impedance boundary condition on $S_2$, and to analyze the expressions of $q_1$ and $p_2$ with $p_e$. Therefore, we show that we have,

- $p_e(r') = p_2(r')$ on the surface of the cavity $S_2$;
- the continuity of $p_e(r')$ through the aperture $S_1$;
- the continuity of $\partial_z p_e(r') - ikgp_e(r')$ through $S_1$;
- $\partial_z p_e(r') - ikgp_e(r') = q_1(r')$ on $S_1$;
- $\partial_n p_e = ikg_c p_2$ on $S_2$,

in the case $p_i \equiv 0$, considered for the uniqueness.

## 6.2 $p_e(r') = p_2(r')$ on $S_2$

Substracting the integral equation (5.2) from (6.2) for $r' \in S_2$, we obtain

$$p_e(r') + (1_{\Omega_2}(r') + 1_{\Omega_2}(r'_{im}) - 1)\, p_2(r') = (1_{\Omega_2}(r') + 1_{\Omega_2}(r'_{im}))p_2(r'), \tag{6.3}$$

on $S_2$, and thus,

$$p_e(r')|_{r'\in S_2} = p_2(r'). \tag{6.4}$$

### 6.3 Continuity of $p_e(r')$ through $S_1$

The integrals in the expressions (6.1) and (6.2) of $p_e(r')$ remain convergent when the point of observation approaches the aperture respectively above and below $S_1$. Morever, $(1_{\Omega_2}(r') + 1_{\Omega_2}(r'_{im})) = 1$ in $\overline{\Omega}_2 \backslash S_2$, and, from the integral equation (5.1) with $p_i \equiv 0$, the expressions (6.1) and (6.2) tend to the same limit, which proves the continuity of $p_e(r')$ through the aperture $S_1$.

### 6.4 Continuity of $\partial_{z'} p_e(r') - ikgp_e(r')$, equal to $q_1(r')$ on $S_1$

Using (3.27) in the expressions (6.1) and (6.2) of $p_e(r')$, we can write

$$\begin{aligned}
\partial_{z'} p_e(r') - ikgp_e(r')|_{z'=h>0} &= \frac{-k}{4\pi}(\frac{\partial.}{\partial z'}) \int_{S_1} 2G^0(r,r')q_1(r)dS|_{z'=h}, \\
\partial_{z'} p_e(r') - ikgp_e(r')|_{z'=-h<0} &= \frac{k}{4\pi}(\frac{\partial.}{\partial z'}) \int_{S_1} 2G^0(r,r')q_1(r)dS|_{z'=-h}+ \\
+ \frac{k}{4\pi}(\frac{\partial.}{\partial z'} - ikg) \int_{S_2} p_2(r)(\partial_n G_b(r,r') &- ikg_c G_b(r,r'))dS|_{z'=-h}, \qquad (6.5)
\end{aligned}$$

We then apply that,

$$\begin{aligned}
&(\frac{\partial.}{\partial z'} - ikg)G_b(r,r') \to 0 \, \text{when} \, z' \to 0, z \neq 0, \\
&(\frac{\partial.}{\partial z'})G^0(r(x,y,0),r')|_{z'=h} = -(\frac{\partial.}{\partial z'})G^0(r(x,y,0),r')|_{z'=-h}, \qquad (6.6)
\end{aligned}$$

This implies that the contribution of the integral term along $S_2$ vanishes when $h \to 0$, and that we have the continuity of $\partial_z p_e(r') - ikgp_e(r')$ through the aperture $S_1$.

Moreover, we notice that

$$\pm (\partial_{z'} p_e(r') - ikgp_e(r'))|_{z'=h>0} \to \frac{-k}{4\pi}(\frac{\partial.}{\partial z'}) \int_{S_1} 2G^0(r,r')q_1(r)dS|_{z'=\pm h}, \qquad (6.7)$$

when $h \to 0$, while, by application of the discontinuity property of the normal derivative of the single-layer potential [20] and substraction of the relations in (6.7) for plus and minus signs, we can write

$$\partial_z p_e(r') - ikgp_e(r') = q_1(r') \text{ on } S_1. \qquad (6.8)$$

### 6.5 $\partial_n p_e(r') = ikg_c p_e$ on $S_2$

The field $p_e(r')$, defined by (6.2), satisfies the Helmholtz equation in $\Omega_2$, and we can write in this domain, from the second Green's theorem,

$$\begin{aligned}
(1_{\Omega_2}(r') + 1_{\Omega_2}(r'_{im}))p_e(r') &= \frac{k}{4\pi} \int_{S_1} G_b(r,r')(\partial_z p_e(r) - ikgp_e(r))dS+ \\
+ \frac{k}{4\pi} \int_{S_2} (p_e(r)\partial_n G_b(r,r') &- G_b(r,r')\partial_n p_e(r))dS. \qquad (6.9)
\end{aligned}$$

We have proved that $\partial_z(p_e(r)) - ikgp_e(r) = q_1(r)$ on $S_1$ and $p_e(r) = p_2(r)$ on $S_2$, and substracting (6.9) from (6.2), we obtain, for $r' \in \Omega_2$,

$$\frac{k}{4\pi}\int_{S_2} G_b(r,r')\mu(r)dS = 0, \tag{6.10}$$

with $\mu(r) \equiv \partial_n p_e(r) - ikg_c p_e(r)$.

The surface $S_2$, bounded by the curve $C_1$, is open, and, considering the domain of analyticity of $G_b(r,r')$, we can use the analytic continuation principle through $S_1$. So, the potential

$$\mathcal{P}(r') = \frac{k}{4\pi}\int_{S_2} G_b(r,r')\mu(r)dS, \tag{6.11}$$

vanishes in the domain $\Omega \equiv \Omega_2 \cup \Omega_2^i$, where $\Omega_2^i$ (resp. $S_2^i$) is the symmetric of $\Omega_2$ (resp. $S_2$) relatively to the plane $z = 0$.

From the properties of $G_b$, $\mathcal{P}$ is also regular in $R^3\backslash(\Omega \cup \Omega_c)$, where $\Omega_c$ is the upper part of the cylinder along $z$-axis bounded by $S_2^i$. It is then possible to prove that $\mu \equiv 0$. For this, two distinct proofs are detailed in appendix A, successively for $g = 0$ or $g \to \infty$, and, for $g \neq 0$, $|g| < \infty$.

# 7 Some simplifications of the integral equations for a shallow cavity

The integrals with $\partial_n(\frac{e^{-ik|r-r'_{im}|}}{k|r-r'_{im}|})$ terms, in the equations (5.1)-(5.2), become difficult to calculate when $|r - r'_{im}| \to 0$ and the depth vanishes. Therefore, we develop our integral equations in a new form, and analytical expressions are derived.

## 7.1 A new form of the integral terms for shallow cavity

For a shallow cavity, we let

$$\begin{aligned}
&G_{bs}(r,r') = G_b(r,r') - G_{st}(r,r'),\\
&G_{st}(r,r') = \frac{1}{k|r-r'|} + \frac{1}{k|r-r'_{im}|},\\
&r'_2(r_1) \in S_2,\ r_1 \in S_1,
\end{aligned} \tag{7.1}$$

where $r'_2(.)$ is a projection of $S_1$ on $S_2$ with $r'_2(r_1) \to r_1$ when $r'_2(r_1) \to S_1$. We then consider the domain $\Omega$ defined so that $1_\Omega(r') = 1_{\Omega_2}(r') + 1_{\Omega_2}(r'_{im})$, and notice that

$$\begin{aligned}
1_\Omega(r')p_2(r') &= \frac{p_2(r')}{4\pi}\int_{\partial\Omega} \widehat{n}\,\mathrm{grad}(\frac{1}{|r-r'|})dS,\\
&= \frac{k}{4\pi}\int_{S_2} \widehat{n}\,\mathrm{grad}(G_{st}(r,r'))p_2(r')dS,
\end{aligned} \tag{7.2}$$

We can use this equality, and derive a new form of integrals along $S_2$ in our system of equations.

So, we obtain, for $r' \in S_1$,

$$\begin{aligned}
p_i(r') - p_2(r'_2(r')) = \frac{k}{4\pi} \int_{S_1} (G_a(r, r') + G_b(r, r')) q_1(r) dS + \\
+ \frac{k}{4\pi} \int_{S_2} p_2(r)(\partial_n(G_{bs}(r, r')) - ikg_c G_b(r, r')) dS + \\
+ \frac{k}{4\pi} \int_{S_2} (p_2(r) - p_2(r'_2(r'))) \partial_n(G_{st}(r, r')) dS,
\end{aligned} \tag{7.3}$$

while, for $r' \in \overline{S}_2$,

$$\begin{aligned}
- k \int_{S_2} (p_2(r) - p_2(r')) \partial_n(G_{st}(r, r')) dS = k \int_{S_1} G_b(r, r') q_1(r) dS + \\
+ k \int_{S_2} p_2(r)(\partial_n(G_{bs}(r, r')) - ikg_c G_b(r, r')) dS.
\end{aligned} \tag{7.4}$$

Comparing with previous integral equations system, we notice that the term $\partial_n(\frac{1}{k|r - r'_{im}|})$ is multiplied by terms that vanish as $|r - r'_{im}| \to 0$, so that the difficulty of calculus for a small cavity depth has disappeared. Let us remark that this modification can be applied whenever a part of $S_2$ is close to $S_1$.

## 7.2 The limit case of an impedance patch

In the limit case where $S_2 \equiv S_1$, the integral with $\partial_n G_{st}(r, r')$ vanishes, and $\partial_n(G_{bs}(r, r')) = ikg\, G_b(r, r')$, so that we obtain, for $r' \in S_1$,

$$\begin{aligned}
k \int_{S_1} G_b(r, r')(q_1(r) + ik(g - g_c) p_2(r)) dS|_{z'=0^-} = 0, \\
p_i(r') - p_2(r') = \frac{k}{4\pi} \int_{S_1} G_a(r, r') q_1(r) dS|_{z'=0^+},
\end{aligned} \tag{7.5}$$

where $q_1(r)$ and $p_2(r)$ are assumed to be continuous on $\overline{S}_1$. The first equation implies $q_1(r) = ik(g_c - g)p_2(r)$ (see appendix C), which leads us to recover the well-known integral equation [21] for an impedance patch,

$$p_2(r') - p_i(r') = \frac{k}{4\pi} \int_{S_1} G_a(r, r') ik(g - g_c) p_2(r) dS. \tag{7.6}$$

**Remark 11.** *Let us notice that*

$$k \int_{S_1} G_a(r, r') \mu(r) dS|_{z'=0^+} = 0, \tag{7.7}$$

*for $r' \in S_1$, $\mu(r)$ continuous on $\overline{S}_1$, implies $\mu(r) \equiv 0$ (see appendix C).*

## 7.3 On some approximations for a small cavity, and validation.

### 7.3.1 Approximate expressions for a small cavity

For small dimensions with $kd_c = k\frac{\int_{\Omega_2} dV}{\int_{S_1} dS} \ll 1$ and $k^2 \int_{S_1} dS \ll 1$, we assume that

$$p_2(r) - p_c = o(kd_c),\ q_1(r) - q_c = o(kd_c),\ p_c = \frac{\int_{S_2} p_2 dS}{\int_{S_2} dS},\ q_c = \frac{\int_{S_1} q_1(r) dS}{\int_{S_1} dS}, \tag{7.8}$$

and that, the terms

$$\begin{aligned}
&\int_{S_2} (p_2(r) - p_c) \int_{S_2} \partial_n (G_{bs(,st)}(r,r')) dS' dS, \\
&\int_{S_1} (q_1(r) - q_c) \int_{S_{2(,1)}} G_{b(,a)}(r,r') dS' dS,
\end{aligned} \tag{7.9}$$

are negligible in our calculus. We can then determinate $q_c$ and $p_c$, after integration over $S_2$ and $S_1$ of integral equations, and obtain an approximate expression of the radiated field.

