Regularity of the Boltzmann Equation in Convex Domains

Y. Guo, C. Kim, D. Tonon, A. Trescases

Abstract

Consider the Boltzmann equation in a strictly convex domain with the diffuse boundary condition. We construct $W^{1,p}$ solutions for $1 < p < 2$. With the aid of a distance function toward the grazing set, we construct weighted $W^{1,p}$ solutions for $2 \leq p \leq \infty$ and classical C^1 solutions away from the grazing set.

1 Introduction

Boundary effects play an important role in the dynamics of Boltzmann solutions of

$$\partial_t F + v \cdot \nabla_x F = Q(F,F),$$

where $F(t,x,v)$ denotes the particle distribution at time t, position $x \in \Omega$ and velocity $v \in \mathbb{R}^3$. Throughout this paper, the collision operator takes the form

$$Q(F_1,F_2) = \int_{\mathbb{R}^3}\int_{\mathbb{S}^2} |v-u|^\gamma q_0(\theta)\Big[F_1(u')F_2(v') - F_1(u)F_2(v)\Big]\mathrm{d}\omega\mathrm{d}u \equiv Q_{\mathrm{gain}}(F_1,F_2) - Q_{\mathrm{loss}}(F_1,F_2),$$

where $u' = u + [(v-u)\cdot\omega]\omega,\ v' = v - [(v-u)\cdot\omega]\omega$ and $0 \leq \gamma \leq 1$ (hard potential) and $0 \leq q_0(\theta) \leq C|\cos\theta|$ (angular cutoff) with $\cos\theta = \frac{v-u}{|v-u|}\cdot\omega$.

Despite many developments in the study of the Boltzmann equation, many basic questions regarding solutions in a bounded domain, such as their regularity, have remained largely open. This is partly due to the characteristic nature of boundary conditions in the kinetic theory. In [7], it was shown that in convex domains, Boltzmann solutions are continuous away from the grazing set. On the other hand, in [11], it is shown that singularity (discontinuity) does occur for Boltzmann solutions in a non-convex domain, and such singularity propagates precisely along the characteristics emanating from the grazing set. In this paper, we establish the first Sobolev regularity for Boltzmann solutions in convex domains, with a diffuse boundary condition :

$$F(t,x,v) = c_\mu \mu(v)\int_{n(x)\cdot u>0} F(t,x,u)\{n(x)\cdot u\}\mathrm{d}u, \qquad \text{for} \quad (x,v)\in\gamma_-,$$

where $c_\mu \int_{n(x)\cdot u>0}\mu(u)\{n(x)\cdot u\}\mathrm{d}u = 1$ and the incoming set $\gamma_- = \{(x,v)\in\partial\Omega\times\mathbb{R}^3 : v\cdot n(x) < 0\}$, the outgoing set $\gamma_+ = \{(x,v)\in\partial\Omega\times\mathbb{R}^3 : v\cdot n(x) > 0\}$ and the grazing set

$$\gamma_0 = \{(x,v)\in\partial\Omega\times\mathbb{R}^3 : v\cdot n(x) = 0\},$$

with $n = n(x)$ the outward normal direction at $x \in \partial\Omega$.

The goal of this paper is to study regularity of solutions near the Maxwellian constructed in [7, 2], for which L^∞ uniform bounds with polynomial and exponential weight in the velocity have been established. We denote $F = \mu + \sqrt{\mu}f$, where $\mu = e^{-\frac{|v|^2}{2}}$ is a steady normalized Maxwellian. The perturbation f satisfies

$$\begin{aligned} \partial_t f + v\cdot\nabla_x f + Lf = \Gamma(f,f)\ , \qquad f|_{t=0} = f_0, \\ f(t,x,v)|_{\gamma_-} = c_\mu\sqrt{\mu(v)}\int_{n(x)\cdot u>0} f(t,x,u)\sqrt{\mu(u)}\ \{n(x)\cdot u\}\mathrm{d}u, \end{aligned} \tag{1}$$

[0] Last update : January 24, 2013

where the linear Boltzmann (symmetric) operator (see [8]) is given by

$$Lf \equiv -\frac{1}{\sqrt{\mu}}\{Q(\mu,\sqrt{\mu}f)+Q(\sqrt{\mu}f,\mu)\},$$

and the nonlinear Boltzmann operator (non-symmetric) is given by

$$\Gamma(f_1,f_2)=\frac{1}{\sqrt{\mu}}Q(\sqrt{\mu}f_1,\sqrt{\mu}f_2)=\Gamma_{\text{gain}}(f_1,f_2)-\Gamma_{\text{loss}}(f_1,f_2).$$

Throughout this paper we assume there exists $\xi:\mathbb{R}^3\to\mathbb{R}$ so that $\Omega = \{x\in\mathbb{R}^3 : \xi(x)<0\}$ and for all $\xi(x)\leq 0$ and $\zeta\in\mathbb{R}^3$

$$\sum_{i,j}\partial_{ij}\xi(x)\zeta_i\zeta_j \geq C_\xi|\zeta|^2. \tag{2}$$

We denote $||\cdot||_p$ the $L^p(\Omega\times\mathbb{R}^3)$ norm, while $|\cdot|_{\gamma,p}$ is the $L^p(\partial\Omega\times\mathbb{R}^3;\mathrm{d}\gamma)$ norm and $|\cdot|_{\gamma_\pm,p}=|\cdot\mathbf{1}_{\gamma_\pm}|_{\gamma,p}$ where $\mathrm{d}\gamma=|n(x)\cdot v|\mathrm{d}S_x\mathrm{d}v$ with the surface measure $\mathrm{d}S_x$ on $\partial\Omega$. Denote $\langle v\rangle=\sqrt{1+|v|^2}$. We define

$$\partial_t f(0)=\partial_t f_0 \equiv -v\cdot\nabla_x f_0 - Lf_0+\Gamma(f_0,f_0). \tag{3}$$

Theorem 1. *Assume that $f_0\in W^{1,p}(\Omega\times\mathbb{R}^3)$ and $||\partial_t f_0||_p<\infty$ for any fixed $1<p<2$, and $||\langle v\rangle^\beta f_0||_\infty \ll 1$ for $\beta\geq 4$, and the compatibility condition on $(x,v)\in\gamma_-$,*

$$f_0(x,v)=c_\mu\sqrt{\mu(v)}\int_{n(x)\cdot u>0} f_0(x,u)\sqrt{\mu(u)}\{n(x)\cdot u\}\mathrm{d}u, \tag{4}$$

then $f\in L^\infty_{loc}([0,\infty);W^{1,p}(\Omega\times\mathbb{R}^3))$ such that for all $t\geq 0$

$$\begin{aligned} &||\partial_t f(t)||_p^p+||\nabla_x f(t)||_p^p+||\nabla_v f(t)||_p^p+\int_0^t\Big\{\ |\partial_t f(s)|_{\gamma,p}^p+|\nabla_x f(s)|_{\gamma,p}^p+|\nabla_v f(s)|_{\gamma,p}^p\Big\}\mathrm{d}s \\ &\lesssim_t\ ||\partial_t f_0||_p^p+||\nabla_x f_0||_p^p+||\nabla_v f_0||_p^p. \end{aligned} \tag{5}$$

We remark that, from [7, 2], the assumption $||\langle v\rangle^\beta f_0||_\infty\ll 1$ without a mass constraint $\iint_{\Omega\times\mathbb{R}^3} f_0\sqrt{\mu}\mathrm{d}v\mathrm{d}x=0$ ensures a uniform-in-time bound as $\sup_{0\leq t\leq\infty}||\langle v\rangle^\beta f(t)||_\infty \lesssim ||\langle v\rangle^\beta f_0||_\infty$ (not a decay). We also remark that this estimate is a global-in-x estimate which includes the grazing set γ_0 and the constant grows exponentially with time, and there is no size restriction on the initial derivatives. Moreover, in Lemma 8, the estimate of (5) in Theorem 1 for $p<2$ is indeed optimal even for the free transport equation $\partial_t f+v\cdot\nabla_x f=0$ with the diffuse boundary condition. In fact, the boundary integral blows up at $p=2$. We therefore conjecture that $f\notin H^1$ in the bulk.

We now illustrate main ideas of the proof. Clearly, both t and v derivatives behave nicely for the diffuse boundary condition as for $(x,v)\in\gamma_-$,

$$\partial_t f(t,x,v) = c_\mu\sqrt{\mu(v)}\int_{n(x)\cdot u>0}\partial_t f(t,x,u)\sqrt{\mu(u)}\{n(x)\cdot u\}\mathrm{d}u, \tag{6}$$

$$\nabla_v f(t,x,v) = c_\mu\nabla_v\sqrt{\mu(v)}\int_{n(x)\cdot u>0} f(t,x,u)\sqrt{\mu(u)}\{n(x)\cdot u\}\mathrm{d}u. \tag{7}$$

Let $\tau_1(x)$ and $\tau_2(x)$ be unit vectors satisfying $\tau_1(x)\cdot n(x)=0=\tau_2(x)\cdot n(x)$ and $\tau_1(x)\times\tau_2(x)=n(x)$. Define the orthonormal transformation from $\{n,\tau_1,\tau_2\}$ to the standard bases $\{\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_3\}$, i.e. $\mathcal{T}(x)n(x)=\mathbf{e}_1,\ \mathcal{T}(x)\tau_1(x)=\mathbf{e}_2,\ \mathcal{T}(x)\tau_2(x)=\mathbf{e}_3$, and $\mathcal{T}^{-1}=\mathcal{T}^t$. Upon a change of variable : $u'=\mathcal{T}(x)u$, we have

$$n(x)\cdot u=n(x)\cdot\mathcal{T}^t(x)u'=n(x)^t\mathcal{T}^t(x)u'=[\mathcal{T}(x)n(x)]^t u'=\mathbf{e}_1\cdot u'=u_1',$$

then

$$c_\mu\sqrt{\mu(v)}\int_{n(x)\cdot u>0} f(t,x,u)\sqrt{\mu(u)}\{n(x)\cdot u\}\mathrm{d}u = c_\mu\sqrt{\mu(v)}\int_{u_1'>0} f(t,x,\mathcal{T}^t(x)u')\sqrt{\mu(u')}\{u_1'\}\mathrm{d}u',$$

so that we can further take tangential derivatives ∂_{τ_i} as, for $(x,v)\in\gamma_-$,

$$\begin{aligned}\partial_{\tau_i} f(t,x,v) &= c_\mu\sqrt{\mu(v)}\int_{u_1'>0}\left\{\partial_{\tau_i} f(t,x,\mathcal{T}^t(x)u') + \nabla_v f(t,x,\mathcal{T}^t(x)u')\frac{\partial\mathcal{T}^t(x)}{\partial\tau_i}u'\right\}\sqrt{\mu(u')}\{u_1'\}\mathrm{d}u' \\ &= c_\mu\sqrt{\mu(v)}\int_{n(x)\cdot u>0}\partial_{\tau_i} f(t,x,u)\sqrt{\mu(u)}\{n(x)\cdot u\}\mathrm{d}u \\ &\quad + c_\mu\sqrt{\mu(v)}\int_{n(x)\cdot u>0}\nabla_v f(t,x,u)\frac{\partial\mathcal{T}^t(x)}{\partial\tau_i}\mathcal{T}(x)u\sqrt{\mu(u)}\{n(x)\cdot u\}\mathrm{d}u.\end{aligned} \tag{8}$$

The difficulty is always the control of the normal spatial derivative of ∂_n. From the general method of proving regularity in PDE with boundary conditions, it is natural to use the Boltzmann equation to solve the normal derivative $\partial_n f$ inside the region, in terms of $\partial_t f$, $\nabla_v f$, and $\partial_\tau f$ as:

$$\partial_n f(t,x,v) = -\frac{1}{n(x)\cdot v}\left\{\partial_t f + \sum_{i=1}^2 (v\cdot\tau_i)\partial_{\tau_i} f + Lf - \Gamma(f,f)\right\}, \tag{9}$$

at least near $\partial\Omega$. Unfortunately, this standard approach encounters a severe difficulty: $\frac{1}{n(x)\cdot v}\notin L^1_{loc}$ in the velocity space (a L^∞ bound is desirable for any $W^{1,p}$ estimate).

The first new ingredient of our approach is to use (9) *not* inside the domain, but at the boundary $\partial\Omega$. Using special feature of the diffuse boundary condition and (6), (7) and (8), we can express $\partial_n f$ at $(x,v)\in\gamma_-$ as

$$\begin{aligned}\partial_n f(t,x,v) = -\frac{1}{n(x)\cdot v}\Bigg\{ &\sqrt{\mu(v)}\int_{n(x)\cdot u>0}\partial_t f(t,x,u)\sqrt{\mu(u)}\{n(x)\cdot u\}\mathrm{d}u \\ &+\sum_{i=1}^2 (v\cdot\tau_i)\sqrt{\mu(v)}\int_{n(x)\cdot u>0}\partial_{\tau_i} f(t,x,u)\sqrt{\mu(u)}\{n(x)\cdot u\}\mathrm{d}u \\ &+\sum_{i=1}^2 (v\cdot\tau_i)\sqrt{\mu(v)}\int_{n(x)\cdot u>0}\nabla_v f(t,x,u)\frac{\partial\mathcal{T}^t(x)}{\partial\tau_i}\mathcal{T}(x)u\sqrt{\mu(u)}\{n(x)\cdot u\}\mathrm{d}u \\ &+ Lf - \Gamma(f,f)\Bigg\},\end{aligned} \tag{10}$$

Due to the additional u integral in (10) and the crucial factor $|n(x)\cdot u|$ in the measure $d\gamma$ on the boundary γ, it is clear that the singularity $|\partial_n f|^p|n\cdot v|$ in (10) is roughly of the order

$$\frac{1}{\{n\cdot v\}^{p-1}},$$

so that its v integration is precisely finite if $1\le p<2$, and indeed its v integration is *uniformly* bounded with respect to x.

However, in order to control $\partial_t f, \nabla_v f$ and $\partial_\tau f$ for $p<2$, a new difficulty arises. It is well-known from [7, 2] that a crucial boundary estimate for diffuse boundary takes the form of a L^2-contraction:

$$\int_{\gamma_-} h^2\mathrm{d}\gamma \le \int_{\gamma_+} h^2\mathrm{d}\gamma.$$

Unfortunately, this is not expected to be valid for $p\neq 2$, so it is impossible to absorb the incoming part $\gamma\_$ solely by the outgoing part γ_+ part.

Our second new ingredient is to split the γ_+ integral into near grazing set γ_+^ε and the rest for $p\neq 2$ for our boundary representation for derivatives (6), (7), (8), and (10). For small $\varepsilon>0$ we define γ_+^ε, the set of almost grazing velocities or large velocities

$$\gamma_+^\varepsilon = \{(x,v)\in\gamma_+ : v\cdot n(x)<\varepsilon \text{ or } |v|>1/\varepsilon\}. \tag{11}$$

Denote $\partial = [\partial_t, \nabla_x, \nabla_v]$. We can roughly obtain

$$
\begin{aligned}
\int_{\gamma_-} |\partial f|^p &\lesssim \int_{\partial\Omega} \left(\int_{n\cdot v>0} |\partial f| \mu^{1/4} \{n\cdot v\} \mathrm{d}v \right)^p + \text{good terms}, \\
&\lesssim \int_{\partial\Omega} \left(\int_{\{v:(x,v)\in\gamma_+^\varepsilon\}} |\partial f| \mu^{1/4} \{n\cdot v\} \right)^p + \int_{\partial\Omega} \left(\int_{\{v:(x,v)\in\gamma_+\setminus\gamma_+^\varepsilon\}} |\partial f| \mu^{1/4} \{n\cdot v\} \right)^p + \text{good terms}, \\
&\lesssim \sup_x \left(\int_{\{v:(x,v)\in\gamma_+^\varepsilon\}} \mu^{q/4} \{n\cdot v\} \mathrm{d}v \right)^{p/q} \int_{\gamma_+^\varepsilon} |\partial f|^p \mathrm{d}\gamma + \int_{\gamma_+\setminus\gamma_+^\varepsilon} |\partial f|^p \mathrm{d}\gamma + \text{good terms}.
\end{aligned}
$$

It is important to realize that $\sup_x \left(\int_{\{v:(x,v)\in\gamma_+^\varepsilon\}} \mu^{q/4}\{n\cdot v\}\mathrm{d}v \right)^{p/q}$ has a small measure of order ε, for $p>1$, so that it can be absorbed by the outgoing part $\int_{\gamma_+}$. Fortunately, the outgoing boundary integral $\int_{\gamma_+\setminus\gamma_+^\varepsilon}$ can be further bounded by the integration in the bulk and initial data by Lemma 4 with a crucial time integration. On the other hand, such a process produces a large constant in the Gronwall estimates and leads to a growth in time. Of course, such an approach breaks down at $p=1$.

We next prove higher regularity away from the grazing set γ_0. Our third new ingredient is the construction of a distance function towards the grazing set γ_0 to achieve this goal. We define a distance function toward γ_0 as

$$
\alpha(x,v) \equiv \Big\{ |v\cdot\nabla\xi(x)|^2 - 2\{v\cdot\nabla^2\xi(x)\cdot v\}\xi(x) \Big\} \geq 0 \tag{12}
$$

by (2) and $\alpha(x,v)$ is zero if and only if (x,v) is at the grazing set γ_0. We observe that

$$
\begin{aligned}
v\cdot\nabla_x\alpha &= \{2v\cdot\nabla\xi(x)[v\cdot\nabla^2\xi\cdot v] - 2v\cdot\nabla\xi(x)[v\cdot\nabla^2\xi\cdot v] - 2v\{v\cdot\nabla^3\xi(x)\cdot v\}\xi(x)\} \\
&= -2v\{v\cdot\nabla^3\xi(x)\cdot v\}\xi(x),
\end{aligned}
$$

which is bounded by $|v|\alpha(x,v)$ since $\{v\cdot\nabla^2\xi(x)\cdot v\} \backsim |v|^2$ from (2). This crucial invariant property of α under operator $v\cdot\nabla_x$ is the key for our approach. On the other hand, unless $\nabla^3\xi\equiv 0$ (for example the domain is a ball or an ellipsoid), a growth factor $|v|$ creates a geometric effect which is out of control for our analysis. We introduce a strong decay factor $e^{-l\langle v\rangle t}$ with sufficiently large $l>0$ to overcome such a geometric effect :

$$
\mathbf{d}^2(t,x,v) \;=\; e^{-l\langle v\rangle t}\alpha(x,v)\chi\left(\frac{1}{\varepsilon}\, e^{-l\langle v\rangle t}\alpha(x,v)\right) + \left[1-\chi\left(\frac{1}{\varepsilon}\, e^{-l\langle v\rangle t}\alpha(x,v)\right)\right], \tag{13}
$$

with a smooth cut-off function $\chi:[0,\infty)\to[0,1]$ such that $-4\leq\chi'(y)\leq 0$ and

$$
\chi(y)=1 \;\text{ for } 0\leq y\leq 1/2, \quad \chi(y)=0 \;\text{ for } y\geq 1.
$$

A direct computation yields

$$
\begin{aligned}
\{\partial_t + v\cdot\nabla_x\}\mathbf{d}^2 &= \{\partial_t + v\cdot\nabla_x\} e^{-l\langle v\rangle t}\alpha \times \{\chi(e^{-l\langle v\rangle t}\alpha/\varepsilon) + \varepsilon^{-1}\chi'(e^{-l\langle v\rangle t}\alpha/\varepsilon)[e^{-l\langle v\rangle t}\alpha - 1]\} \\
&= \Big\{ -l\langle v\rangle e^{-l\langle v\rangle t}\alpha - e^{-l\langle v\rangle t} 2v\{v\cdot\nabla^3\xi(x)\cdot v\}\xi(x) \Big\} \Big\{ \chi(e^{-l\langle v\rangle t}\alpha/\varepsilon) + \varepsilon^{-1}\chi'(e^{-l\langle v\rangle t}\alpha/\varepsilon)[e^{-l\langle v\rangle t}\alpha - 1] \Big\} \\
&\lesssim (-l + O_\xi(1))\langle v\rangle e^{-l\langle v\rangle t}\alpha \Big\{ \chi(e^{-l\langle v\rangle t}\alpha/\varepsilon) + \varepsilon^{-1}\chi'(e^{-l\langle v\rangle t}\alpha/\varepsilon)[e^{-l\langle v\rangle t}\alpha - 1] \Big\} \\
&= (-l + O_\xi(1))\langle v\rangle \mathbf{d}^2 \times e^{-l\langle v\rangle t}\alpha \Big\{ \chi(e^{-l\langle v\rangle t}\alpha/\varepsilon) + \varepsilon^{-1}\chi'(e^{-l\langle v\rangle t}\alpha/\varepsilon)[e^{-l\langle v\rangle t}\alpha - 1] \Big\} / \mathbf{d}^2 \\
&\sim_\varepsilon (-l + O_\xi(1))\langle v\rangle \mathbf{d}^2,
\end{aligned}
$$

from our cutoff function χ. Here $O_\xi(1) = \frac{2v\{v\cdot\nabla^3\xi(x)\cdot v\}\xi}{\alpha\langle v\rangle}$ represents the geometric effect. Throughout this paper we choose

$$
l > \max \frac{2v\{v\cdot\nabla^3\xi(x)\cdot v\}\xi}{\alpha\langle v\rangle}. \tag{14}
$$

Remark that if ξ is quadratic (for example the domain is a ball or an ellipsoid) then we are able to set $l=0$ and $\{\partial_t + v\cdot\nabla_x\}\mathbf{d}^2 = 0$.

Theorem 2. *Assume the compatibility condition (4) and recall (3). For any fixed* $2 \leq p < \infty$ *and* $\frac{p-2}{p} < \lambda < \frac{p-1}{p}$*, if* $||\mathbf{d}^\lambda \partial_t f_0||_p + ||\mathbf{d}^\lambda \nabla_x f_0||_p + ||\mathbf{d}^\lambda \nabla_v f_0||_p < \infty$ *and* $||e^{\zeta|v|^2} f_0||_\infty \ll 1$ *for some* $0 < \zeta < \frac{1}{4}$ *then* $\mathbf{d}^\lambda \partial_t f, \mathbf{d}^\lambda \nabla_x f, \mathbf{d}^\lambda \nabla_v f \in L^\infty_{loc}([0,\infty); L^p(\Omega \times \mathbb{R}^3))$ *such that for all* $t \geq 0$,

$$||\mathbf{d}^\lambda \partial_t f(t)||_p^p + ||\mathbf{d}^\lambda \nabla_x f(t)||_p^p + ||\mathbf{d}^\lambda \nabla_v f(t)||_p^p + \int_0^t \Big\{ |\mathbf{d}^\lambda \partial_t f(s)|_{\gamma,p}^p + |\mathbf{d}^\lambda \nabla_x f(s)|_{\gamma,p}^p + |\mathbf{d}^\lambda \nabla_v f(s)|_{\gamma,p}^p \Big\} \mathrm{d}s$$
$$\lesssim_t \ ||\mathbf{d}^\lambda \nabla_t f_0||_p^p + ||\mathbf{d}^\lambda \nabla_x f_0||_p^p + ||\mathbf{d}^\lambda \nabla_v f_0||_p^p + ||e^{\zeta|v|^2} f_0||_\infty^p.$$

If $||\mathbf{d}\nabla_t f_0||_\infty + ||\mathbf{d}\nabla_x f_0||_\infty + ||\mathbf{d}\nabla_v f_0||_\infty < +\infty$ *and* $||e^{\zeta|v|^2} f_0||_\infty \ll 1$ *for some* $0 < \zeta < \frac{1}{4}$*, then* $\mathbf{d}\partial_t f, \mathbf{d}\nabla_x f$, $\mathbf{d}\nabla_v f \in L^\infty_{loc}([0,\infty); L^\infty(\Omega \times \mathbb{R}^3))$ *such that for all* $t \geq 0$,

$$||\mathbf{d}\partial_t f(t)||_\infty + ||\mathbf{d}\nabla_v f(t)||_\infty + ||\mathbf{d}\nabla_x f(t)||_\infty \ \lesssim_t \ ||\mathbf{d}\partial_t f_0||_\infty + ||\mathbf{d}\nabla_x f_0||_\infty + ||\mathbf{d}\nabla_v f_0||_\infty + ||e^{\zeta|v|^2} f_0||_\infty.$$