We notice first that we have

$$\begin{aligned}
&(\Delta + k^2) G_{bs}(r,r') = -k^2 G_{st}(r,r'),\ \ r' \in \Omega_2,\ r \in \Omega_2, \\
&\int_{\partial\Omega_2 \equiv S_2 \cup S_1} \partial_n G_{bs}(r,r') dS = k^2 \int_{\Omega_2} G_{st}(r,r') dV,\ r' \in \Omega_2, \\
&\partial_z G_{bs}(r,r') = ikg G_b(r,r'),\ r' \in \Omega_2,\ r \in S_1,
\end{aligned} \tag{7.10}$$

and thus,

$$\int_{S_2} \partial_n G_{bs}(r,r') dS = \int_{S_1} ikg\, G_b(r,r') dS + k^2 \int_{\Omega_2} G_{st}(r,r') dV,\ r' \in \Omega_2. \tag{7.11}$$

Then, summing the integral equation (7.4) over $S_2$ and using (7.11), we obtain

$$\begin{aligned}
&q_c \int_{S_1} \int_{S_2} G_b(r,r') dS' dS = (ikp_c( \int_{S_2} g_c \int_{S_2} G_b(r,r') dS' dS - \\
&- \int_{S_1} g \int_{S_2} G_b(r,r') dS' dS) - k^2 p_c \int_{\Omega_2} \int_{S_2} G_{st}(r,r') dS' dV)(1 + o(1)),
\end{aligned} \tag{7.12}$$

and deduce that

$$q_c = ikp_c[(r_c(g_c) - g) + ikl_c](1 + o(1)), \tag{7.13}$$

where

$$\begin{aligned}
r_c(g_c) &= \frac{\int_{S_2} g_c \int_{S_2} G_b(r,r') dS' dS}{\int_{S_1} \int_{S_2} G_b(r,r') dS' dS} \sim \frac{\int_{S_2} g_c dS}{\int_{S_1} dS}, \\
l_c &= \frac{\int_{\Omega_2} \int_{S_2} G_{st}(r,r') dS' dV}{\int_{S_1} \int_{S_2} G_b(r,r') dS' dS} \sim \frac{\int_{\Omega_2} dV}{\int_{S_1} dS}.
\end{aligned} \tag{7.14}$$

We then consider the integral equation (7.3), sum it over $S_1$, and use (7.11). This gives us,

$$\int_{S_1} p_i\,(r)dS - p_c \int_{S_1} dS \;= \frac{kq_c}{4\pi} \int_{S_1} \int_{S_1} G_a(r,r')dS'dS(1+o(1)). \tag{7.15}$$

which leads us, from (7.13), to the approximate expressions of $p_c$ and $q_c$,

$$p_c = \frac{\int_{S_1} p_i\,(r)dS / \int_{S_1} dS}{1 + \frac{ik}{4\pi}[(r_c(g_c) - g) + ikl_c]\frac{k\int_{S_1}\int_{S_1} G_a(r,r')dS'dS}{\int_{S_1} dS}},$$
$$q_c = ikp_c[(r_c(g_c) - g) + ikl_c], \tag{7.16}$$

where $p_i$ is the field radiated in presence of a plane without perturbation.

Therefore, we can use the expression of $q_c$ in (4.6), and obtain, for the field diffracted by a shallow cavity above the plane,

$$p(r') - p_i\,(r') = \frac{-k}{4\pi}\; q_c \int_{S_1} G_a(r,r')dS\,(1+o(1)), \tag{7.17}$$

in particular for the far field.

**Remark 12.** *In the case of a cavity $\Omega_2$ filled with a homogenous material of wavenumber $k_2$, and the conditions of continuity $p|_{z=0^-} = p|_{z=0^+}$ and $\partial_z p|_{z=0^-} = a_2\partial_z p|_{z=0^+}$, we can consider $G_b$ with $k_2$ instead of $k$, and $g_2 = a_2kg/k_2$ in place of $g$. In this case, we have $q_1|_{z=0^-} = (\partial_z p(r') - ik_2g_2p(r'))|_{z=0^-} = a_2(\partial_z p(r') - ikgp(r'))|_{z=0^+} = a_2q_1|_{z=0^+}$.*

*We can then modify the integral equations and obtain*

$$q_c|_{z=0^-} = ik_2p_c[(r_c(g_c') - g_2) + ik_2l_c],\; q_c|_{z=0^-} = a_2q_c|_{z=0^+},$$
$$p(r') - p_i\,(r') = \frac{-k}{4\pi}q_c|_{z=0^+} \int_{S_1} G_a(r,r')dS(1+o(1)), \tag{7.18}$$

*when $(\frac{\partial}{\partial n} - ik_2g_c')p|_{S_2} = 0$. Generally, the relation between $g_c'$ and $g_c$ (surface impedance in free space) is $g_c' = a_2g_c/k_2$, and thus,*

$$q_c|_{z=0^+} = ip_c[k(r_c(g_c) - g) + ik_2^2l_c/a_2]. \tag{7.19}$$

**Remark 13.** *A similar demonstration can be used for a small protuberance $\Omega_2$ of surface $S_2$ above the plane. By the argument of analytic continuation, we consider that the radiation is equivalent to a fictitious $q_1$ over $S_1$. In this case, (7.17) applies with*

$$-\,q_c \int_{S_1}\int_{S_2} G_a(r,r')dS'dS = -ikp_c\Big(\int_{S_2} g_c \int_{S_2} G_a(r,r')dS'dS -$$
$$-\int_{S_1} g \int_{S_2} G_a(r,r')dS'dS\Big) - k^2p_c \int_{\Omega_2}\int_{S_2} G_{st}(r,r')dV + o(1),$$
$$\int_{S_1} p_i\,(r)dS - p_c \int_{S_1} dS = \frac{kq_c}{4\pi} \int_{S_1}\int_{S_1} G_a(r,r')dS'dS(1+o(1)), \tag{7.20}$$

*and,*

$$p_c = \frac{\int_{S_1} p_i(r)dS / \int_{S_1} dS}{1 + \frac{ik}{4\pi}[(r'_c(g_c) - g) - ikl'_c]\frac{k\int_{S_1}\int_{S_1} G_a(r,r')dS'dS}{\int_{S_1} dS}},$$

$$q_c = ikp_c[(r'_c(g_c) - g) - ikl'_c]\,,\; r'_c(g_c) \sim r_c(g_c),\, l'_c \sim l_c. \tag{7.21}$$

*Let us notice that $+ikl_c$ is replaced by $-ikl'_c$ when we compare with (7.16).*

**Remark 14.** *To our knowledge, our approximate expressions are original, but a similar low frequency analysis could also be done with the integral equations given in [1].*

### 7.3.2 Validation in the case of a small cylindrical cavity with impedance wall

For the validation, we choose to verify the expression of the impedance on the aperture, given, from our results (7.13)-(7.15), by

$$\eta_a = \frac{\int_{S_1} \frac{\partial p}{\partial z} dS / \int_{S_1} dS}{ikp_c} = \frac{q_c}{ikp_c} + g = r_c(g_c) + ikl_c \sim \frac{\int_{S_2} g_c dS}{\int_{S_1} dS} + ik\frac{\int_{\Omega_2} dV}{\int_{S_1} dS}, \tag{7.22}$$

in some particular case with well-tabulated results.

For this, we consider the delicate problem of a cylindrical cavity of radius $a$ and depth $d$ with an imperfectly reflective surface, characterized by impedances $g_{cw}$ on the wall and $g_{ce}$ on the bottom, with $ka = o(1)$ and $d/a = O(1)$.

So, from (7.22), we have,

$$\eta_a \sim \frac{g_{ce}\pi a^2 + g_{cw} 2\pi a d}{\pi a^2} + ik\frac{\pi a^2 d}{\pi a^2} = g_{ce} + \frac{2 g_{cw} d}{a} + ikd, \tag{7.23}$$

while, from the modal expansion of the field [22],

$$\eta_m = \frac{\frac{\partial p}{\partial z}}{ikp}|_{S_1} \simeq \frac{\alpha_1}{k}\frac{(1 + \frac{g_{ce} - \frac{\alpha_1}{k}}{g_{ce} + \frac{\alpha_1}{k}} e^{-2i\alpha_1 d})}{(1 - \frac{g_{ce} - \frac{\alpha_1}{k}}{g_{ce} + \frac{\alpha_1}{k}} e^{-2i\alpha_1 d})} \sim g_{ce} + i\alpha_1^2 \frac{d}{k} \simeq g_{ce} + \frac{2 g_{cw} d}{a} + ikd,$$

$$-ikag_{cw} J_0(\xi_1) + \xi_1 J_1(\xi_1) = 0,\; \alpha_1^2 = k^2 - (\frac{\xi_1}{a})^2 \simeq k^2 - \frac{2ikg_{cw}}{a}\,. \tag{7.24}$$

As expected for a small cavity, $\eta_m$ perfectly recovers $\eta_a$, and the expression (7.22) is validated.

**Remark 15.** *For a perfectly rigid small cavity, we have $g_c = 0$ and thus $\eta_a = ikl_c$, and we recover the result given in [14, equ.(3)-(6)].*

**Remark 16.** *Similar developments can be made in electromagnetism, from the use of tensors for the expression of potentials (see appendix E). In this case, for the E (electric) and H (magnetic) fields on the aperture of a small cavity, we derive the approximations,*

$$-Z_0(J_c \wedge \widehat{z})g^e + M_c \sim Z_0(J_c \wedge \widehat{z})((\frac{\int_{S_2} g_c dS}{\int_{S_1} dS} - g^e) + ik\frac{\int_{\Omega_2} dV}{\int_{S_1} dS}),$$

$$M_c = -(\widehat{z} \wedge E)|_{S_1}, J_c = (\widehat{z} \wedge H)|_{S_1}, \tag{7.25}$$

*when impedance boundary conditions are considered on the perturbated plane following*

$$E - \widehat{n}(\widehat{n}E)|_{S_0\backslash S_1} = Z_0 g^e(\widehat{n} \wedge H)|_{S_0\backslash S_1}, E - \widehat{n}(\widehat{n}E)|_{S_2} = Z_0 g_c^e(\widehat{n} \wedge H)|_{S_2}, \tag{7.26}$$

*where* $\widehat{n}$ *is the outward normal to the surface,* $Z_0$ *is the free space impedance [3, 4],[12]. From (7.25), the problem for a shallow cavity is then reduced to the one of the scattering by a patch of relative impedance* $(\frac{\int_{S_2} g_c dS}{\int_{S_1} dS} g_c^e + ik\frac{\int_{\Omega_2} dV}{\int_{S_1} dS})$ *on* $S_1$ *inserted in an impedance plane. Equations, similar to (4.6) and (7.6), can be then derived.*

# 8 Conclusion

We have developed novel integral equations which permit to simplify the calculus of the field scattered by a cavity in an impedance plane. For this, a new Green's function is used for the expression of the field in the cavity which leads to reduce the number of unknowns. Moreover, a particular attention is paid to the uniqueness of the solution. In the case of a small cavity, our equations are detailed and developed in a new form. In this case, analytical results are derived and our expression for approximate aperture impedance is validated.

# A $\int_{S_2\textbf{(open)}} G_b(r, r')\mu(r)dS|_{r'\in\Omega_2\cup\Omega_2^i} = 0$ implies $\mu \equiv 0$

This appendix concerns the study of the solution $\mu(r)$ of

$$\mathcal{P}(r') = 0 \text{ in } \Omega \equiv \Omega_2 \cup \Omega_2^i, \tag{A.1}$$

where $\mathcal{P}(r') = \frac{k}{4\pi}\int_{S_2} G_b(r, r')\mu(r)dS$, and the proof that $\mu(r)$ (in some functions class) vanishes. $S_2$ is the surface of an open cavity in the plane $z = 0$, and the domain $\Omega_2^i$ (resp. $S_2^i$) is the symmetric of $\Omega_2$ (resp. $S_2$) relatively to $z = 0$.

## A.1 $\mu \equiv 0$ in the cases $g = 0$ (Neumann) or $g \to \infty$ (Dirichlet)

In the respective cases $g = 0$ (Neumann boundary condition) and $g \to \infty$ (Dirichlet boundary condition), we have

$$\begin{aligned} G_b(r, r')|_{g=0} &= [G^0(r - r') + G^0(r - r'_{im})] \\ G_b(r, r')|_{g\to\infty} &= [G^0(r - r') - G^0(r - r'_{im})] \end{aligned} \tag{A.2}$$

and thus,

$$\begin{aligned} \mathcal{P}(r')|_{g=0} &= \frac{k}{4\pi}\int_{\partial\Omega} G^0(r - r')\Xi_0(r)dS, \\ \mathcal{P}(r')|_{g\to\infty} &= \frac{k}{4\pi}\int_{\partial\Omega} G^0(r - r')\Xi_\infty(r)dS \end{aligned} \tag{A.3}$$

where $\Xi_0(r_{im}) = \Xi_0(r) = \mu(r)$ and $\Xi_\infty(r_{im}) = -\Xi_\infty(r) = -\mu(r)$. We assume that $\mu$ is a function, piecewise continuous (except possibly for singularities of $\mu$ at the edge of $\partial\Omega$), so that $\mathcal{P}$ is continuous on $\partial\Omega$. We can then use a proof similar to the ones given by Colton and Kress in [20] to prove that $\mu(r) \equiv 0$.

The potential $\mathcal{P}$ vanishes in $\Omega$, and thus, by continuity, on $\partial\Omega$. Moreover, $\mathcal{P}$ satisfies the Helmholtz equation and the Sommerfeld radiation condition at infinity in $R^3$. Hence by Rellich's uniqueness theorem generalized by Levine for non smooth domain [15], $\mathcal{P}(r')$ also vanishes outside $\Omega$. We can then conclude, from the discontinuity property of the normal derivative of the single layer potential [20],

$$\frac{\partial \mathcal{P}(r')}{\partial n}|_+ - \frac{\partial \mathcal{P}(r')}{\partial n}|_- = -\Xi(r'), \tag{A.4}$$

at any non singular points of $S_2$, where $\Xi$ is $\Xi_0$ (resp. $\Xi_\infty$) when $g = 0$ (resp. $g \to \infty$). that we have $\Xi \equiv 0$ and thus $\mu \equiv 0$.

## A.2 A proof that $\mu \equiv 0$ for $g \neq 0$, $|g| < \infty$

In the definition taken when $g \neq 1$ in section 3.2.2, we notice that $G_b(r, r')|_{g=v} = G_a(r_{im}, r'_{im})|_{g=-v}$, and the problem is then equivalent to a boundary value problem in the upper half-space, concerning a perturbation in relief (image of the cavity) on a plane of impedance $-g$, with a field $u(r') = \mathcal{P}(r'_{im})$ vanishing inside and on the surface of the perturbation, and verifying the Sommerfeld conditions at infinity. For $\mathrm{Re}(-g) > 0$, we can use for this problem the uniqueness theorem of Levine [15, sect.7] and the discontinuity property of the normal derivative of the single layer potential [20], and deduce that $\mu \equiv 0$.

For $\mathrm{Re}(g) > 0$, this demonstration is no more valid, and we develop here a more general proof which uses that $S_2$ is an open surface.

### A.2.1 Definition of the function $\mathcal{P}_1$

For this, we begin to define new functions $\mathcal{P}_0$ and $\mathcal{P}_1$, and we write,

$$\begin{aligned}
\mathcal{P}(r') &= (\mathcal{P}_0 + 2ig\mathcal{P}_1) \\
\mathcal{P}_0(r') &= \frac{k}{4\pi}\int_{S_2} (G^0(r - r') + G^0(r - r'_{im}))\mu(r)dS, \\
\mathcal{P}_1(r') &= \frac{k}{4\pi}\int_{S_2} \mathcal{V}_b(r - r'_{im})\mu(r)dS
\end{aligned} \tag{A.5}$$

where, from (3.8), the function $\mathcal{V}_b(r) = -e^{ikgz}\mathcal{J}_{-g}(\rho, z)$ satisfies

$$\begin{aligned}
&\frac{\partial \mathcal{V}_b(r - r'_{im})}{\partial z} = \frac{e^{-ik|r - r'_{im}|}}{|r - r'_{im}|} + ikg\mathcal{V}_b(r - r'_{im}) \\
&= kG^0(r - r'_{im}) + ikg\mathcal{V}_b(r - r'_{im}).
\end{aligned} \tag{A.6}$$

with $r'_{im} \equiv (x', y', -z')$. We notice that $\mathcal{V}_b(r)$ is regular for $z < 0$, and has a weak singularity, like $\ln \rho$, at $\rho = 0$ for $z \geq 0$. Thus, the potential $\mathcal{P}_1(r')$ is an analytic function in $R^3 \backslash \Omega_c$, where $\Omega_c$ is the upper part of the cylinder along $z$-axis bounded by $S_2^i$, which is the image of $S_2$.

### A.2.2 A problem for $\mathcal{P}_1$ equivalent to the problem $\mathcal{P} \equiv 0$

Since $\mathcal{P}$ vanishes in $\Omega$, and $\mathcal{P}_0(r'_{im}) = \mathcal{P}_0(r')$, we can write that $\mathcal{P}_1(r'_{im}) = \mathcal{P}_1(r')$ in this domain. So, we have

$$\begin{aligned}
&\mathcal{P}_1(r') = \frac{k}{4\pi} \int_{S_2} \mathcal{V}_b(r - r'_{im}) \mu(r) dS \\
&\mathcal{P}_1(r'_{im}) = \mathcal{P}_1(r'),\ r' \in \Omega \equiv \Omega_2 \cup \Omega_2^i \\
&(\Delta + k^2)\mathcal{P}_1 = 0 \text{ in } R^3 \backslash \Omega_c,
\end{aligned} \tag{A.7}$$

where $S_2$ is an open surface. This implies reciprocally that $\mathcal{P} = 0$ in $\Omega$. Indeed, from (A.6), we have

$$\begin{aligned}
\partial_{z'} \mathcal{P}_1(r') - ikg\mathcal{P}_1(r') &= \frac{k}{4\pi} \int_{S_2} kG^0(r - r'_{im}) \mu(r) dS, \\
-\partial_{z'} \mathcal{P}_1(r'_{im}) - ikg\mathcal{P}_1(r'_{im}) &= \frac{k}{4\pi} \int_{S_2} kG^0(r - r') \mu(r) dS.
\end{aligned} \tag{A.8}$$

Adding both equations and using that $\mathcal{P}_1(r'_{im}) = \mathcal{P}_1(r')$, $\partial_{z'}(\mathcal{P}_1(r') - \mathcal{P}_1(r'_{im}))$ vanishes and $\mathcal{P}_1(r'_{im}) + \mathcal{P}_1(r') = 2\mathcal{P}_1(r')$, and we conclude, by definition of $\mathcal{P}$, that $\mathcal{P} = 0$ in $\Omega$.

### A.2.3 A proof that $\mu \equiv 0$, by the analysis of the singularities at the ends of $S_2$

The singularities of the field at the ends of $S_2$, i.e. the singular behaviour in vicinity of the curve $C_1$, depends on the geometry. For this, we denote $\widehat{n}_0$, the unit vector, normal to $C_1$ at $r_0$ and orthogonal to the normal $\widehat{n}$ to $S_2$, and $\widehat{c}$ the unit vector tangent to $C_1$, so that $(\widehat{c}, \widehat{n}, \widehat{n}_0)$ is an orthonormal basis (figure 4), and $(\rho, \varphi)$ the cylindrical coordinates associated to $(\widehat{n}, \widehat{n}_0)$, with $\rho \cos\varphi = \widehat{n}_0.(r - r_0)$, $\rho \sin\varphi = -\widehat{n}.(r - r_0)$. We also denote $\widehat{y}$ the unit vector perpendicular to $\widehat{z}$ and to $\widehat{c}$ so that $(\widehat{c}, \widehat{y}, \widehat{z})$ is an orthonormal basis.