Furthermore, if $\mathbf{d} f_0 \in C^0(\bar{\Omega} \times \mathbb{R}^3)$ *and*

$$v \cdot \nabla_x f_0 + L f_0 - \Gamma(f_0, f_0) = c_\mu \sqrt{\mu} \int_{n \cdot u > 0} \{u \cdot \nabla_x f_0 + L f_0 - \Gamma(f_0, f_0)\} \sqrt{\mu} \{n \cdot u\} \mathrm{d}u, \tag{15}$$

is valid for $\gamma_- \cup \gamma_0$*, then* $\mathbf{d} f \in C^0([0,\infty) \times \bar{\Omega} \times \mathbb{R}^3)$ *with the same estimate.*

There is again no size restriction on initial derivatives. Remark that the assumption $||e^{\zeta|v|^2} f_0||_\infty \ll 1$ ensures a uniform-in-time bound $\sup_{0 \leq t \leq \infty} ||e^{\zeta|v|^2} f(t)||_\infty \lesssim ||e^{\zeta|v|^2} f_0||_\infty$ due to [7, 2]. We remark for $l \neq 0$, $\mathbf{d}(t) \backsim e^{-l\langle v \rangle t} \mathbf{d}(0)$ so that in terms of solution $f(t)$, such an estimate not only creates an exponential growth in time, but also creates less integrability in velocity. Furthermore, when $l \neq 0$, we crucially need a strong weight function $e^{\zeta|v|^2}$ to balance such a factor $e^{-l\langle v \rangle t}$, which produces a super exponential growth e^{t^2} in time in controls of the non-local collision operator. We suspect that it is impossible to obtain a uniform in time estimate especially when $l \neq 0$. Distance functions similar to $\mathbf{d}$ play an important role in the study of regularity in convex domains for Vlasov equations ([6, 10]), which can be controlled along the characteristics via so-called velocity lemma. However, such an approach has not been successful in the study of Boltzmann equation due to the non-local nature of the Boltzmann collision operator, which mixes up different velocities so that their distance towards γ_0 can not be controlled. In addition of the key boundary representation, we establish a delicate estimate for interaction of $\mathbf{d}$ and the collision kernel $\mathbf{d}^\lambda K(\frac{f}{\mathbf{d}^\lambda})$ in (85) for $\lambda < \frac{p-1}{p}$. An additional requirement $\lambda > \frac{p-2}{p}$ is needed to control the boundary singularity in (89). These estimates are sufficient to treat the case for $\lambda < 1$, but unfortunately these fail for the case $\lambda = 1$, which accounts for the important C^1 estimate. In order to establish the C^1 estimate, we employ the Lagrangian view point, estimating along so-called stochastic cycles [7, 2].

Our fourth new ingredient is a key estimate in Lemma 12. Even though $\mathbf{d}K(\frac{f}{\mathbf{d}})$ is impossible to estimate directly due to severe singularity of $\frac{1}{\mathbf{d}}$ in the velocity space, along the characteristics, $\frac{1}{\mathbf{d}(s,x-st,v)}$ is integrable in time for a convex domain. Therefore, the integral

$$\int_0^t \int_{\mathbb{R}^3} \mathbf{d}K(\frac{f}{\mathbf{d}})$$

can be controlled by first integrating over time, and we can close the estimate.

Our paper is organized as follows: In Section 2, we establish some preliminary estimates. Section 3 is devoted to in-flow problems with the prescribed incoming data. Theorem 1 is established in Section 4, and weighted $W^{1,p}$ estimate in Theorem 2 is established in Section 5. We finally conclude the proof of weighted C^1 estimate in Theorem 2 in Section 6.

2 Preliminary

It is standard to write the linear Boltzmann operator as

$$Lf \equiv \nu(v) f - Kf,$$

with the collision frequency $\nu(v) \equiv \int_{\mathbb{S}^2}\int_{\mathbb{R}^3} |v-u|^\gamma q_0(\frac{v-u}{|v-u|}\cdot\omega)\mu(u)\mathrm{d}u\mathrm{d}\omega \sim \langle v\rangle^\gamma$ for the hard potential $0\le\gamma\le 1$ and

$$Kf = \int_{\mathbb{R}^3} \mathbf{k}(u,v) f(u)\mathrm{d}u, \qquad \mathbf{k} = \mathbf{k}_2 - \mathbf{k}_1, \tag{16}$$

where

$$\begin{aligned}
\mathbf{k}_1(u,v) &= \sqrt{\mu(u)}\sqrt{\mu(v)}|u-v|^\gamma \int_{\mathbb{S}^2} q_0(\frac{v-u}{|v-u|}\cdot\omega)\mathrm{d}\omega, \qquad (17)\\
\mathbf{k}_2(u,v) &= \frac{2}{|u-v|^2} e^{-\frac18|u-v|^2 - \frac18\frac{(|u|^2-|v|^2)^2}{|u-v|^2}} \int_{w\cdot(u-v)=0} q_0^*(|w|,|u-v|)e^{-|w+\varsigma|^2}(|w|^2+|u-v|^2)^{\gamma/2}\mathrm{d}w,
\end{aligned}$$

where q_0^* is bounded, and $\varsigma = \frac14(u+v) - \frac14\frac{|u|^2-|v|^2}{|u-v|^2}(u-v)$, so that $\varsigma\cdot(u-v)=0$ and $|\varsigma|^2 = \frac18|u+v|^2 - \frac18\frac{(|u|^2-|v|^2)^2}{|u-v|^2}$. See page 44 of [8] for details.

Lemma 1. *Recall (17) and the Grad estimate [9, 8] for hard potential, $0\le\gamma\le1$,*

$$|\mathbf{k}(u,v)| \;\lesssim\; \{|v-u|^\gamma + |v-u|^{-2+\gamma}\} e^{-\frac18|v-u|^2 - \frac18\frac{(|v|^2-|u|^2)^2}{|v-u|^2}}. \tag{18}$$

For $\varrho>0$ and $-2\varrho<\theta<2\varrho$ and $\beta\in\mathbb{R}$, we have for $0<\gamma\le1$,

$$\int_{\mathbb{R}^3}\{|v-u|^\gamma + |v-u|^{-2+\gamma}\} e^{-\varrho|v-u|^2 - \varrho\frac{(|v|^2-|u|^2)^2}{|v-u|^2}} \frac{\langle v\rangle^\beta e^{\theta|v|^2}}{\langle u\rangle^\beta e^{\theta|u|^2}}\mathrm{d}u \;\lesssim\; \langle v\rangle^{-1}, \tag{19}$$

and, for $\gamma=0$,

$$\int_{\mathbb{R}^3}\{|v-u| + |v-u|^{-2}\} e^{-\varrho|v-u|^2 - \varrho\frac{(|v|^2-|u|^2)^2}{|v-u|^2}} \frac{\langle v\rangle^\beta e^{\theta|v|^2}}{\langle u\rangle^\beta e^{\theta|u|^2}}\mathrm{d}u \;\lesssim\; \langle v\rangle^{-1+\delta}, \quad \textit{for any } 0<\delta\ll1. \tag{20}$$

Moreover for $p\in[1,\infty)$, if $h, \nabla_v h$ in $L^p(\mathbb{R}^3)$, we have

$$\|Kh\|_p \;\lesssim_p\; \|h\|_p, \qquad \|\nabla_v Kh\|_p \;\lesssim_p\; \|h\|_p + \|\nabla_v h\|_p. \tag{21}$$

Proof. The proof is based on [7]. Notice that

$$\frac{\langle v\rangle^\beta e^{\theta|v|^2}}{\langle u\rangle^\beta e^{\theta|u|^2}} \;\lesssim\; [1+|v-u|^2]^{\frac{\beta}{2}} e^{-\theta(|u|^2-|v|^2)}.$$

Set $v-u=\eta$ and $u=v-\eta$ in the integration of (19) and (20). Now we compute the total exponent of the integrand of (19) and (20) as

$$\begin{aligned}
&-\varrho|\eta|^2 - \varrho\frac{||\eta|^2-2v\cdot\eta|^2}{|\eta|^2} - \theta\{|v-\eta|^2-|v|^2\} = -2\varrho|\eta|^2 + 4\varrho\{v\cdot\eta\} - 4\varrho\frac{|v\cdot\eta|^2}{|\eta|^2} - \theta\{|\eta|^2 - 2v\cdot\eta\}\\
&= (-\theta-2\varrho)|\eta|^2 + (4\varrho+2\theta)\, v\cdot\eta - 4\varrho\frac{|v\cdot\eta|^2}{|\eta|^2}.
\end{aligned}$$

Since $-2\varrho<\theta<2\varrho$, the discriminant of the above quadratic form of $|\eta|$ and $\frac{v\cdot\eta}{|\eta|}$ is negative : $(4\varrho+2\theta)^2 + 16\varrho(-\theta-2\varrho) = 4\theta^2 - 16\varrho^2 < 0$. We thus have

$$-\varrho|\eta|^2 - \varrho\frac{||\eta|^2-2v\cdot\eta|^2}{|\eta|^2} - \theta\{|v-\eta|^2-|v|^2\} \;\lesssim_{\varrho,\theta}\; \frac{|\eta|^2}{2} + |v\cdot\eta|.$$

Therefore, for $0\le\gamma\le1$

$$\int_{\mathbb{R}^3}\Big\{|v-u|^\gamma + |v-u|^{-2+\gamma}\Big\} e^{-\varrho|v-u|^2 - \varrho\frac{(|v|^2-|u|^2)^2}{|v-u|^2}} \frac{\langle v\rangle^\beta e^{\theta|v|^2}}{\langle u\rangle^\beta e^{\theta|u|^2}}\mathrm{d}u \;\lesssim\; \int_{\mathbb{R}^3}\Big\{|\eta|^\gamma + |\eta|^{-2+\gamma}\Big\}\langle\eta\rangle^\beta e^{-C_{\varrho,\theta}|\eta|^2} \;\lesssim_{\varrho,\theta,\gamma}\; 1.$$

Therefore, in order to show (19) and (20) it suffices to consider the case $|v| \geq 1$. We make another change of variables $\eta_{\|} = \left\{\eta \cdot \frac{v}{|v|}\right\} \frac{v}{|v|}$ and $\eta_\perp = \eta - \eta_{\|}$, so that $|v \cdot \eta| = |v||\eta_{\|}|$ and $|v-u| \geq |\eta_\perp|$. We can absorb $\langle\eta\rangle^\beta,\ |\eta|\langle\eta\rangle^\beta$ by $e^{-C_{\varrho,\theta}|\eta|^2}$, and bound the integral of (19) and (20) by, for $\delta \geq 0$

$$
\begin{aligned}
&\int_{\mathbb{R}^3} \{1+|\eta|^{-2+\gamma}\} e^{-C_{\varrho,\theta}\left\{\frac{|\eta|^2}{2}+|v\cdot\eta|\right\}} \mathrm{d}\eta \ \leq\ \int_{\mathbb{R}^3} \{1+|\eta|^{-2+\gamma+\delta}\} e^{-\frac{C_{\varrho,\theta}}{2}|\eta|^2} |\eta|^{-\delta} e^{-C_{\varrho,\theta}|v\cdot\eta|} \mathrm{d}\eta \\
&\leq \int_{\mathbb{R}^2} \{1+|\eta_\perp|^{-2+\gamma+\delta}\} e^{-\frac{C_{\varrho,\theta}}{2}|\eta_\perp|^2} \left\{ \int_{\mathbb{R}} |\eta_{\|}|^{-\delta} e^{-C_{\varrho,\theta}|v|\times|\eta_{\|}|} \mathrm{d}|\eta_{\|}| \right\} \mathrm{d}\eta_\perp \\
&\lesssim \langle v\rangle^{-1+\delta} \int_{\mathbb{R}^2} \{1+|\eta_\perp|^{-2+\gamma+\delta}\} e^{-\frac{C_{\varrho,\theta}}{2}|\eta_\perp|^2} \left\{ \int_0^\infty y^{-\delta} e^{-C_{\varrho,\theta} y} \mathrm{d}y \right\} \mathrm{d}\eta_\perp, \quad (y = |v||\eta_{\|}|).
\end{aligned}
$$

If $\gamma > 0$ we can set $\delta = 0$ to conclude (19) since $|\eta_\perp|^{-2+\gamma} \in L^1(\mathbb{R}^2)$. On the other hand if $\gamma = 0$ we conclude (20) since $|\eta_\perp|^{-2+\delta} \in L^1(\mathbb{R}^2)$ for $\delta > 0$.