Let us consider $S'_2$, a part of $S_2$ bounded by an analytic arc $C'_1$ of $C_1$, and consider to simplify, without losing generality, that the function $\mu(r)$ satisfies

$$\begin{aligned}
\mu(r) &= \mu_f(r) + \mu_a(r) \\
\mu_f(r) &= \sum_{p \geq 1} a_p J_{\alpha_p}(k\rho), \\
\mu_a(r) &= \sum_{m \geq 0} b_m \rho^m,
\end{aligned} \tag{A.9}$$

on $S'_2$ where the $\alpha_p$ are not entire numbers, $\alpha_p < \alpha_{p+1}$, $\alpha_1 > -1$, $a_1 \neq 0$ except if $\mu_f \equiv 0$, and $J_\nu(z) = (\frac{z}{2})^\nu \sum_{k \geq 0} \frac{(-z/4)^k}{k!\Gamma(\nu+k+1)}$ is the bessel function of order $\nu$ [19]. The terms

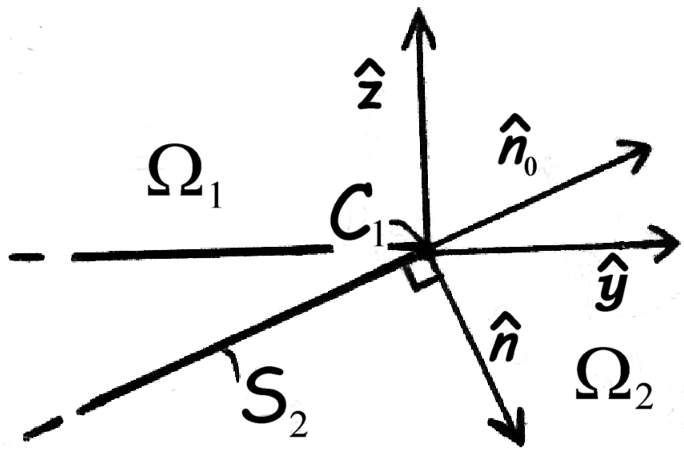


Figure 4: definitions of unit vectors on the curve $C_1$ defining the aperture

with powers of $\ln \rho$ could be considered in the method but are omitted for simplification. Thereafter, we prove that the conditions (A.7) on $\mathcal{P}_1$ imply the vanishing of $\mu_f$ and $\mu_a$ on $S_2'$, and that, by the continuation principle through a hole and the nullity of $\mathcal{P}$ in $\Omega_2$, $\mu \equiv 0$ on $S_2$. To simplify the analysis, we will only detail the demonstration in the case where $(\widehat{y}.\widehat{n}_0) = \cos\Phi' \neq 0$.

$\mu_a = 0$ **on** $S_2'$ **when** $\widehat{y}.\widehat{n}_0 = \cos\Phi' \neq 0$ **on** $C_1'$

Let us consider the analytic part of $\mu$ in vicinity of $C_1'$ and the singularities of $\mathcal{P}_1$ induced by it. Since we have

$$(\partial_z(\partial_y \mathcal{P}_1)(r) - ikg(\partial_y \mathcal{P}_1)(r))|_{r=r'_{im}} = \partial_y \frac{k}{4\pi}\int_{S_2} kG^0(r-r')\mu(r)dS, \qquad \text{(A.10)}$$

a singularity appears (see [23] or appendix B), following

$$\partial_y \frac{k}{4\pi}\int_{S_2} kG^0(r-r')\mu(r)dS = -\frac{k\cos\Phi'}{2\pi}\mu(r_0)\ln|r'-r_0| + O(1), \qquad \text{(A.11)}$$

as $r'$ tends normally to $r_0 \in C_1'$. This implies, from $\partial_y \mathcal{P}_1(r') = O(1)$, that

$$(\partial_z(\partial_y \mathcal{P}_1(r))|_{r=r'_{im}} = -\frac{k\cos\Phi'}{2\pi}\mu(r_0)\ln|r'-r_0| + O(1) \qquad \text{(A.12)}$$

Considering the parity of $\mathcal{P}_1(r')$ (see (A.7)), and thus of $\partial_y \mathcal{P}_1(r')$, $\partial_z\partial_y\mathcal{P}_1$ is odd with respects to the plane $z=0$, and (A.12) implies that $\mu$ vanishes on $C_1'$ so that $b_0 = 0$ in (A.9). In the same manner, the case of higher order terms of $\mu_a$, $b_1\rho^1$, $b_2\rho^2$, ... can be considered successively with higher order $y$-derivatives of $\mathcal{P}_1(r')$, so that $b_m = 0$, $m \geq 0$.

**$\mu_f = 0$ on $S_2'$ for arbitrary $\cos\Phi'$ on $C_1'$**

Let us consider the fractional part $\mu_f$ of $\mu$, and the (single layer) potential induced (or radiated) by it, which corresponds, to the expression of $\partial_z \mathcal{P}_1(r) - ikg\mathcal{P}_1(r)$ presented in (A.8). $S_2'$ is assumed to simplify with null curvature, and the results obtained in appendix B are used. Under this hypothesis, the potential has a fractional part of order $1+\alpha_1$ ($\sim \rho^{1+\alpha_1}$ as $\rho \to 0$), which is thus the fractional order of $\partial_z \mathcal{P}_1$. We then deduce that $\mathcal{P}_1$ has a fractional order $2+\alpha_1$. Considering the results in appendix B, the term $J_{\alpha_1}(k\rho)$ of $\mu_f$ radiates, like $\frac{4\pi}{k\sin(\nu\pi)} J_{1+\alpha_1}(k\rho')\cos((1+\alpha_1)\varphi') + O(J_{3+\alpha_1}(k\rho'))$, which does not contains $\rho^{2+\alpha_1}$ terms in its expansion, and thus the order $2+\alpha_1$ of $\mathcal{P}_1$ comes from the next term $a_2 J_{\alpha_2}(k\rho)$ in the expansion of $\mu_f$. This implies $\alpha_2 = \alpha_1 + 1$, and $a_2 \neq 0$ if $a_1 \neq 0$. Consequently, when $a_1 \neq 0$, we can write,

$$\begin{aligned}
&\frac{k}{4\pi}\int_{S_2} kG^0(r-r')\mu(r)dS = \\
&= \frac{1}{\sin(\alpha_1\pi)}(a_1 J_{1+\alpha_1}(k\rho')\cos((1+\alpha_1)\varphi') - a_2 J_{2+\alpha_1}(k\rho')\cos((2+\alpha_1)\varphi')) \\
&+ O(J_{3+\alpha_1}(k\rho')) + O(J_{1+\alpha_3}(k\rho')) + \text{entire function of } \rho'
\end{aligned} \tag{A.13}$$

as $\rho' \to 0$, with $\rho'\cos\varphi' = \widehat{n}_0.(r'-r_0)$, $\rho'\sin\varphi' = -\widehat{n}.(r'-r_0)$, $\alpha_3 > \alpha_2 = \alpha_1 + 1$. Thus, from (A.8), we have

$$\begin{aligned}
&(\partial_z \mathcal{P}_1(r)) - ikg\mathcal{P}_1(r))|_{r=r'_{im}} \\
&= \frac{1}{\sin(\alpha_1\pi)}(a_1 J_{1+\alpha_1}(k\rho')\cos((1+\alpha_1)\varphi') - a_2 J_{2+\alpha_1}(k\rho')\cos((2+\alpha_1)\varphi')) \\
&+ O(J_{3+\alpha_1}(k\rho')) + O(J_{1+\alpha_3}(k\rho')) + \text{entire function of } \rho'
\end{aligned} \tag{A.14}$$

Therefore, from (A.7), and the parity of $\mathcal{P}_1$ and $\partial_z\mathcal{P}_1$, we derive that

$$\begin{aligned}
a_1\cos((1+\alpha_1)\Phi' + \varphi) &= -a_1\cos((1+\alpha_1)\Phi' - \varphi), \\
a_2\cos((2+\alpha_1)\Phi' + \varphi) &= a_2\cos((2+\alpha_1)\Phi' - \varphi).
\end{aligned} \tag{A.15}$$

Consequently, when $a_1 \neq 0$, we can write,

$$\begin{aligned}
\cos((1+\alpha_1)\Phi') &= 0 \\
\sin((2+\alpha_1)\Phi') &= 0
\end{aligned} \tag{A.16}$$

This implies $\cos\Phi' = 0$, and $\alpha_1$ is entire, which is impossible by definition. We then deduce that the first order coefficient $a_1$ of $\mu_f$ is null, which induces, by definition, that $\mu_f \equiv 0$.

**$\mu$ vanishes on $S_2'$ implies $\mu \equiv 0$**

From the previous results, it exists a subdomain $S_2'$ of $S_2$ where $\mu = 0$, that we can substract of the support of $\mu$, assuming without losing generality, that $|\cos\Phi'| \neq 1$ along $C_1'$. In this case, we can use the continuation principle through the hole $S_2'$, and the field $\mathcal{P}(r')$, null in $\overline{\Omega}_2$, also vanishes outside the cavity below the plane $z = 0$.

Noticing the regularity of $\mathcal{P}_1(r')$ for $z' < 0$, and thus the continuity of the normal derivative of $\mathcal{P}_1(r')$ through $S_2$, we can apply the discontinuity property of the normal derivative of single-layer potentials with free space Green's function [20],

$$\frac{\partial \mathcal{P}(r')}{\partial n}|_+ - \frac{\partial \mathcal{P}(r')}{\partial n}|_- = -\mu(r), \tag{A.17}$$

at any non singular points of $S_2$, which implies, from the vanishing of the left side, that $\mu \equiv 0$.

### A.2.4 Elements of proof for the particular case $\widehat{y}.\widehat{n}_0 = \cos\Phi' = 0$ on $C_1$

From the previous analysis, the fractional part $\mu_f$ vanishes for any $\Phi'$, and we can then assume that $\mu$ is analytic. In the case where $\widehat{y}.\widehat{n}_0 = \cos\Phi' = 0$ on $C_1$, we choose to study the function,

$$\mathcal{P}'(r') = (\partial_{z'} - ikg)\mathcal{P}(r') = \frac{k}{4\pi}\int_{S_2} G_b'(r, r')\mu(r)dS \tag{A.18}$$

where, from (3.26),

$$\begin{aligned} G_b'(r, r') &= \partial_{z'}(G^0(r - r') + G^0(r - r_{im}')) - ikg(G^0(r - r') - G^0(r - r_{im}')) \\ &= (-\partial_z - ikg)(G^0(r - r') - G^0(r - r_{im}')) \end{aligned} \tag{A.19}$$

From $G_b'(r, r') = -G_b'(r, r_{im}')$, we have $\mathcal{P}'(r') = -\mathcal{P}'(r_{im}')$ . The function $\mathcal{P}'$ satisfies the Helmholtz equation in $R^3\backslash\Omega$, with Sommerfeld conditions at infinity. Since $\mathcal{P} = 0$ in $\Omega$, $\mathcal{P}'$ vanishes, like its derivatives, in $\Omega$. Let us show that it is also the case for $\mathcal{P}'$ outside $\Omega$, then for $\mathcal{P}$, and thus for $\mu$.

Since $\mathcal{P}'$ vanishes along the plane $z = 0$, $(\widehat{z}.\text{grad})^{2n}\mathcal{P}' = 0$ along $C_1$, $n \in N$. Using integration by parts and continuity for odd derivatives of $\mathcal{P}'$, we derive that $(\widehat{z}.\text{grad})^{2n+1}\mathcal{P}' = 0$ along $C_1$, $n \in N$. We then consider to simplify that the surface $S_c$ of the cylinder $\Omega_c'$ along $z$-axis, defined with a section $C_1$, does not have common points with $S_2$, except on $C_1$, and that we have no essential sigularity. We then deduce that $\mathcal{P}'$ vanishes on $S_c$. From the first Green's theorem and the properties of $\mathcal{P}'$, we have

$$\text{Re}(\int_{R^3\backslash\Omega_c'} -ik|\mathcal{P}'(r)|^2 + \frac{|\text{grad}\mathcal{P}'(r)|^2}{-ik}dV) = \lim_{a\to\infty}\int_{r=a}|\mathcal{P}'(r)|^2 dS, \tag{A.20}$$

Since $|\arg(ik)| \leq \pi/2$, both members have opposite signs, and thus vanish. We then derive, from Rellich's theorem and continuation principle [12], that $\mathcal{P}'(r) = 0$ in $R^3\backslash\Omega$. Since $\mathcal{P}$ and all $z$-derivatives of $\mathcal{P}'$ vanishes on $S_2$, we derive that $\mathcal{P} = 0$ below $S_2$, and, by continuation principle, everywhere below the plane $z = 0$.

We can then use the discontinuity property of the normal derivative of single layer potential (A.17), and deduce that $\mu \equiv 0$.

# B Behaviour of single layer potentials on open surfaces

Let $S$ be an open analytic, orientable surface in three-dimensional space bounded by a Jordan curve $C$, and $C'$ an arc belonging to it. Let $r'$ and $r$ be two points, and $\mu(r)$ an analytical function defined for all $r \in S$ except possibly for a singularity on the edge $C'$. We study the behaviour of single layer potentials

$$U_0(r') = \int_S \frac{\mu(r)}{|r-r'|} dS,\; U_k(r') = \int_S \frac{\mu(r) e^{-ik|r-r'|}}{|r-r'|} dS \tag{B.1}$$

## B.1 When $\mu(r) = O(1)$ on $C'$

If $\mu(r)$ is finite on $C'$, we can write, from Rolf Leis [23], in vicinity of $C'$

$$\begin{aligned}\operatorname{grad}(U_0(r')) = &- \int_S \mu(r)\widehat{n}\partial_n \frac{1}{|r-r'|} dS + \int_S \frac{\operatorname{grad}_S(\mu(r))}{|r-r'|} dS + \\ &+ 2\int_S \frac{\widehat{n} H \mu(r)}{|r-r'|} dS - \int_C \frac{\widehat{n}_0 \mu(r)}{|r-r'|} dc\end{aligned} \tag{B.2}$$

where $\widehat{n}_0$ is a unit vector, normal to $C$ and orthogonal to the normal $\widehat{n}$, $\operatorname{grad}_S$ is the surface gradient, $H$ is a function depending on the characteristics of the surface. The line integral becomes logarithmically singular, while the other surface integrals are regular. The singularity, as $r' \notin C \to r_0$, $r_0$ being the projection of $r'$ on $C'$, can be described by

$$\int_C \frac{\widehat{n}_0 \mu(r)}{|r-r'|} dc = -2\widehat{n}_0(r_0)\mu(r_0) \ln|r'-r_0| + O(1) \tag{B.3}$$

where $\widehat{c}$ is the unit vector, tangent to $C'$ at $r_0$, and $(\widehat{c}, \widehat{n}, \widehat{n}_0)$ is an orthonormal basis [23]. More generally we notice from [23], that analytic $\mu$ does not induce singularities of fractional order in the expansion of $U_0$ and $U_k$.

## B.2 When $\mu(r)$ is of fractional order

In the case of $\mu(r)$ of fractional order (with fractional power of $|r - r_0|$ near $r_0 \in C'$), it is possible to analyze the fractional part of the field, letting the curvature of the edge $C'$ tending to 0, and $\mu(r)$ depending only on the distance $\rho$ to the edge. In this case, we can write,

$$\begin{aligned}U_k(r') &\sim -i\pi \int_L \mu_f(\rho) H_0^{(2)}(k|\overline{\rho} - \overline{\rho}'|) d\rho \\ &\sim -i \int_{-i\infty}^{+i\infty} \int_0^{\infty} \mu_f(\rho) e^{-ik\rho\cos\alpha} d\rho e^{-ik\rho'\cos(\alpha-\varphi')} d\alpha,\end{aligned} \tag{B.4}$$

when $\rho' \to 0$, $\rho'$ denoting the radial distance to the edge of the point $r'$, with $\rho'\cos(\varphi') = \widehat{n}_0(r'-r_0)$, $\rho'\sin(\varphi') = -\widehat{n}(r'-r_0)$, $r_0 \in C'$, and $\mu_f(\rho) = \mu(r)$.

So, for $\mu_f(\rho) = J_\nu(k\rho) \sim (\frac{\beta}{k})^{-\nu} \lim_{\beta\to 0} J_\nu(\beta\rho)$, $\nu = \alpha_1 > -1$, $\nu \neq 0, 1, 2, ...$, we obtain $U_k(r')$, from [24, eq. 6.611.1] and [19, eq. 9.1.22], following

$$\begin{aligned}
U_k(r') &\sim -\frac{ie^{-i(1+\nu)\pi/2}}{k\,2^\nu}\int_{-i\infty}^{+i\infty}\frac{1}{(\cos\alpha)^{1+\nu}}e^{-ik\rho'\cos(\alpha-\varphi')}d\alpha \\
&\sim -\frac{ie^{-i(1+\nu)\pi/2}}{k\,2^\nu}\int_{0}^{+i\infty}(\frac{1}{(\cos(\alpha+\varphi'))^{1+\nu}}+\frac{1}{(\cos(\alpha-\varphi'))^{1+\nu}})e^{-ik\rho'\cos\alpha}d\alpha \\
&\sim -\frac{i\,4e^{-i(1+\nu)\pi/2}}{k}\cos((1+\nu)\varphi')\int_{0}^{+i\infty}e^{i(1+\nu)\alpha}e^{-ik\rho'\cos\alpha}d\alpha \\
&\sim \frac{4\pi}{k\sin(\nu\pi)}\cos((1+\nu)\varphi')(J_{1+\nu}(k\rho') + \text{an entire function of } \rho').
\end{aligned} \tag{B.5}$$

Then, using the discontinuity property of the normal derivative of $U_k$ through $S$ [20], and $2(1+\nu)J_{1+\nu}(k\rho')/k\rho' = J_\nu(k\rho') + J_{2+\nu}(k\rho')$ [19], we can rewrite (B.5) following

$$U_k(r') = \frac{4\pi}{k\sin(\nu\pi)}J_{1+\alpha_1}(k\rho')\cos((1+\alpha_1)\varphi') + D_a(r') + O(J_{3+\alpha_1}(k\rho')) \tag{B.6}$$

where $D_a(r')$ is an entire function of $\rho'$.

**Remark 17.** *In the case of logarithmic behaviour, we can let* $\mu(\rho) = \frac{1}{2}\ln(\frac{\rho}{2}) = \lim_{\nu\to 0^+}\partial_\nu(K_\nu(\rho)/\Gamma(\nu))$, *derive, from [24, eq. 6.611.3],*

$$\begin{aligned}
U_k(p) &\sim -i\int_{-i\infty}^{+i\infty}\partial_\nu(\frac{\Gamma(1-\nu)\sin\nu\alpha}{\sin\alpha})|_{\nu=0}e^{-ik\rho'\cos(\alpha-\varphi')}d\alpha, \\
&\sim -i\int_{-i\infty}^{+i\infty}(\gamma\alpha+\frac{\alpha}{\sin\alpha})e^{-ik\rho'\cos(\alpha-\varphi')}d\alpha,\ \gamma = .577... \\
&\sim -i\gamma\varphi'\int_{-i\infty}^{+i\infty}e^{-ik\rho'\cos\alpha}d\alpha + o(\ln\rho) = 2\gamma\varphi' K_0(ik\rho) + o(\ln\rho).
\end{aligned} \tag{B.7}$$

**Remark 18.** *Let* $t_0(r_0) = a\widehat{n}_0(r_0) + b\widehat{n}(r_0)$ *when* $t_0\widehat{n}_0 \neq 0$. *Considering higher derivatives of* $U_0$, *we can write*

$$\begin{aligned}
(t_0.grad)^n(U_0(r')) = &-\int_S (t_0.grad_S)^{n-1}(\mu(r))(t_0.\widehat{n})\frac{\partial}{\partial n}(\frac{1}{|r-r'|})dS + \\
&+ 2\int_S \frac{(t_0.\widehat{n})H(t_0.grad_S)^{n-1}(\mu(r))}{|r-r'|}dS + \int_S \frac{(t_0.grad_S)^n(\mu(r))}{|r-r'|}dS - \\
&- t_0\int_C \frac{(t_0.\widehat{n}_0)(t_0.grad_S)^{n-1}(\mu(r))}{|r-r'|}dc + \mathcal{R}_n
\end{aligned} \tag{B.8}$$

*when* $(\widehat{n}_0 grad_S)^j(\mu(r_0)) = 0$ *(or* $(t_0(r_0)grad_S)^j(\mu(r_0)) = 0$ *when* $t_0\widehat{n}_0 \neq 0$*) on* $C'$ *for* $j < n-1$, *and* $(\widehat{n}_0 grad_S)^{n-1}(\mu(r_0)) = O(1)$, *with, in this case,* $\mathcal{R}_n$ *which is continuous on* $C'$. *This result also applies if we replace* $U_0$ *by* $U_k$ *since the behaviour of highest rank is the same for* $U_k$ *and* $U_0$.

# C About $\int_{S_1} G_{b(a)}(r,r')\mu(r)dS = 0$ on the aperture

Let us show that

$$\mathcal{U}_{b(a)}(r') \equiv \int_{S_1} G_{b(a)}(r,r')\mu(r)dS = 0 \text{ on } S_1 \tag{C.1}$$

implies $\mu(r) \equiv 0$, when $\mu(r) = A_0 + o(1)$ as $r \to r_c \in \partial S_1 \equiv C_1$, $A_0$ is a constant.

## C.1 The case with $G_b$

From the analysis of Rolf Leis (see [23] or appendix $B$), $\mu(r) = A_0 + o(1)$ (as $r \to r_c \in \partial S_1 = C_1$) induces a singularity of derivative in vicinity of $C_1$ of the form $A_0 \ln|r - r_c|$. This implies, from (C.1) (i.e. $\mathcal{U}_b(r')|_{S_1} = 0$), that $A_0 = 0$.