For the first estimate of (21) we have $|Kh(v)| \leq \left(\int |\mathbf{k}(u,v)|\mathrm{d}u\right)^{1/q} \times \left(\int |\mathbf{k}(u,v)||h(u)|^p \mathrm{d}u\right)^{1/p}$ to conclude

$$
\|Kh\|_p \ \leq\ \sup_v \left(\int |\mathbf{k}(u,v)|\mathrm{d}u \right)^{1/q} \sup_u \left(\int |\mathbf{k}(u,v)|\mathrm{d}v \right)^{1/p} \|h\|_p \ \lesssim\ ||h||_p,
$$

where we have used (19) and (20).

For the second estimate of (21) we use a change of variable $u = v - u$ in the integral (16) : $Kf(v) = \int \mathbf{k}(v-u,v) f(v-u)\mathrm{d}u$, so that

$$
\begin{aligned}
\nabla_v(Kh) &= \int_{\mathbb{R}^3} \{\nabla_u \mathbf{k}(v-u,v) + \nabla_v \mathbf{k}(v-u,v)\} h(v-u)\mathrm{d}u + \int_{\mathbb{R}^3} \mathbf{k}(v-u,v) \nabla_v h(v-u)\mathrm{d}u \\
&= \int_{\mathbb{R}^3} \{\nabla_u \mathbf{k}(u,v) + \nabla_v \mathbf{k}(u,v)\} h(u)\mathrm{d}u + Kh_v \equiv K_v h + Kh_v.
\end{aligned} \tag{22}
$$

From (17) we have

$$
|\nabla_u \mathbf{k}_1| + |\nabla_v \mathbf{k}_1| \leq C\sqrt{\mu(u)}\sqrt{\mu(v)}(|u|^{\gamma+1} + |v|^{\gamma+1}), \tag{23}
$$

so that $\sup_v \left(\int |\nabla_u \mathbf{k}_1| + |\nabla_v \mathbf{k}_1| \mathrm{d}u\right)$ and $\sup_u \left(\int |\nabla_u \mathbf{k}_1| + |\nabla_v \mathbf{k}_1| \mathrm{d}v\right)$ are finite. For $\mathbf{k}_2$, we absorb the terms in $|w+\zeta|$, in $|u-v|$ and in $\frac{||u|^2-|v|^2|}{|u-v|}$ in the associated exponential terms in (17) to have

$$
|\nabla_u \mathbf{k}_2| + |\nabla_v \mathbf{k}_2| \leq C \frac{2}{|u-v|^2} e^{-\frac{1}{10}|u-v|^2 - \frac{1}{10}\frac{(|u|^2-|v|^2)^2}{|u-v|^2}} \int_{w\cdot(u-v)=0} q_0^*(|w|,|u-v|) e^{-\frac{1}{2}|w+\zeta|^2} (|w|^2 + |u-v|^2)^{\gamma/2} \mathrm{d}w.
$$

This expression is of the same shape as the expression of $\mathbf{k}_2$ in (17) up to some constants in the exponential terms. The proof of the Grad estimate (18) is also valid for this expression, so that

$$
|\nabla_u \mathbf{k}_2(v,u)| + |\nabla_v \mathbf{k}_2(v,u)| \ \lesssim\ \{|v-u|^\gamma + |v-u|^{-2+\gamma}\} e^{-\frac{1}{16}|v-u|^2 - \frac{1}{16}\frac{(|v|^2-|u|^2)^2}{|v-u|^2}}, \tag{24}
$$

and we deduce that $\sup_v \left(\int |\nabla_u \mathbf{k}_2| + |\nabla_v \mathbf{k}_2| \mathrm{d}u\right)$ and $\sup_u \left(\int |\nabla_u \mathbf{k}_2| + |\nabla_v \mathbf{k}_2| \mathrm{d}v\right)$ are finite from (19) and (20). Then we conclude the second estimate of (21) as the first estimate. □

Lemma 2. *There exists a constant $C > 0$ such that for any g_1, g_2, g_3, we have for $(i,j) = (1,2)$ and for $(i,j) = (2,1)$*

$$
\begin{aligned}
\left| \iint_{\Omega\times\mathbb{R}^3} \Gamma(g_1,g_2) g_3 \mathrm{d}v\mathrm{d}x \right| &\leq C \sup_\Omega \left\{ \int_{\mathbb{R}^3} \nu |g_i|^p \mathrm{d}v \right\}^{1/p} \|\nu^{1/p} g_j\|_p \|\nu^{1/q} g_3\|_q, \\
|\Gamma(g_1,g_2)| &\leq ||e^{\zeta|v|^2} g_1||_\infty \left(|\langle v\rangle^\gamma g_2| + \int_{\mathbb{R}^3} \mathbf{k}_\zeta(v,u)|g_2|\mathrm{d}u \right),
\end{aligned} \tag{25}
$$

where for some $C_\zeta > 0$,

$$
|\mathbf{k}_\zeta(v,u)| \ \lesssim_\zeta\ \left\{ |v-u|^\gamma + \frac{1}{|v-u|^{2-\gamma}} \right\} e^{-C_\zeta\left[|v-u|^2 + \frac{||v|^2-|u|^2|^2}{|v-u|^2}\right]}. \tag{26}
$$

For $\beta > 0,\ \theta > 0,$

$$\big|\Gamma(g_1,g_2)(v)\big| \lesssim \langle v\rangle^{\gamma-\beta}||\langle v\rangle^{\beta}g_1||_\infty||\langle v\rangle^{\beta}g_2||_\infty, \quad \big|\Gamma(g_1,g_2)(v)\big| \lesssim e^{-\frac{\theta}{4}|v|^2}||e^{\theta|v|^2}g_1||_\infty||e^{\theta|v|^2}g_2||_\infty. \tag{27}$$

Proof. Recall the definition of Γ:

$$\Gamma(g_1,g_2) \;=\; \int_{\mathbb{R}^3}\int_{\mathbb{S}^2}\sqrt{\mu(u)}\Big[g_1(u')g_2(v')-g_1(u)g_2(v)\Big]|u-v|^\gamma q_0(\theta)\mathrm{d}\omega\mathrm{d}u = \Gamma_{\mathrm{gain}}(g_1,g_2)-\Gamma_{\mathrm{loss}}(g_1,g_2).$$

First notice that (for $p>1$)

$$\begin{aligned}\int |u|^{\gamma q}e^{-|u-v|^2q/2}\mathrm{d}u &= \int_{|u-v|<|u|/2}|u|^{\gamma q}e^{-|u-v|^2q/2}\mathrm{d}u + \int_{|u-v|\geq|u|/2}|u|^{\gamma q}e^{-|u-v|^2q/2}\mathrm{d}u\\ &\leq 2^{\gamma q}|v|^{\gamma q}\int e^{-|u|^2q/2}\mathrm{d}u + 2^{\gamma q}\int |u|^{\gamma q}e^{-|u|^2q/2}\mathrm{d}u \lesssim 1+|v|^{\gamma q} \;\lesssim\; \nu(v)^q,\end{aligned}$$

and similarly, for $p=1$,

$$\sup_{u\in\mathbb{R}^3}\{|u|^\gamma e^{-|u-v|^2/2}\} \;\lesssim\; 1+|v|^\gamma \;\lesssim\; \nu(v).$$

For the loss term, we compute

$$\begin{aligned}&\left|\int_{\mathbb{R}^3}\Gamma_{\mathrm{loss}}(g_1,g_2)g_3\mathrm{d}v\right| \lesssim \left|\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\sqrt{\mu(u)}g_1(u)g_2(v)g_3(v)|u-v|^\gamma\mathrm{d}u\mathrm{d}v\right|\\ \lesssim\;& \left|\int_{\mathbb{R}^3}|g_1|^p\mathrm{d}v\right|^{1/p}\int_{\mathbb{R}^3}\left|\int_{\mathbb{R}^3}\sqrt{\mu(u)}^q|u-v|^{q\gamma}\mathrm{d}u\right|^{1/q}|g_2(v)||g_3(v)|\mathrm{d}v\\ \lesssim\;& \left|\int_{\mathbb{R}^3}|g_1|^p\mathrm{d}v\right|^{1/p}\int_{\mathbb{R}^3}\nu(v)|g_2(v)||g_3(v)|\mathrm{d}v \lesssim \left|\int_{\mathbb{R}^3}\nu|g_1|^p\mathrm{d}v\right|^{1/p}\left|\int_{\mathbb{R}^3}\nu|g_2|^p\mathrm{d}v\right|^{1/p}\left|\int_{\mathbb{R}^3}\nu|g_3|^q\mathrm{d}v\right|^{1/q},\end{aligned}$$

where we have used Hölder's inequality in u, and Hölder's inequality in v. We then integrate over the domain Ω and apply Hölder's inequality in space.

For the gain term, using successively Hölder's inequality in u, Hölder's inequality in v, the inequality $|v|\leq|v'|+|u'|$ and the change of variable $(u,v):=(u',v')$, we obtain

$$\begin{aligned}&\left|\int_{\mathbb{R}^3}\Gamma_{\mathrm{gain}}(g_1,g_2)g_3\mathrm{d}v\right| = \left|\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\int_{\mathbb{S}^2}\sqrt{\mu(u)}g_1(u')g_2(v')g_3(v)|u-v|^\gamma q_0(\theta)\mathrm{d}\omega\mathrm{d}u\mathrm{d}v\right|\\ \lesssim\;& \int_{\mathbb{S}^2}q_0(\theta)\int_{\mathbb{R}^3}|g_3(v)|\nu(v)\left|\int_{\mathbb{R}^3}|g_1(u')|^p|g_2(v')|^p\mathrm{d}u\right|^{1/p}\mathrm{d}v\mathrm{d}\omega\\ \lesssim\;& \left|\int_{\mathbb{R}^3}\nu|g_3|^q\mathrm{d}v\right|^{1/q}\int_{\mathbb{S}^2}q_0(\theta)\left|\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\nu(v)|g_1(u')|^p|g_2(v')|^p\mathrm{d}u\mathrm{d}v\right|^{1/p}\mathrm{d}\omega\\ \lesssim\;& \left|\int_{\mathbb{R}^3}\nu|g_3|^q\mathrm{d}v\right|^{1/q}\int_{\mathbb{S}^2}q_0(\theta)\left|\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\nu(v')\nu(u')|g_1(u')|^p|g_2(v')|^p\mathrm{d}u\mathrm{d}v\right|^{1/p}\mathrm{d}\omega\\ \lesssim\;& \left|\int_{\mathbb{R}^3}\nu|g_3|^q\mathrm{d}v\right|^{1/q}\left|\int_{\mathbb{R}^3}\nu|g_1|^p\mathrm{d}v\right|^{1/p}\left|\int_{\mathbb{R}^3}\nu|g_2|^p\mathrm{d}v\right|^{1/p}.\end{aligned}$$

We then integrate over the domain Ω and apply Hölder's inequality in space.