We then choose to define the following functions $u$ and $w$,

$$\begin{aligned} u(r') &= \frac{-k}{4\pi}\int_{S_1} G_b(r,r')\mu(r)dS \text{ with } u(r') = 0 \text{ on } S_1, \\ w(r') &= (\frac{\partial .}{\partial z'} - ikg)u(r') = \frac{-2k}{4\pi}\int_{S_1}(\frac{\partial .}{\partial z'})G^0(r-r')\mu(r)dS, \end{aligned} \tag{C.2}$$

where we have used that

$$\begin{aligned} &(\frac{\partial .}{\partial z'} - ikg)G_b(r,r') \\ &= (\frac{\partial .}{\partial z'})(G^0(r-r') + G^0(r-r'_{im})) - ikg(G^0(r-r') - G^0(r-r'_{im})). \end{aligned} \tag{C.3}$$

Considering the property of the double layer potential with free space Green's function $G^0$, and $F$ the radiation pattern (or scattering diagram) of $w$, we can write

$$\begin{aligned} w(r') &= -\mu(r') = O(1) \text{ on } S_1,\ w(r') = 0 \text{ on } S_0\backslash\overline{S}_1, \\ w(r') &= \frac{e^{-ik|r'|}}{|r'|}(F(\frac{r'}{|r'|}) + o(1)) \text{ when } r' \to \infty. \end{aligned} \tag{C.4}$$

Moreover, we have, from Leis's second theorem [23],

$$\mathrm{grad}(w(r)) = o(1/|r - r_c|), \tag{C.5}$$

when $r \to r_c \in C_1$, and, from $u(r) = 0$ on $S_1$,

$$\lim_{z\to 0^-}\frac{\partial w(r)}{\partial z} = (k^2 - \frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2})u(r) - ikgw(r) \to -ikg\,w(r) \text{ on } S_1. \tag{C.6}$$

Thus, we can apply the Green's first theorem on the domain $z < 0$, and we obtain

$$\begin{aligned} &\mathrm{Re}\,(\int_{z\le 0^-} -ik|w(r)|^2 + \frac{|\mathrm{grad} w(r)|^2}{-ik}dV) = (\int_{z=0^-}\mathrm{Re}(g)|w(r)|^2 dS + \\ &+ \mathrm{Re}\,(\int_0^{2\pi}\int_{\pi/2}^{\pi}|F(\Theta,\phi)|^2 \sin\Theta d\Theta d\phi). \end{aligned} \tag{C.7}$$

For $\mathrm{Re}(g) \geq 0$ and $|\arg(ik)| \leq \pi/2$, the left-hand term is $\leq 0$, while the right-hand term is $\geq 0$, and thus both terms vanish. So, we have,

$$\begin{aligned} &w(r) = 0 \text{ as } z < 0, \text{ when } |\arg(ik)| < \pi/2, \mathrm{Re}(g) \geq 0, \\ &w(r) = 0 \text{ as } z = 0^-, \text{ when } |\arg(ik)| = \pi/2,\ \mathrm{Re}(g) > 0, \end{aligned} \tag{C.8}$$

which implies in these cases, from $w(r') = -\mu(r')$ on $S_1$, that $\mu$ vanishes.

In the case $g = 0$, $G_b$ can be replaced by $2G_0$ in the definition of $u$, and the demonstration of Colton and Kress [20, sect. 2] can be directly used to conclude that $\mu \equiv 0$.

**Remark 19.** *the same property can be deduced for Reg $< 0$, except along the branch-cut of $G_b$ with Re($ik\cos\theta_1$)$= 0$, $g = \sin\theta_1$. For this, we can directly use the first Green's theorem with $u$ instead of $w$, and deduce that $\mu \equiv 0$.*

## C.2 The case with $G_a$

If we consider in the definitions of $u$, $G_a$ instead of $G_b$, and the domain $z > 0$ instead of the domain $z < 0$, we can directly use the first Green's theorem with $u$ instead of $w$, and deduce that $\mu(r') = 0$ when $\mathrm{Re}(g) > 0$ or $g = 0$.

# D On some analytical applications for the 2D case

Let us consider our developments in [11], for a scatterer illuminated by a plane wave coming from the direction $\varphi' = \varphi'_o$, for the geometry given in figure below,

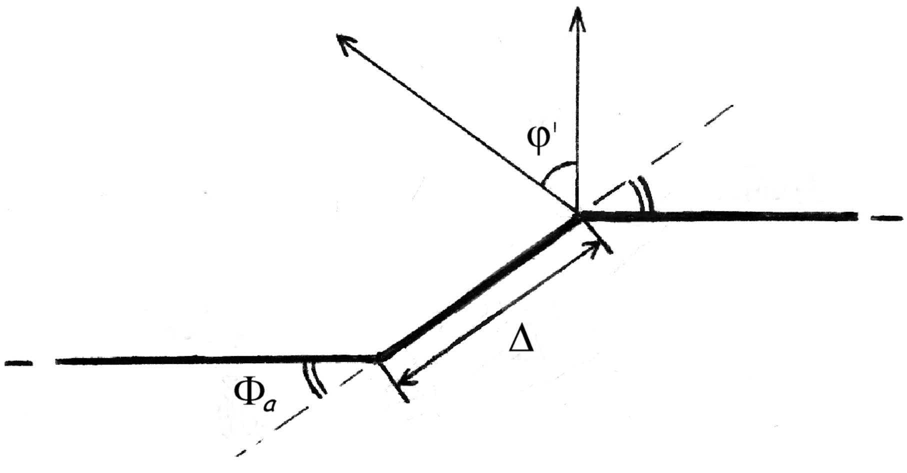


Figure 5: impedance skew step geometry

Impedance boundary conditions are assumed,

$$(\frac{\partial}{\partial n} - ikg_1)p|_{\mathcal{L}_1} = 0,\ (\frac{\partial}{\partial n} - ikg)p|_{\mathcal{L}_\pm} = 0, \tag{D.1}$$

where $\mathcal{L}_1$ is the strip of length $\Delta$, between both singularities, and the $\mathcal{L}_\pm$ are the half planes going to infinity. From [11], the diffracted field given by,

$$u_d(\varphi') = \frac{-e^{-i\pi/4 - ik\rho_a}}{\sqrt{2\pi k\rho_a}}[F(\varphi')] + O(1/(k\rho)^{3/2}), \tag{D.2}$$

can be approximated, when $|k\Delta| \ll 1$, with

$$\begin{aligned} F(\varphi') \sim & \frac{2D_0\, ik\Delta \cos\varphi'_\circ \cos\varphi'}{(\cos\varphi'_\circ + \sin\theta_+)(\cos\varphi' + \sin\theta_+)}(-g\cos\Phi_a - \sin\Phi_a \sin\varphi' + \\ & + g_1 - \frac{ik\Delta}{4}\sin(2\Phi_a)) + \frac{C_0(\cos\varphi' - g)\cos\varphi'_\circ}{(\cos\varphi'_\circ + g)(\sin\varphi' + \sin\varphi'_\circ)}(e^{-2ik\Delta\sin\Phi_a\cos\varphi'} - 1) \end{aligned} \tag{D.3}$$

where

$$\begin{aligned} D_0 &= \frac{1 + (iB_0/\pi)\sin\varphi'_\circ \sin\Phi_a}{1 + (iB_0/\pi)(g_1 - g\cos\Phi_a)} \\ C_0 &= 1 + (iB_0/\pi)(\sin\Phi_a(\sin\varphi'_\circ + \sin\varphi')/2) \\ B_0 &= -k\Delta(\ln(k\Delta/2) + \gamma_0 - 1 + i\pi/2),\ \gamma_0 \approx .577, \end{aligned} \tag{D.4}$$

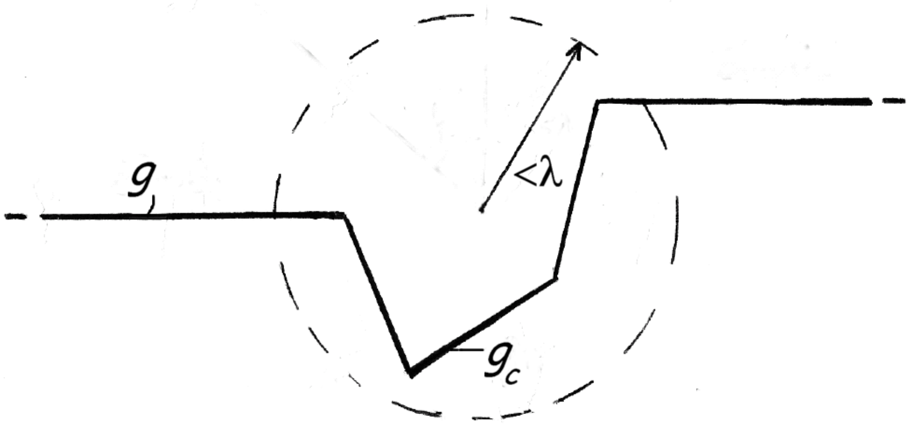


Figure 6: 2D cavity geometry

From the results of our present paper, it is possible to consider a general curved lines $\mathcal{L}_c$ instead of the straight line $\mathcal{L}_1$, with $(\frac{\partial}{\partial n} - ikg_c)p|_{\mathcal{L}_c} = 0$, if we write

$$g_1 \sim \int_{\mathcal{L}_c} g_c\, dl/\Delta + ik(S_+ - S_-)/\Delta \tag{D.5}$$

where $S_+$ is the total surface of the cavity under the straight strip $\mathcal{L}_1$, and $S_-$ is the total surface of the protuberences above $\mathcal{L}_1$. Let us notice that, in the above approximated expression of $F$, the first order in $k$ is exact.

# E The Green's tensors for an impedance plane in electromagnetism

In our method, a key point is the use of the 'below' Green's functions in the cavity which derives from our solution for an arbitrary impedance plane (passive or active). In a similar manner, an extension of our present work to electromagnetism needs the Green's tensors for an arbitrary impedance plane, that we now develop from [3]-[4]. For this, the electromagnetic field $(E, H)$ that satisfies the Maxwell equation,

$$\mathrm{curl}(E) = -ik(Z_0 H) - M,\ \mathrm{curl}(Z_0 H) = ikE + Z_0 J \tag{E.1}$$

above the plane, and the impedance boundary conditions,

$$\widehat{z} \wedge E|_{z=0} = g^e(\widehat{z} \wedge (\widehat{z} \wedge (Z_0 H)))|_{z=0}, \tag{E.2}$$

or

$$(\partial_z - ikg^e)E_z|_{z=0} = 0,\ (\partial_z - ik/g^e)H_z|_{z=0} = 0, \tag{E.3}$$

is considered.

## E.1 The field radiated by bounded sources $J$ and $M$ in free space

The incident field, radiated by the sources $J$ and $M$ in free space, is given by

$$\begin{aligned}
E_{inc} &= \mathrm{curl}(G * M) + \frac{i}{k}(\mathrm{grad}(\mathrm{div}(.)) + k^2)(G * Z_0 J) \\
&= \frac{1}{8\pi k^2}(-M * [\underline{\mathcal{D}}_{e,i}(r', r)] + Z_0 J * [\underline{\mathcal{F}}_{h,i}(r', r)]) \\
Z_0 H_{inc} &= -\mathrm{curl}(G * Z_0 J) + \frac{i}{k}(\mathrm{grad}(\mathrm{div}(.)) + k^2)(G * M) \\
&= \frac{1}{8\pi k^2}(Z_0 J * [\underline{\mathcal{D}}_{h,i}(r', r)] + M * [\underline{\mathcal{F}}_{e,i}(r', r)])
\end{aligned} \tag{E.4}$$

where $G = -\frac{e^{-ik|r|}}{4\pi|r|}$, $|r| = \sqrt{x^2 + y^2 + z^2}$, and $*$ is the convolution product.

## E.2 The field scattered by the impedance plane

### E.2.1 The tensors

Developing the expressions of potentials given in [3]-[4] for the scattered field $(E_s, H_s)$, we can write, when $M = M_{r'}\delta(r - r')$ and $J = J_{r'}\delta(r - r')$,

$$\begin{aligned}
E_s(r) &= -ik\mathrm{curl}(\mathcal{H}_s\, \widehat{z}) + (\mathrm{grad}(\mathrm{div}(.)) + k^2)(\mathcal{E}_s \widehat{z}) \\
&= \frac{1}{8\pi k^2}([\underline{\mathcal{F}}_{he}(r, r')].Z_0 J_{r'} - [\underline{\mathcal{D}}_{he}(r, r')].M_{r'}) \\
&= \frac{1}{8\pi k^2}(Z_0 J_{r'}.[\underline{\mathcal{F}}_{he}(r', r)] - M_{r'}.[\underline{\mathcal{D}}_{eh}(r', r)]) \\
&= \frac{1}{8\pi k^2}(Z_0 J_{r'}.[\frac{1}{ik}\mathrm{curl}_{r'}([\underline{\mathcal{D}}_{eh}(r', r)])] - M_{r'}.[\underline{\mathcal{D}}_{eh}(r', r)])
\end{aligned} \tag{E.5}$$

and

$$
\begin{aligned}
Z_0 H_s(r) &= ik\mathrm{curl}(\mathcal{E}_s\,\widehat{z}) + (\mathrm{grad}(\mathrm{div}(.)) + k^2)(\mathcal{H}_s\,\widehat{z})\\
&= \frac{1}{8\pi k^2}([\underline{\mathcal{D}}_{eh}(r,r')].Z_0 J_{r'} + [\underline{\mathcal{F}}_{eh}(r,r')].M_{r'})\\
&= \frac{1}{8\pi k^2}(Z_0 J_{r'}.[\underline{\mathcal{D}}_{he}(r',r)] + M_{r'}.[\underline{\mathcal{F}}_{eh}(r',r)])\\
&= \frac{1}{8\pi k^2}(Z_0 J_{r'}.[\underline{\mathcal{D}}_{he}(r',r)] + M_{r'}.[\frac{1}{ik}\mathrm{curl}_{r'}([\underline{\mathcal{D}}_{he}(r',r)])]
\end{aligned}
\tag{E.6}
$$

where $\underline{\mathcal{F}}_{he(,eh)}(r',r)$ and $\underline{\mathcal{D}}_{eh(,he)}(r',r)$ are dyadic tensors. In these notations, we have $D.[\widehat{a}\,\widehat{b}] = (D.\,\widehat{a})\,\widehat{b}$ , $[\widehat{a}\,\widehat{b}].D = \widehat{a}\,(\,\widehat{b}.D)$ and

$$
[\underline{\mathcal{G}}(r,r')] \;\rightarrow [\underline{\mathcal{G}}(r',r)] \text{ if } (x,y,z) \leftrightarrow (x',y',z') \text{ and } (\widehat{x},\widehat{y},\,\widehat{z}) \leftrightarrow (\widehat{x}',\widehat{y}',\,\widehat{z}'). \tag{E.7}
$$

The tensors verify the impedance boundary conditions,

$$
\begin{aligned}
&\widehat{z}\wedge[(\underline{\mathcal{D}}_{he} + \underline{\mathcal{D}}_{h,i})(r,r')]|_{z=0} = -g^e(\widehat{z}\wedge\widehat{z}\wedge[(\underline{\mathcal{F}}_{eh} + \underline{\mathcal{F}}_{e,i})(r,r')]|_{z=0},\\
&\widehat{z}\wedge[(\underline{\mathcal{F}}_{he} + \underline{\mathcal{F}}_{h,i})(r,r')]|_{z=0} = g^e(\widehat{z}\wedge\widehat{z}\wedge[(\underline{\mathcal{D}}_{eh} + \underline{\mathcal{D}}_{e,i})(r,r')])|_{z=0},
\end{aligned}
\tag{E.8}
$$

and can be written,

$$
\begin{aligned}
\underline{\mathcal{F}}_{he(,eh)} &\equiv -\mathcal{B}(\underline{B}_{h(,e)}) + \mathcal{A}(\underline{A}_{e(,h)})\\
\underline{\mathcal{D}}_{he(,eh)} &\equiv \mathcal{B}(\underline{A}_{h(,e)}) + \mathcal{A}(\underline{B}_{e(,h)})
\end{aligned}
\tag{E.9}
$$

where

$$
\begin{aligned}
&[\mathcal{A}(\underline{B}_{e(,h)})(r,r')] =\\
&= [ik(\widehat{x}\partial_x + \,\widehat{y}\partial_y + \widehat{z}\partial_z)(\widehat{y}'\partial_x - \widehat{x}'\partial_y)(\partial_z\mathcal{S}_{e(,h)}(r,r'))+\\
&+ ik^3\widehat{z}(\,\widehat{y}'\partial_x - \widehat{x}'\partial_y)(\mathcal{S}_{e(,h)}(r,r'))]
\end{aligned}
\tag{E.10}
$$

$$
\begin{aligned}
&[\mathcal{B}(\underline{A}_{e(,h)})(r,r')] =\\
&= [ik(\widehat{x}\partial_y - \,\widehat{y}\partial_x)(\widehat{x}'\partial_x + \widehat{y}'\partial_y + \widehat{z}'\epsilon\partial_z)(\epsilon\partial_z\mathcal{S}_{e(,h)}(r,r'))+\\
&+ ik^3(\widehat{x}\partial_y - \,\widehat{y}\partial_x)\widehat{z}'(\mathcal{S}_{e(,h)}(r,r'))]
\end{aligned}
\tag{E.11}
$$

$$
\begin{aligned}
&[\mathcal{A}(\underline{A}_{e(,h)})(r,r')] =\\
&= [(\widehat{x}\partial_x + \,\widehat{y}\partial_y + \widehat{z}\partial_z)(\widehat{x}'\partial_x + \widehat{y}'\partial_y + \widehat{z}'\epsilon\partial_z)(\epsilon\partial_{z^2}\mathcal{S}_{e(,h)}(r,r'))+\\
&+ k^2\,\widehat{z}(\widehat{x}'\partial_x + \widehat{y}'\partial_y + \widehat{z}'\epsilon\partial_z)(\epsilon\partial_z\mathcal{S}_{e(,h)}(r,r'))+\\
&+ k^2(\widehat{x}\partial_x + \,\widehat{y}\partial_y + \widehat{z}\partial_z)(\widehat{z}')(\partial_z\mathcal{S}_{e(,h)}(r,r'))+\\
&+ \,\widehat{z}\,\widehat{z}'k^4(\mathcal{S}_{e(,h)}(r,r'))]
\end{aligned}
\tag{E.12}
$$

$$
\begin{aligned}
&[\mathcal{B}(\underline{B}_{e(,h)})(r,r')] =\\
&= [-\,k^2(-\widehat{x}\partial_y + \,\widehat{y}\partial_x)(\widehat{x}'\partial_y - \widehat{y}'\partial_x)(\mathcal{S}_{e(,h)}(r,r'))],
\end{aligned}
\tag{E.13}
$$

with $\epsilon = -1$, $\widehat{x}' \equiv \,\widehat{x}$, $\widehat{y}' \equiv \,\widehat{y}$ , $\widehat{z}' \equiv \,\widehat{z}$. The functions $\mathcal{S}_{e(,h)}$ verify the conditions [3],

$$
(\partial_z - ikg^{e(,h)})\mathcal{S}_{e(,h)}(r,r') = (\partial_z + ikg^{e(,h)})\mathcal{S}_i(r_{im},r'))|_{z=0}, \tag{E.14}
$$

where $g^h = 1/g^e$, $r_{im} - r = 2\widehat{z}.r$, $\mathcal{S}_i(r, r'_{im}) = \mathcal{S}_i(r_{im}, r')$, and

$$\begin{aligned}
&\mathcal{S}_i(r,r') = (e^{ik|\widehat{z}.(r-r')|}E_1(ik(|(r-r')| + |\widehat{z}.(r-r')|)+\\
&+ e^{-ik|\widehat{z}.(r-r')|}(E_1(ik(|(r-r')| - |\widehat{z}.(r-r')|)) + 2\ln|\widehat{z}\wedge(r-r')|)),
\end{aligned} \tag{E.15}$$

In a similar manner, the functions $\underline{\mathcal{F}}_{h,i(e,i)}$ and $\underline{\mathcal{D}}_{h,i(e,i)}$ can be also expressed like $\underline{\mathcal{F}}_{he(,eh)}$ and $\underline{\mathcal{D}}_{he(,eh)}$, if we take $\mathcal{S}_i(r,r')$ in place of $\mathcal{S}_{e(,h)}(r,r')$ in (E.9)-(E.13) and $\epsilon = 1$ (instead of $\epsilon = -1$).