For the second estimate of (25), we first define for some $C_\zeta>0$,

$$\nu_\zeta(v) = \int_{\mathbb{R}^3}\int_{\mathbb{S}^2}|v-u|^\gamma q_0(\theta)\sqrt{\mu(u)}e^{-\zeta|u|^2}\mathrm{d}\omega\mathrm{d}u, \quad \mathbf{k}_{\zeta,1}(v,u) = \int_{\mathbb{S}^2}|v-u|^\gamma q_0(\theta)e^{-\zeta|u|^2}e^{-\zeta|v|^2}\mathrm{d}\omega,$$

$$\mathbf{k}_{\zeta,2}(v,u) = \frac{2}{|v-u|^2}e^{-C_\zeta\left[|u-v|^2+\frac{(|u|^2-|v|^2)^2}{|u-v|^2}\right]}\int_{w\cdot(u-v)=0}q_0^*(|w|,|u-v|)e^{-C_\zeta|w+\varsigma|}(|w|^2+|u-v|^2)^{\gamma/2}\mathrm{d}w,$$

and $\mathbf{k}_\zeta = \mathbf{k}_{\zeta,2} - \mathbf{k}_{\zeta,1}$. Clearly the estimate of (26) is valid for $\mathbf{k}_\zeta$ by the same proof in Lemma 1. Furthermore, if $||e^{\zeta|v|^2} g_1||_\infty < \infty$,

$$
\begin{aligned}
|\Gamma(g_1, g_2)| &\leq \Big\{ \big|\Gamma_{\text{gain}}(e^{-\zeta|v|^2}, g_2)\big| + \big|\Gamma_{\text{loss}}(e^{-\zeta|v|^2}, g_2)\big| \Big\} \times ||e^{\zeta|v|^2} g_1||_\infty \\
&\leq \Big\{ \big|\mu^{-1/2}(v) Q_{\text{gain}}(\mu^{1/2} e^{-\zeta|v|^2}, \mu^{1/2} g_2)\big| + \big|\mu^{-1/2}(v) Q_{\text{loss}}(\mu^{1/2} e^{-\zeta|v|^2}, \mu^{1/2} g_2)\big| \Big\} \times ||e^{\zeta|v|^2} g_1||_\infty \\
&\leq ||e^{\zeta|v|^2} g_1||_\infty \times \mu(v)^{-1/2} \int_{\mathbb{R}^3} \int_{\mathbb{S}^2} |v-u|^\gamma q_0(\theta) \Big\{ \mu(v')^{1/2} |g_2(v')| \mu^{1/2}(u') e^{-\zeta|u'|^2} \\
&\qquad + \mu(u')^{1/2} |g_2(u')| \mu^{1/2}(v') e^{-\zeta|v'|^2} + \mu(u)^{1/2} |g_2(u)| \mu^{1/2}(v) e^{-\zeta|v|^2} + \mu(v)^{1/2} |g_2(v)| \mu^{1/2}(u) e^{-\zeta|u|^2} \Big\} \\
&\leq ||e^{\zeta|v|^2} g_1||_\infty \times \Big\{ \int_{\mathbb{R}^3} \big\{ \mathbf{k}_{\zeta,1}(v,u) + \mathbf{k}_{\zeta,2}(v,u) \big\} |g_2(u)| \mathrm{d}u + \nu_\zeta(v) |g_2(v)| \Big\}.
\end{aligned}
$$

We therefore conclude second estimates of (25). In order to prove (27) we plug in $g_1 = g_2 = e^{-\zeta|v|^2}$ to have

$$
\begin{aligned}
&|\Gamma(g_1, g_2)| \leq \mu^{-1/2}(v) \big| Q_{\text{gain}}(\mu^{1/2} e^{-\zeta|v|^2}, \mu^{1/2} e^{-\zeta|v|^2}) + Q_{\text{loss}}(\mu^{1/2} e^{-\zeta|v|^2}, \mu^{1/2} e^{-\zeta|v|^2}) \big| \times ||e^{-\zeta|v|^2} g_1||_\infty ||e^{-\zeta|v|^2} g_2||_\infty \\
&= \int_{\mathbb{R}^3} \int_{\mathbb{S}^2} |v-u|^\gamma q_0(\theta) \big[e^{-\zeta|v'|^2 - \zeta|u'|^2} + e^{-\zeta|v|^2 - \zeta|u|^2} \big] ||e^{-\zeta|v|^2} g_1||_\infty ||e^{-\zeta|v|^2} g_2||_\infty \lesssim e^{-\frac{\zeta}{4}|v|^2} ||e^{-\zeta|v|^2} g_1||_\infty ||e^{-\zeta|v|^2} g_2||_\infty.
\end{aligned}
$$

The first estimate of (27) is a direct consequence of Lemma 5 of [7]. □

Lemma 3. *For any g_1, g_2, g_3, we have for $(i,j) = (1,2)$ and for $(i,j) = (2,1)$*

$$
\begin{aligned}
&\Big| \iint_{\Omega \times \mathbb{R}^3} \nabla_v \Gamma(g_1, g_2) g_3 \mathrm{d}v \mathrm{d}x \Big| \lesssim \sup_\Omega \Big\{ \int_{\mathbb{R}^3} \nu |g_i|^p \mathrm{d}v \Big\}^{1/p} \|\nu^{1/p} g_j\|_p \|\nu^{1/q} g_3\|_q \\
&\quad + \sup_\Omega \Big\{ \int_{\mathbb{R}^3} \nu |g_2|^p \mathrm{d}v \Big\}^{1/p} \|\nu^{1/p} \nabla_v g_1\|_p \|\nu^{1/q} g_3\|_q + \sup_\Omega \Big\{ \int_{\mathbb{R}^3} \nu |g_1|^p \mathrm{d}v \Big\}^{1/p} \|\nu^{1/p} \nabla_v g_2\|_p \|\nu^{1/q} g_3\|_q, \\
&|\Gamma_v(g_1, g_2)(v)| \lesssim \langle v \rangle^{\gamma - \beta} ||\langle v \rangle^\beta g_1||_\infty ||\langle v \rangle^\beta g_2||_\infty, \qquad |\Gamma_v(g_1, g_2)(v)| \lesssim e^{-\frac{\theta}{4}|v|^2} ||e^{\theta|v|^2} g_1||_\infty ||e^{\theta|v|^2} g_2||_\infty.
\end{aligned}
\tag{28}
$$

Proof. We compute the velocity derivative of Γ after the change of variable $u := v - u$:

$$
\begin{aligned}
\nabla_v \Gamma(g_1, g_2) &= \Gamma(g_1, \nabla_v g_2) + \Gamma(\nabla_v g_1, g_2) + \Gamma_v(g_1, g_2) \equiv \Gamma(g_1, \nabla_v g_2) + \Gamma(\nabla_v g_1, g_2) \\
&\quad + \int_{\mathbb{R}^3} \int_{\mathbb{S}^2} \nabla_v(\sqrt{\mu})(v-u) g_1(v - u_\perp) g_2(v - u_\parallel) |u|^\gamma q_0(\theta) \mathrm{d}\omega \mathrm{d}u \\
&\quad - g_2(v) \int_{\mathbb{R}^3} \int_{\mathbb{S}^2} \nabla_v(\sqrt{\mu})(v-u) g_1(v-u) |u|^\gamma q_0(\theta) \mathrm{d}\omega \mathrm{d}u,
\end{aligned}
\tag{29}
$$

where $u_\parallel = (u \cdot \omega)\omega$ and $u_\perp = u - (u \cdot \omega)\omega$. The two first terms are estimated directly with lemma 2. Since $|\nabla_v(\sqrt{\mu})(v-u)| \leq C \sqrt{\mu}^{1/2}(v-u)$, the two remaining terms are estimated as in the proof of Lemma 2. Following the same proof of Lemma 2 we have the last two estimates. □

3 Traces and the In-flow Problems

Let $t_{\mathbf{b}}(x,v)$ be the the backward exit time as defined in [7]:

$$
t_{\mathbf{b}}(x,v) = \sup\{\tau > 0 : x - sv \in \Omega \text{ for all } 0 \leq s \leq \tau\}, \qquad x_{\mathbf{b}}(v) = x - t_{\mathbf{b}} v. \tag{30}
$$

Recall the almost grazing set γ_+^ε defined in (11). We first estimate the outgoing trace on $\gamma_+ \setminus \gamma_+^\varepsilon$. We remark that for the outgoing part, our estimate is global in time without cut-off, in contrast to the general trace theorem.

Lemma 4. *Assume that $\varphi = \varphi(v)$ is $L^\infty_{loc}(\mathbb{R}^3)$. For any small parameter $\varepsilon > 0$, there exists a constant $C_{\varepsilon,T,\Omega} > 0$ such that for any h in $L^1([0,T], L^1(\Omega \times \mathbb{R}^3))$ with $\partial_t h + v \cdot \nabla_x h + \varphi h$ in $L^1([0,T], L^1(\Omega \times \mathbb{R}^3))$, we have for all $0 \leq t \leq T$,*

$$
\int_0^t \int_{\gamma_+ \setminus \gamma_+^\varepsilon} |h| \mathrm{d}\gamma \mathrm{d}s \leq C_{\varepsilon,T,\Omega} \Big[||h_0||_1 + \int_0^t \big\{ \|h(s)\|_1 + \big\| [\partial_t + v \cdot \nabla_x + \varphi] h(s) \big\|_1 \big\} \mathrm{d}s \Big].
$$

Furthermore, for any (s,x,v) in $[0,T]\times\Omega\times\mathbb{R}^3$ the function $h(s+s',x+s'v,v)$ is absolutely continuous in s' in the interval $[-\min\{t_{\mathbf{b}}(x,v),s\},\min\{t_{\mathbf{b}}(x,-v),T-s\}]$.

Proof. With a proper change of variables (e.g. Page 247 in [1]) we have

$$\begin{aligned}
&\int_0^T\iint_{\Omega\times\mathbb{R}^3}h(t,x,v)\mathrm{d}v\mathrm{d}x\mathrm{d}t \qquad (31)\\
&=\int_{-\min\{T,t_{\mathbf{b}}(x,v)\}}^0\iint_{\Omega\times\mathbb{R}^3}h(T+s,x+sv,v)\mathrm{d}v\mathrm{d}x\mathrm{d}s+\int_0^{\min\{T,t_{\mathbf{b}}(x,-v)\}}\iint_{\Omega\times\mathbb{R}^3}h(0+s,x+sv,v)\mathrm{d}v\mathrm{d}x\mathrm{d}s\\
&\quad+\int_0^T\int_{\gamma_+}\int_{-\min\{t,t_{\mathbf{b}}(x,v)\}}^0 h(t+s,x+sv,v)\mathrm{d}s\mathrm{d}\gamma\mathrm{d}t+\int_0^T\int_{\gamma_-}\int_0^{\min\{T-t,t_{\mathbf{b}}(x,-v)\}}h(t+s,x+sv,v)\mathrm{d}s\mathrm{d}\gamma\mathrm{d}t.
\end{aligned}$$

For $(t,x,v)\in[0,T]\times\gamma_+$ and $0\le s\le\min\{t,t_{\mathbf{b}}(x,v)\}$,

$$h(t,x,v)=h(t-s,x-sv,v)e^{-\varphi(v)s}+\int_{-s}^0 e^{\varphi(v)\tau}[\partial_t h+v\cdot\nabla_x h+\varphi(v)h](t+\tau,x+\tau v,v)\mathrm{d}\tau.$$

Now for $(t,x,v)\in[\varepsilon_1,T]\times\gamma_+\setminus\gamma_+^\varepsilon$, we integrate over $\int_{\varepsilon_1}^T\int_{\gamma_+\setminus\gamma_+^\varepsilon}\int_{\min\{t,t_{\mathbf{b}}(x,v)\}}^0$ to get

$$\begin{aligned}
&\min\{\varepsilon_1,\varepsilon^3\}\times\int_{\varepsilon_1}^T\int_{\gamma_+\setminus\gamma_+^\varepsilon}|h(t,x,v)|\mathrm{d}\gamma\mathrm{d}t\lesssim\min_{[\varepsilon_1,T]\times[\gamma_+\setminus\gamma_+^\varepsilon]}\{t,t_{\mathbf{b}}(x,v)\}\times\int_{\varepsilon_1}^T\int_{\gamma_+\setminus\gamma_+^\varepsilon}|h(t,x,v)|\mathrm{d}\gamma\mathrm{d}t\\
&\lesssim\int_0^T\int_{\gamma_+}\int_{-\min\{t,t_{\mathbf{b}}(x,v)\}}^0|h(t+s,x+sv,v)|\mathrm{d}s\mathrm{d}\gamma\mathrm{d}t\\
&\quad+T\int_0^T\int_{\gamma_+}\int_{-\min\{t,t_{\mathbf{b}}(x,v)\}}^0|\partial_t h+v\cdot\nabla_x h+\varphi h|(t+\tau,x+\tau v,v)\mathrm{d}\tau\mathrm{d}\gamma\mathrm{d}t\\
&\lesssim\int_0^T||h(t)||_1\mathrm{d}t+\int_0^T||[\partial_t+v\cdot\nabla_x+\varphi]h(t)||_1\mathrm{d}t,
\end{aligned}$$

where we have used the integration identity (31), and (40) of [7] to obtain $t_{\mathbf{b}}(x,v)\ge C_\Omega|n(x)\cdot v|/|v|^2\ge C_\Omega\varepsilon^3$ for $(x,v)\in\gamma_+\setminus\gamma_+^\varepsilon$. Now we choose $\varepsilon_1=\varepsilon_1(\Omega,\varepsilon)$ as

$$\varepsilon_1\le C_\Omega\varepsilon^3\le\inf_{(x,v)\in\gamma_+\setminus\gamma_+^\varepsilon}t_{\mathbf{b}}(x,v).$$