### E.2.2 Expressions of $\mathcal{S}_{e(h)}(r, r')$ and some properties

The expressions of $\mathcal{S}_{e(h)}(r,r')$ are given [3]-[4] by,

$$\begin{aligned}
\mathcal{S}_e(r,r') &= (\mathcal{S}_i(r_{im},r') + \sum_{\epsilon'=-1,1} \frac{-2g^e}{(g^e-\epsilon')}(\mathcal{V}_{\epsilon'} + \epsilon'\mathcal{K}_{g^e}))(x-x', y-y', -z-z'),\\
\mathcal{S}_h(r,r') &= (-\mathcal{S}_i(r_{im},r') + \sum_{\epsilon'=-1,1} \frac{2g^e}{(g^e-\epsilon')}(\mathcal{V}_{\epsilon'} + \epsilon'\mathcal{K}_{g^h}))(x-x', y-y', -z-z'),
\end{aligned} \tag{E.16}$$

for $z \geq 0$, $z' \geq 0$, $r_{im} \equiv (x, y, -z)$. The functions $\mathcal{V}_{\epsilon'}$ and $\mathcal{K}_g$, which satisfy the Helmholtz equation above the plane, are given by

$$\begin{aligned}
\mathcal{V}_{\epsilon'}(x,y,-z) &= e^{\epsilon' ikz}(E_1(ik(|r| + \epsilon' z)) + (1-\epsilon')\ln\rho),\\
\mathcal{K}_g(x,y,-z) &= e^{ikgz}\mathcal{J}_g(\rho, -z),
\end{aligned} \tag{E.17}$$

for $z \geq 0$, $\rho = \sqrt{x^2+y^2}$, $g = g^e$ or $g = g^h$, $g^h = 1/g^e$.

Let us notice that we have

$$\begin{aligned}
&\frac{\partial}{\partial z}\mathcal{S}_i(r_{im},r') = ik(e^{ik(z+z')}E_1(ik(|r_{im}-r'| + (z+z')))-\\
&- e^{-ik(z+z')}(E_1(ik(|r_{im}-r'| - (z+z'))) + 2\ln\rho)),\\
&\frac{\partial^2}{\partial z^2}\mathcal{S}_i(r_{im},r') = -2ik\frac{e^{-ik|r_{im}-r'|}}{|r_{im}-r'|} - k^2\mathcal{S}_i(r_{im},r'),
\end{aligned} \tag{E.18}$$

and

$$\begin{aligned}
&\frac{\partial}{\partial z}(\mathcal{V}_{\epsilon'} + \epsilon'\mathcal{K}_g)(x,y,-z) = ik\epsilon'(\mathcal{V}_{\epsilon'} + g\,\mathcal{K}_g)(x,y,-z),\\
&\frac{\partial^2}{\partial z^2}(\mathcal{V}_{\epsilon'} + \epsilon'\mathcal{K}_g) = -ik\epsilon'((\epsilon'-g)\frac{e^{-ik|r|}}{|r|} - ik(\epsilon'\mathcal{V}_{\epsilon'} + g^2\,\mathcal{K}_g)),\\
&\sum_{\epsilon'=-1,1}\frac{-2g^e}{(g^e-\epsilon')}\frac{\partial^2}{\partial z^2}(\mathcal{V}_{\epsilon'} + \epsilon'\mathcal{K}_{g^e}) = 2k^2\sum_{\epsilon'=-1,1}\frac{g^e(\mathcal{V}_{\epsilon'} + \epsilon'(g^e)^2\,\mathcal{K}_{g^e})}{(g^e-\epsilon')},\\
&\sum_{\epsilon'=-1,1}\frac{2g^e}{(g^e-\epsilon')}\frac{\partial^2}{\partial z^2}(\mathcal{V}_{\epsilon'} + \epsilon'\mathcal{K}_{g^h}) = -4ik\frac{e^{-ik|r|}}{|r|} - 2k^2\sum_{\epsilon'=-1,1}\frac{(g^e\mathcal{V}_{\epsilon'} + \epsilon' g^h\mathcal{K}_{g^h})}{(g^e-\epsilon')},
\end{aligned} \tag{E.19}$$

for $z \geq 0$. The term $\ln \rho$ does not contribute to the field, except to suppress a singularity due to $E_1(ik(|r|-|z|))$ at $\rho = 0$ [3]. From the behaviour of $\mathcal{J}_g$, $\mathcal{S}_{e(,h)}(r',r)$ remains definite for $g^e = 1$ because $\mathcal{V}_{\epsilon'=1} + \mathcal{K}_{g^e} \to 0$ when $g^e \to 1$, while it is singular for $g^e = -1$. Moreover, when $g^h = (g^e)^{-1} \to \infty$, we have $g^h \mathcal{K}_{g^h} \to -\frac{e^{-ik|r|}}{ik|r|}$.

**Remark 20.** *In the case of the radiation of surface sources [12],*

$$M = -\widehat{n} \wedge E\, \delta_S,\ J = \widehat{n} \wedge H\, \delta_S, \tag{E.20}$$

*where $E$ and $H$ satisfy the equations of Maxwell, it is important to notice that,*

$$\begin{aligned} &Z_0\, div(J) = Z_0\, div_S(\widehat{n} \wedge H\delta_S) = -ik\widehat{n}.E\, \delta_S - Z_0(\widehat{n} \wedge H).v\delta_{\partial S}, \\ &div(M) = -div_S(\widehat{n} \wedge E\delta_S) = -ikZ_0\widehat{n}.H\, \delta_S + (\widehat{n} \wedge E).v\delta_{\partial S}, \end{aligned} \tag{E.21}$$

*where $\widehat{n}$ is the normal to $S$, $v$ is the geodesic normal to $\partial S$ directed outside $S$, and $\delta_S$ (resp. $\delta_{\partial S}$) is the Dirac surface (resp. line) function (see in particular [25, (A.15) in appendix of section 6]).*

**Remark 21.** *We notice that*

$$\begin{aligned} &curl_r([\underline{\mathcal{D}}_{he(,eh)}(r,r')].C_{r'}) = ik([\underline{\mathcal{F}}_{eh(,he)}(r,r')].C_{r'}), \\ &curl_r([\underline{\mathcal{F}}_{he(,eh)}(r,r')].C_{r'}) = -ik([\underline{\mathcal{D}}_{eh(,he)}(r,r')].C_{r'}), \end{aligned} \tag{E.22}$$

*and*

$$\begin{aligned} &D_r.[\underline{\mathcal{F}}_{he(,eh)}(r,r')].C_{r'} = C_{r'}.[\underline{\mathcal{F}}_{he(,eh)}(r',r)].D_r, \\ &D_r.[\underline{\mathcal{D}}_{he(,eh)}(r,r')].C_{r'} = C_{r'}.[\underline{\mathcal{D}}_{eh(,he)}(r',r)].D_r, \end{aligned} \tag{E.23}$$

*with $C_{r'} = c_x\widehat{x}' + c_y\widehat{y}' + c_z\widehat{z}'$, $D_r = d_x\widehat{x} + d_y\widehat{y} + d_z\widehat{z}$ being two constant vectors.*

**Remark 22.** *The tensors also satisfy,*

$$\begin{aligned} &[\mathcal{A}(\underline{B}_{e(,h)})(r,r')].C_{r'} = \\ &= ik(grad(div(\,\widehat{z}.)) + k^2\,\widehat{z}.)((C^t_{r'} \wedge \widehat{z})grad(\mathcal{S}_{e(,h)}(r,r')) \\ &= [ik(\widehat{x}\partial_x + \widehat{y}\partial_y)(\widehat{y}'\partial_x - \widehat{x}'\partial_y)(\partial_z\mathcal{S}_{e(,h)}(r,r')) + \\ &+ ik\widehat{z}(\widehat{y}'\partial_x - \widehat{x}'\partial_y)(\partial_{z^2} + k^2)\mathcal{S}_{e(,h)}(r,r')].C_{r'}, \end{aligned} \tag{E.24}$$

$$\begin{aligned} &[\mathcal{B}(\underline{A}_{e(,h)})(r,r')].C_{r'} = \\ &=\ ik\, curl(\widehat{z}(\epsilon\partial_z(C^t_{r'}grad(\mathcal{S}_{e(,h)}(r,r'))) + \\ &+ c_z((\partial_{z^2} + k^2)\mathcal{S}_{e(,h)}(r',r))) \\ &= [ik((\widehat{x}\partial_y -\ \widehat{y}\partial_x)(\widehat{x}'\partial_x +\ \widehat{y}'\partial_y)\epsilon\partial_z\mathcal{S}_{e(,h)}(r,r') + \\ &+ (\widehat{x}\partial_y -\ \widehat{y}\partial_x)\widehat{z}'(\partial_{z^2} + k^2)\mathcal{S}_{e(,h)}(r,r'))].C_{r'}, \end{aligned} \tag{E.25}$$

$$\begin{aligned}
&[\mathcal{A}(\underline{A}_{e(,h)})(r,r')].C_{r'} = \\
&= (grad(div(\,\widehat{z}.)) + k^2\,\widehat{z}.)(C^t_{r'}(\epsilon\partial_z grad(\mathcal{S}^{\epsilon}_{e(,h)}(r,r'))) + \\
&+ c_z((\partial_{z^2} + k^2)\mathcal{S}_{e(,h)}(r,r'))) \\
&= [\epsilon\partial_{z^2}(\widehat{x}\partial_x + \,\widehat{y}\partial_y)(\widehat{x}'\partial_x + \widehat{y}'\partial_y)\,\mathcal{S}_{e(,h)}(r,r') + \\
&+ \partial_z(\widehat{x}\partial_x + \,\widehat{y}\partial_y)(\widehat{z}')(\partial_{z^2} + k^2)\mathcal{S}_{e(,h)}(r,r') + \\
&+ \,\widehat{z}(\widehat{x}'\partial_x + \widehat{y}'\partial_y)(\partial_{z^2} + k^2)(\epsilon\partial_z\mathcal{S}_{e(,h)}(r,r')) + \\
&+ \widehat{z}\widehat{z}'(\partial_{z^2} + k^2)(\partial_{z^2} + k^2)\mathcal{S}_{e(,h)}(r,r')].C_{r'},
\end{aligned} \tag{E.26}$$

$$\begin{aligned}
&[\mathcal{B}(\underline{B}_{e(,h)})(r,r')].C_{r'} = \\
&= \;ik\,curl(\widehat{z}(ik(C^t_{r'} \wedge \widehat{z})grad(\mathcal{S}_{e(,h)}(r,r')))) \\
&= [-\,k^2(-\widehat{x}\partial_y + \,\widehat{y}\partial_x)(\widehat{x}'\partial_y - \widehat{y}'\partial_x)(\mathcal{S}_{e(,h)}(r,r'))].C_{r'},
\end{aligned} \tag{E.27}$$

*where* $C_{r'} = C^t_{r'} + c_z\widehat{z}'$, *and, from the Helmholtz equation satisfied by* $\mathcal{S}_{e(,h)}$,

$$\begin{aligned}
&[\mathcal{B}(\underline{B}_{e(,h)})(r,r')].C_{r'} = \\
&= -\,k^2((c_x\partial_x + c_y\partial_y)(\widehat{x}\partial_x + \,\widehat{y}\partial_y) + \\
&+ (c_x\widehat{x} + c_y\,\widehat{y})(\partial_{z^2} + k^2))(\mathcal{S}_{e(,h)}(r,r')).
\end{aligned} \tag{E.28}$$

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