We only need to show, for $\varepsilon_1\le C_\Omega\varepsilon^3$,

$$\int_0^{\varepsilon_1}\int_{\gamma_+\setminus\gamma_+^\varepsilon}|h(t,x,v)|\mathrm{d}\gamma\mathrm{d}t\lesssim_{\Omega,\varepsilon,\varepsilon_1}||h_0||_1+\int_0^{\varepsilon_1}||[\partial_t+v\cdot\nabla_x+\varphi]h(t)||_1\mathrm{d}t.$$

Because of our choice ε and ε_1, $t_{\mathbf{b}}(x,v)>t$ for all $(t,x,v)\in[0,\varepsilon_1]\times\gamma_+\setminus\gamma_+^\varepsilon$. Then

$$|h(t,x,v)|\lesssim|h_0(x-tv,v)|+\int_0^t\Big|[\partial_t+v\cdot\nabla_x+\varphi(v)]h(s,x-(t-s)v,v)\Big|\mathrm{d}s,$$

where the second contribution is bounded, from (31), by

$$\int_0^{\varepsilon_1}\int_{\gamma_+\setminus\gamma_+^\varepsilon}\int_0^t\Big|[\partial_t+v\cdot\nabla_x+\varphi(v)]h(s,x-(t-s)v,v)\Big|\mathrm{d}s\mathrm{d}\gamma\mathrm{d}t\lesssim\int_0^{\varepsilon_1}||[\partial_t+v\cdot\nabla_x+\varphi(v)]h(t)||_1\mathrm{d}t.$$

Consider the initial datum contribution of $|h_0(x-tv,v)|$: Assume $\partial_{x_3}\xi(x_0)\neq0$. By the implicit function theorem $\partial\Omega$ can be represented locally by the graph $\eta=\eta(x_1,x_2)$ satisfying $\xi(x_1,x_2,\eta(x_1,x_2))=0$ and $(\partial_{x_1}\eta(x_1,x_2),\partial_{x_2}\eta(x_1,x_2))=(-\partial_{x_1}\xi/\partial_{x_3}\xi,-\partial_{x_2}\xi/\partial_{x_3}\xi)$ at $(x_1,x_2,\eta(x_1,x_2))$. We define the change of variables

$$(x,t)\in\partial\Omega\cap\{x\sim x_0\}\times[0,\varepsilon_1]\mapsto y=x-tv\in\bar\Omega,$$

where $\left|\frac{\partial y}{\partial(x,t)}\right| = -v_1\frac{\partial_{x_1}\xi}{\partial_{x_3}\xi} - v_2\frac{\partial_{x_2}\xi}{\partial_{x_3}\xi} - v_3$. Therefore

$$|n(x)\cdot v|\mathrm{d}S_x\mathrm{d}t = (n(x)\cdot v)\Big[1+(\frac{\partial_{x_1}\xi}{\partial_{x_3}\xi})^2+(\frac{\partial_{x_2}\xi}{\partial_{x_3}\xi})^2\Big]^{1/2}\mathrm{d}x_1\mathrm{d}x_2\mathrm{d}t = \Big[-v_1\frac{\partial_{x_1}\xi}{\partial_{x_3}\xi} - v_2\frac{\partial_{x_2}\xi}{\partial_{x_3}\xi} - v_3\Big]\mathrm{d}x_1\mathrm{d}x_2\mathrm{d}t = \mathrm{d}y,$$

and $\int_0^{\varepsilon_1}\int_{\gamma_+\setminus\gamma_+^\varepsilon\cap\{x\sim x_0\}}|h_0(x-tv,v)|\mathrm{d}\gamma\mathrm{d}t \ \lesssim_{\varepsilon,\varepsilon_1,x_0} \ \iint_{\Omega\times\mathbb{R}^3}|h_0(y,v)|\mathrm{d}y\mathrm{d}v$. Since $\partial\Omega$ is compact we can choose a finite covers of $\partial\Omega$ and repeat the same argument for each piece to conclude

$$\int_0^{\varepsilon_1}\int_{\gamma_+\setminus\gamma_+^\varepsilon}|h_0(x-tv,v)|\mathrm{d}\gamma\mathrm{d}t \ \lesssim_{\Omega,\varepsilon,\varepsilon_1} \ \iint_{\Omega\times\mathbb{R}^3}|h_0(y,v)|\mathrm{d}y\mathrm{d}v.$$

□

Lemma 5 (Green's Identity)**.** *For $p\in[1,\infty)$ assume that $f,\partial_t f+v\cdot\nabla_x f\in L^p([0,T];L^p(\Omega\times\mathbb{R}^3))$ and $f_{\gamma_-}\in L^p([0,T];L^p(\gamma))$. Then $f\in C^0([0,T];L^p(\Omega\times\mathbb{R}^3))$ and $f_{\gamma_+}\in L^p([0,T];L^p(\gamma))$ and for almost every $t\in[0,T]$:*

$$||f(t)||_p^p + \int_0^t |f|_{\gamma_+,p}^p = ||f(0)||_p^p + \int_0^t |f|_{\gamma_-,p}^p + \int_0^t\iint_{\Omega\times\mathbb{R}^3}\{\partial_t f + v\cdot\nabla_x f\}|f|^{p-2}f.$$

Lemma 6 (Velocity Lemma)**.** *Let Ω be strictly convex as defined in (2). Recall (12) and define along the trajectories for $x\in\bar{\Omega}$ and $s\in[t_1,t_2]$ with $t_1=t-t_{\mathbf{b}}(x,v),\ t_2=t+t_{\mathbf{b}}(x,-v)$,*

$$\alpha(s;t,x,v)\equiv|v\cdot\nabla\xi(x-(t-s)v)|^2 - 2\{v\cdot\nabla^2\xi(x-(t-s)v)\cdot v\}\xi(x-(t-s)v).$$

Then there exists constant $C_\Omega>0$ such that, for $t_1\le s_1\le s_2\le t_2$,

$$e^{C_\Omega\langle v\rangle s_1}\alpha(s_1) \ \le \ e^{C_\Omega\langle v\rangle s_2}\alpha(s_2), \quad e^{-C_\Omega\langle v\rangle s_1}\alpha(s_1) \ \ge \ e^{-C_\Omega\langle v\rangle s_2}\alpha(s_2).$$

See [7] for the proof. Now we state and prove following propositions for the in-flow problems:

$$\{\partial_t + v\cdot\nabla_x + \nu(v)\}f = H, \quad f(0,x,v)=f_0(x,v), \quad f(t,x,v)|_{\gamma_-}=g(t,x,v). \tag{32}$$

For notational simplicity, we define

$$\partial_t f_0 \ \equiv \ -v\cdot\nabla_x f_0 - \nu f_0 + H(0,x,v), \tag{33}$$

$$\nabla_x g \ \equiv \ \frac{n}{n\cdot v}\Big\{-\partial_t g - \sum_{i=1}^2(v\cdot\tau_i)\partial_{\tau_i}g - \nu g + H\Big\} + \sum_{i=1}^2\tau_i\partial_{\tau_i}g. \tag{34}$$

We remark that $\partial_t f_0$ is obtained from formally solving (32), and (34) leads to the usual tangential derivatives of $\partial_{\tau_i}g$, while defines new 'normal derivative' $\partial_n g$ from the equation (32).

Proposition 1. *Assume a compatibility condition*

$$f_0(x,v) = g(0,x,v) \quad \mathit{for} \ \ (x,v)\in\gamma_-. \tag{35}$$

For any fixed $p\in[1,\infty)$, assume

$$\begin{aligned} \nabla_x f_0, \nabla_v f_0, -v\cdot\nabla_x f_0 - \nu f_0 + H(0,x,v) \ &\in \ L^p(\Omega\times\mathbb{R}^3), \\ g,\partial_t g,\nabla_v g,\partial_{\tau_i}g,\frac{1}{n(x)\cdot v}\{-\partial_t g - \sum_i(v\cdot\tau_i)\partial_{\tau_i}g - \nu(v)g + H\} \ &\in \ L^p([0,T]\times\gamma_-), \end{aligned}$$

and, assume $1/p+1/q=1$ there exist $TC_T\sim O(T)$ and $\varepsilon\ll 1$ such that for all $t\in[0,T]$

$$\left|\iint_{\Omega\times\mathbb{R}^3}\partial H(t)h(t)\mathrm{d}x\mathrm{d}v\right| \le C_T\big\{||h(t)||_q + \varepsilon||\nu^{1/q}h(t)||_q\big\}.$$

Then for sufficiently small $T>0$ there exists a unique solution f to (32) such that $f,\partial_t f,\nabla_x f,\nabla_v f\in C^0([0,T];L^p(\Omega\times\mathbb{R}^3))$ and the traces satisfy

$$\begin{aligned}\partial_t f|_{\gamma_-}=\partial_t g,\ \nabla_v f|_{\gamma_-}=\nabla_v g,\ \nabla_x f|_{\gamma_-}=\nabla_x g,\quad on\quad \gamma_-,\\ \nabla_x f(0,x,v)=\nabla_x f_0,\ \nabla_v f(0,x,v)=\nabla_v f_0,\ \partial_t f(0,x,v)=\partial_t f_0,\quad in\quad \Omega\times\mathbb{R}^3,\end{aligned}\tag{36}$$

where $\partial_t f_0$ and $\nabla_x g$ are given by (33) and (34). Moreover

$$\begin{aligned}&||\partial_t f(t)||_p^p+\int_0^t|\partial_t f|^p_{\gamma_+,p}+p\int_0^t||\nu^{1/p}\partial_t f||_p^p \qquad (37)\\ &\quad=||\partial_t f_0||_p^p+\int_0^t|\partial_t g|^p_{\gamma_-,p}+p\int_0^t\iint_{\Omega\times\mathbb{R}^3}\partial_t H|\partial_t f|^{p-2}\partial_t f,\\ &||\nabla_x f(t)||_p^p+\int_0^t|\nabla_x f|^p_{\gamma_+,p}+p\int_0^t||\nu^{1/p}\nabla_x f||_p^p \qquad (38)\\ &\quad=||\nabla_x f_0||_p^p+\int_0^t|\nabla_x g|^p_{\gamma_-,p}+p\int_0^t\iint_{\Omega\times\mathbb{R}^3}\nabla_x H|\nabla_x f|^{p-2}\nabla_x f,\\ &||\nabla_v f(t)||_p^p+\int_0^t|\nabla_v f|^p_{\gamma_+,p}+p\int_0^t||\nu^{1/p}\nabla_v f||_p^p \qquad (39)\\ &\quad=||\nabla_v f_0||_p^p+\int_0^t|\nabla_v g|^p_{\gamma_-,p}+p\int_0^t\iint_{\Omega\times\mathbb{R}^3}\{\nabla_v H-\nabla_x f-\nabla_v\nu f\}|\nabla_v f|^{p-2}\nabla_v f.\end{aligned}$$

Proof. We apply the trace theorem to the derivatives of f by explicit computations. First we assume f_0,g and H have compact support in $v\in\mathbb{R}^3$. We integrate the equation (32) along the backward trajectories. If the initial condition is reached before hitting the boundary (case $t<t_{\mathbf{b}}$), we have

$$f(t,x,v)=e^{-t\nu(v)}f_0(x-tv,v)+\int_0^t e^{-s\nu(v)}H(t-s,x-vs,v)\mathrm{d}s.$$

If the boundary is first reached (case $t>t_{\mathbf{b}}$), we have

$$f(t,x,v)=e^{-t_{\mathbf{b}}\nu(v)}g(t-t_{\mathbf{b}},x_{\mathbf{b}},v)+\int_0^{t_{\mathbf{b}}}e^{-s\nu(v)}H(t-s,x-vs,v)\mathrm{d}s.$$

Let us rewrite it

$$\begin{aligned}f(t,x,v)=&\mathbf{1}_{\{t\le t_{\mathbf{b}}\}}e^{-t\nu(v)}f_0(x-tv,v)+\mathbf{1}_{\{t>t_{\mathbf{b}}\}}e^{-t_{\mathbf{b}}\nu(v)}g(t-t_{\mathbf{b}},x_{\mathbf{b}},v)\\ &+\int_0^{\min(t,t_{\mathbf{b}})}e^{-s\nu(v)}H(t-s,x-vs,v)\mathrm{d}s.\end{aligned}\tag{40}$$

We take derivative of f with respect to time, space and velocity for $t\neq t_{\mathbf{b}}$. Recall the following derivatives of $x_{\mathbf{b}}$ and $t_{\mathbf{b}}$ (see lemma 2 in [7]) :

$$\nabla_x t_{\mathbf{b}}=\frac{n(x_{\mathbf{b}})}{v\cdot n(x_{\mathbf{b}})},\quad \nabla_v t_{\mathbf{b}}=-\frac{t_{\mathbf{b}}n(x_{\mathbf{b}})}{v\cdot n(x_{\mathbf{b}})},\quad \nabla_x x_{\mathbf{b}}=I-\frac{n(x_{\mathbf{b}})}{v\cdot n(x_{\mathbf{b}})}\otimes v,\quad \nabla_v x_{\mathbf{b}}=-t_{\mathbf{b}}I+\frac{t_{\mathbf{b}}n(x_{\mathbf{b}})}{v\cdot n(x_{\mathbf{b}})}\otimes v.\tag{41}$$

Since g is defined on a surface, we cannot define its space gradient. We then use directly the gradient in space of $g(x_{\mathbf{b}})$. Regarding $g(t-t_{\mathbf{b}},x_{\mathbf{b}}(x,v),v)$ as function on $[0,T]\times\bar\Omega\times\mathbb{R}^3$ we obtain from (41)

$$\begin{aligned}\nabla_x[g(t-t_{\mathbf{b}},x_{\mathbf{b}},v)]&=-\nabla_x t_{\mathbf{b}}\partial_t g+\nabla_x x_{\mathbf{b}}\nabla_\tau g=-\frac{n(x_{\mathbf{b}})}{v\cdot n(x_{\mathbf{b}})}\partial_t g+\left(I-\frac{n\otimes v}{n\cdot v}\right)\nabla_\tau g\\ &=\tau_1\partial_{\tau_1}g+\tau_2\partial_{\tau_2}g-\frac{n(x_{\mathbf{b}})}{v\cdot n(x_{\mathbf{b}})}\left\{\partial_t g+v\cdot\tau_1\partial_{\tau_1}g+v\cdot\tau_2\partial_{\tau_2}g\right\},\\ \nabla_v[g(t-t_{\mathbf{b}},x_{\mathbf{b}},v)]&=-t_{\mathbf{b}}\nabla_x[g(t-t_{\mathbf{b}},x_{\mathbf{b}},v)]+\nabla_v g,\end{aligned}$$

where $\tau_1(x)$ and $\tau_2(x)$ are unit vectors satisfying $\tau_1(x)\cdot n(x)=0=\tau_2(x)\cdot n(x)$ and $\tau_1(x)\times\tau_2(x)=n(x)$.

Therefore by direct computation for $t \neq t_{\mathbf{b}}$, we deduce

$$\begin{aligned}
\partial_t f(t,x,v)\mathbf{1}_{\{t\neq t_{\mathbf{b}}\}} &= -\mathbf{1}_{\{t<t_{\mathbf{b}}\}} e^{-t\nu(v)}[\nu f_0 + v\cdot\nabla_x f_0 - H_{|t=0}](x-tv,v) + \mathbf{1}_{\{t>t_{\mathbf{b}}\}} e^{-t_{\mathbf{b}}\nu}\partial_t g(t-t_{\mathbf{b}},x_{\mathbf{b}},v) \quad (42)\\
&\quad + \int_0^{\min(t,t_{\mathbf{b}})} e^{-s\nu}\partial_t H(t-s,x-vs,v)\mathrm{d}s,\\
\nabla_x f(t,x,v)\mathbf{1}_{\{t\neq t_{\mathbf{b}}\}} &= \mathbf{1}_{\{t<t_{\mathbf{b}}\}} e^{-t\nu}\nabla_x f_0(x-tv,v) \quad (43)\\
&\quad + \mathbf{1}_{\{t>t_{\mathbf{b}}\}} e^{-t_{\mathbf{b}}\nu}\Big\{\sum_{i=1}^2 \tau_i\partial_{\tau_i} g - \frac{n(x_{\mathbf{b}})}{v\cdot n(x_{\mathbf{b}})}\big\{\partial_t g + \sum_{i=1}^2 (v\cdot\tau_i)\partial_{\tau_i} g + \nu g - H\big\}\Big\}(t-t_{\mathbf{b}},x_{\mathbf{b}},v)\\
&\quad + \int_0^{\min(t,t_{\mathbf{b}})} e^{-s\nu}\nabla_x H(t-s,x-vs,v)\mathrm{d}s,\\
\nabla_v f(t,x,v)\mathbf{1}_{\{t\neq t_{\mathbf{b}}\}} &= \mathbf{1}_{\{t<t_{\mathbf{b}}\}} e^{-t\nu}[-t\nabla_x f_0 + \nabla_v f_0 - t\nabla_v\nu(v) f_0](x-tv,v) \quad (44)\\
&\quad - \mathbf{1}_{\{t>t_{\mathbf{b}}\}} t_{\mathbf{b}} e^{-t_{\mathbf{b}}\nu}\Big\{\sum_{i=1}^2 \tau_i\partial_{\tau_i} g - \frac{n(x_{\mathbf{b}})}{v\cdot n(x_{\mathbf{b}})}\big\{\partial_t g + \sum_{i=1}^2 (v\cdot\tau_i)\partial_{\tau_i} g + \nu g - H\big\}\Big\}(t-t_{\mathbf{b}},x_{\mathbf{b}},v)\\
&\quad + \mathbf{1}_{\{t>t_{\mathbf{b}}\}} e^{-t_{\mathbf{b}}\nu}\Big\{\nabla_v g(t-t_{\mathbf{b}},x_{\mathbf{b}},v) - t_{\mathbf{b}}\nabla_v\nu(v) g(t-t_{\mathbf{b}},x_{\mathbf{b}},v)\Big\}\\
&\quad + \int_0^{\min(t,t_{\mathbf{b}})} e^{-s\nu}\{\nabla_v H - s\nabla_x H - s\nabla\nu H\}(t-s,x-vs,v)\mathrm{d}s.
\end{aligned}$$

We, first, show that $\partial f\mathbf{1}_{\{t>t_{\mathbf{b}}\}} \in L^p$ and $\partial f\mathbf{1}_{\{t<t_{\mathbf{b}}\}}$ separately. Now we take L^p norms above with the changes of variables in Lemma 2.1 of [6] and using Jensen's inequality in $[0,t]$. More precisely, for $\phi \in L^1$ with $\phi \geq 0$,

$$\begin{aligned}
&\iint_{\Omega\times\mathbb{R}^3} \mathbf{1}_{\{x-tv\in\Omega\}}\phi(x-tv,v) = \int_{\mathbb{R}^3}\left[\int_\Omega \mathbf{1}_{\{x-tv\in\Omega\}}\phi(x-tv,v)\mathrm{d}x\right]\mathrm{d}v \leq \iint_{\Omega\times\mathbb{R}^3}\phi(x,v),\\
&\iint_{\{\Omega\times\mathbb{R}^3\}\cap B((x_0,v_0);\delta)} \mathbf{1}_{\{t\geq t_{\mathbf{b}}\}}\phi(t-t_{\mathbf{b}}(x,v),x_{\mathbf{b}}(x,v),v) \leq \int_0^t\int_{\partial\Omega\times\mathbb{R}^3}\phi(s,x,v)|n(x)\cdot v|\mathrm{d}S_x\mathrm{d}v\mathrm{d}s,
\end{aligned} \tag{45}$$

where for the second inequality we have used the change of variables for fixed t,v,

$$x \mapsto (t-t_{\mathbf{b}}(x,v), x_{\mathbf{b}}(x,v)). \tag{46}$$

In fact, without the loss of generality we may assume $\partial_{x_3}\xi(x_{\mathbf{b}}(x,v)) \neq 0$ for $(x,v) \in B((x_0,v_0);\delta)$ so that $x_{\mathbf{b}}(x,v) = (x_{\mathbf{b},1}, x_{\mathbf{b},2}, \eta(x_{\mathbf{b},1},x_{\mathbf{b},2}))$. Using (41), we compute the Jacobian

$$\det\begin{pmatrix} -\nabla_x t_{\mathbf{b}} \\ -\nabla_x x_{\mathbf{b},1} \\ -\nabla_x x_{\mathbf{b},2}\end{pmatrix} = \det\begin{pmatrix} -(v\cdot n)^{-1} n \\ -\nabla_x x_{\mathbf{b},1} \\ -\nabla_x x_{\mathbf{b},2}\end{pmatrix} = \left|-v_1\frac{\partial_{x_1}\xi}{\partial_{x_3}\xi} - v_2\frac{\partial_{x_2}\xi}{\partial_{x_3}\xi} + v_3\right|^{-1}.$$

Therefore $\mathrm{d}x\mathrm{d}v = \left|-v_1\frac{\partial_{x_1}\xi}{\partial_{x_3}\xi} - v_2\frac{\partial_{x_2}\xi}{\partial_{x_3}\xi} + v_3\right|\mathrm{d}x_1\mathrm{d}x_2\mathrm{d}v\mathrm{d}t = |n\cdot v|\mathrm{d}S_x\mathrm{d}v\mathrm{d}t = \mathrm{d}\gamma\mathrm{d}t$. Using these changes of variables, we obtain

$$\begin{aligned}
\|f(t)\mathbf{1}_{\{t\neq t_{\mathbf{b}}\}}\|_p &\leq \|f_0\|_p + \left[\int_0^t\int_{\gamma_-}|g|^p\mathrm{d}\gamma\mathrm{d}s\right]^{1/p} + t^{(p-1)/p}\left[\int_0^t\|H\|_p^p\mathrm{d}s\right]^{1/p},\\
\|\partial_t f(t)\mathbf{1}_{\{t\neq t_{\mathbf{b}}\}}\|_p &\leq \|v\cdot\nabla_x f_0 + \nu f_0 - H(0,\cdot,\cdot)\|_p\\
&\quad + \left[\int_0^t\int_{\gamma_-}|\partial_t g|^p\mathrm{d}\gamma\mathrm{d}s\right]^{1/p} + t^{(p-1)/p}\left[\int_0^t\|\partial_t H\|_p^p\mathrm{d}s\right]^{1/p},\\
\|\nabla_x f(t)\mathbf{1}_{\{t\neq t_{\mathbf{b}}\}}\|_p &\leq \|\nabla_x f_0\|_p + t^{(p-1)/p}\left[\int_0^t\|\nabla_x H(s)\|_p^p\mathrm{d}s\right]^{1/p}\\
&\quad + \left[\int_0^t\int_{\gamma_-}\left|\Big\{\sum_{i=1}^2\tau_i\partial_{\tau_i} g - \frac{n(x_{\mathbf{b}})}{v\cdot n(x_{\mathbf{b}})}\big\{\partial_t g + \sum_{i=1}^2(v\cdot\tau_i)\partial_{\tau_i} g + \nu g - H(0,x,v)\big\}\Big\}(t-t_{\mathbf{b}},x_{\mathbf{b}},v)\right|^p\mathrm{d}\gamma\mathrm{d}s\right]^{1/p},
\end{aligned} \tag{47}$$

$$
\begin{aligned}
\|\nabla_v f(t)\mathbf{1}_{\{t\neq t_{\mathbf{b}}\}}\|_p &\leq t\|\nabla_x f_0\|_p + \|\nabla_v f_0\|_p + C\|f_0\|_p + Ct\left[\int_0^t\int_{\gamma_-}|g|^p\mathrm{d}\gamma\mathrm{d}s\right]^{1/p}\\
&+t\left[\int_0^t\int_{\gamma_-}\left|\Big\{\sum_{i=1}^2\tau_i\partial_{\tau_i}g - \frac{n(x_{\mathbf{b}})}{v\cdot n(x_{\mathbf{b}})}\big\{\partial_t g + \sum_{i=1}^2(v\cdot\tau_i)\partial_{\tau_i}g + \nu g - H(0,x,v)\big\}\Big\}(t-t_{\mathbf{b}},x_{\mathbf{b}},v)\right|^p\mathrm{d}\gamma\mathrm{d}s\right]^{1/p}\\
&+\left[\int_0^t\int_{\gamma_-}|\nabla_v g|^p\mathrm{d}\gamma\mathrm{d}s\right]^{1/p} + t\left[\int_0^t\int_{\gamma_-}|\langle v\rangle g|^p\mathrm{d}\gamma\mathrm{d}s\right]^{1/p} + t^{(p-1)/p}\left[\int_0^t\|\nabla_x H\|_p^p\mathrm{d}s\right]^{1/p}\\
&+t^{(p-1)/p}\left[\int_0^t\|\nabla_v H\|_p^p\mathrm{d}s\right]^{1/p} + Ct^{(p-1)/p}\left[\int_0^t\|H\|_p^p\mathrm{d}s\right]^{1/p}.
\end{aligned}
$$

From our hypothesis and assumption on f_0, g and H to have compact supports, these terms are bounded, therefore

$$
\partial f\mathbf{1}_{\{t\neq t_{\mathbf{b}}\}} \equiv \left[\partial_t f\mathbf{1}_{\{t\neq t_{\mathbf{b}}\}}, \nabla_x f\mathbf{1}_{\{t\neq t_{\mathbf{b}}\}}, \nabla_v f\mathbf{1}_{\{t\neq t_{\mathbf{b}}\}}\right] \ \in \ L^\infty([0,T];L^p(\Omega\times\mathbb{R}^3)).
$$

On the other hand, thanks to the compatibility condition, we need to show f has the same trace on the set

$$
\mathcal{M} \equiv \{t = t_{\mathbf{b}}(x,v)\} \equiv \{(t_{\mathbf{b}}(x,v),x,v)\in[0,T]\times\Omega\times\mathbb{R}^3\}. \tag{48}
$$

We claim : Let $\phi(t,x,v)\in C_c^\infty((0,T)\times\Omega\times\mathbb{R}^3)$ and we have

$$
\int_0^T\iint_{\Omega\times\mathbb{R}^3} f\partial\phi = -\int_0^T\iint_{\Omega\times\mathbb{R}^3}\partial f\mathbf{1}_{\{t\neq t_{\mathbf{b}}\}}\phi,
$$

so that $f\in W^{1,p}$ with weak derivatives given by $\partial f\mathbf{1}_{\{t\neq t_{\mathbf{b}}\}}$.

Proof of claim. We first fix the test function $\phi(t,x,v)$. There exists $\delta=\delta_\phi>0$ such that $\phi\equiv 0$ for $t\geq\frac{1}{\delta}$, or $\text{dist}(x,\partial\Omega)<\delta$, or $|v|\geq\frac{1}{\delta}$. Let $\phi(t,x,v)\neq 0$ and $(t,x,v)\in\mathcal{M}$. If follows that $t=t_{\mathbf{b}}(x,v)$ so that $x_{\mathbf{b}} = x - t_{\mathbf{b}}v$. Hence $|x-x_{\mathbf{b}}| = t_{\mathbf{b}}|v|$ and

$$
\text{dist}(x,\Omega)\leq|x-x_{\mathbf{b}}| = t_{\mathbf{b}}|v|.
$$

Since $t_{\mathbf{b}}\leq\frac{1}{\delta}$, this implies that

$$
|v|\geq\frac{\delta}{t_{\mathbf{b}}}\geq\delta^2.
$$

Otherwise $\text{dist}(x,\partial\Omega)\leq\delta$ so that $\phi(t,x,v)=0$. Furthermore, by the Velocity lemma (Lemma 6) and this lower bound of $|v|$, we conclude that there exists $\delta'(\delta,\Omega)>0$ such that

$$
\begin{aligned}
|v\cdot n(x_{\mathbf{b}})|^2 &\gtrsim_\Omega |v\cdot\nabla_x\xi(x_{\mathbf{b}})|^2 = \alpha(t-t_{\mathbf{b}};t,x,v)\geq e^{-C_\Omega\langle v\rangle t_{\mathbf{b}}}\alpha(t;t,x,v)\geq e^{-C_\Omega\langle v\rangle t_{\mathbf{b}}}C_\xi|v|^2|\xi(x)|\\
&\geq e^{-C_\Omega\delta^{-2}}C_\xi\delta^4\min_{\text{dist}(x,\partial\Omega)\geq\delta}|\xi(x)| = 2\delta'(\delta,\Omega)>0.
\end{aligned}
$$

In particular, this lower bound and a direct computation of (41) imply that $\{\phi\neq 0\}\cap\mathcal{M}$ is a smooth 6D hypersurface.

We next take C^1 approximation of f_0^l, H^l, and g^l (by partition of unity and localization) such that

$$
||f_0^l - f_0||_{W^{1,p}}\to 0,\quad ||g^l - g||_{W^{1,p}([0,T]\times\gamma_-\setminus\gamma_-^{\delta'})}\to 0,\quad ||H^l - H||_{W^{1,p}([0,T]\times\Omega\times\mathbb{R}^3)}\to 0.
$$

This implies, from the trace theorem, that

$$
f_0^l(x,v)\to f_0(x,v)\quad\text{and}\quad g^l(0,x,v)\to g(0,x,v)\quad\text{in}\ \ L^1(\gamma_-\setminus\gamma_-^{\delta'}).
$$

We define accordingly, for $(t,x,v)\in[0,T]\times\Omega\times\mathbb{R}^3$,

$$
f^l(t,x,v) = \mathbf{1}_{\{t<t_{\mathbf{b}}\}}e^{-t\nu(v)}f_0^l(x-tv,v) + \mathbf{1}_{\{t>t_{\mathbf{b}}\}}e^{-t_{\mathbf{b}}\nu(v)}g^l(t-t_{\mathbf{b}},x_{\mathbf{b}},v) + \int_0^{\min\{t,t_{\mathbf{b}}\}}e^{-s\nu(v)}H^l(t-s,x-sv,v)\mathrm{d}s, \tag{49}
$$

and $f^l_\pm(t,x,v) \equiv \mathbf{1}_{\{t \gtrless t_{\mathbf{b}}\}} f^l$. Therefore for all $(x,v) \in \gamma_-$,

$$f^l_+(s, x+sv, v) - f^l_-(s, x+sv, v) = e^{-s\nu(v)} g^l(0,x,v) - e^{-s\nu(v)} f^l_0(x,v).$$

Since $\{\phi \neq 0\} \cap \mathcal{M}$ is a smooth hypersurface, we apply the Gauss theorem to f^l to obtain

$$\iiint \partial_{\mathbf{e}} \phi f^l \mathrm{d}x\mathrm{d}v\mathrm{d}t = \iint [f^l_+ - f^l_-]\phi \, \mathbf{e} \cdot \mathbf{n}_{\mathcal{M}} \mathrm{d}\mathcal{M} - \left\{ \iiint_{t > t_{\mathbf{b}}} \phi \, \partial_{\mathbf{e}} f^l_+ \mathrm{d}x\mathrm{d}v\mathrm{d}t + \iiint_{t < t_{\mathbf{b}}} \phi \, \partial_{\mathbf{e}} f^l_- \mathrm{d}x\mathrm{d}v\mathrm{d}t \right\}, \quad (50)$$

where $\partial_{\mathbf{e}} = [\partial_t, \nabla_x, \nabla_v] = [\partial_t, \partial_{x_1}, \partial_{x_2}, \partial_{x_3}, \partial_{v_1}, \partial_{v_2}, \partial_{v_3}]$ and $\mathbf{n}_{\mathcal{M}} = \mathbf{e}_1 \in \mathbb{R}^7$. We have used $(s, x+sv, v)$ and $(x,v) \in \gamma_-$ as our parametrization for the manifold $\mathcal{M} \cap \{\phi \neq 0\}$, so that $n(x_{\mathbf{b}}(x,v)) \cdot v \geq 2\delta'$ is equivalent to $n(x) \cdot v \geq 2\delta'$. Therefore the above hypersurface integration over $\{t \neq t_{\mathbf{b}}\}$ is bounded by

$$\begin{aligned} &C_{\phi\delta} \int_0^{\frac{1}{\delta}} \int_{n(x)\cdot v \geq 2\delta'} |f^l_+(s, x+sv, v) - f^l_-(s, x+sv, v)| \mathrm{d}S_x \mathrm{d}v \mathrm{d}s \\ &\lesssim_{\phi,\delta} \int_{n(x)\cdot v \geq 2\delta'} |g^l(0,x,v) - f^l_0(s,v)| \mathrm{d}S_x \mathrm{d}v \to 0, \quad \text{as } l \to \infty, \end{aligned}$$

since the compatibility condition $f_0(x,v) = g(0,x,v)$ for $(x,v) \in \gamma_-$. Clearly, taking difference of (49) and (40), we deduce $f^l \to f$ strongly in $L^p(\{\phi \neq 0\})$ due to the first estimate of (47). Furthermore, due to (47), we have a uniform-in-l bound of $f^l_\pm$ in $W^{1,p}(\{t \gtrless t_{\mathbf{b}}, \ \phi \neq 0\})$ such that, up to subsequence,

$$\partial_{\mathbf{e}} f^l_+ \rightharpoonup \partial_{\mathbf{e}} f \mathbf{1}_{\{t > t_{\mathbf{b}}\}}, \quad \partial_{\mathbf{e}} f^l_- \rightharpoonup \partial_{\mathbf{e}} f \mathbf{1}_{\{t < t_{\mathbf{b}}\}}, \quad \text{weakly in } L^p(\{\phi \neq 0\}).$$

Finally we conclude the claim by letting $l \to \infty$ in (50).

Now notice that from its explicit form (40), and since all the data are compactly supported in velocity, f is itself compactly supported in velocity. Recall $\partial = [\partial_t, \nabla_x, \nabla_v]$. From this and the L^p bounds above, we conclude

$$\{\partial_t + v \cdot \nabla_x + \nu(v)\} \partial f = \partial H - \partial v \cdot \nabla_x f - \partial \nu f \in L^p. \quad (51)$$

By the trace theorem (Lemma 4), traces of $\partial_t f, \nabla_x f, \nabla_v f$ exist. To evaluate these traces, we take derivatives along characteristics. Letting $t \to t_{\mathbf{b}}$ and $t \to 0$, we deduce (36). From the Green's identity, Lemma 5, we have (37), (38) and (39), and therefore we conclude $\partial f \in C^0([0,T]; L^p)$.

In order to remove the compact support assumption we employ the cut-off function χ used in (13). Define $f^m = \chi(|v|/m) f$ then f^m satisfies

$$\{\partial_t + v \cdot \nabla_x + \nu(v)\} f^m = \chi(|v|/m) H, \quad (52)$$

$$f^m(0,x,v) = \chi(|v|/m) f_0, \quad f^m|_{\gamma_-} = \chi(|v|/m) g. \quad (53)$$

Note that $\nabla_v[\chi(|v|/m) g] = \chi(|v|/m) \nabla_v g + g \nabla_v \chi(|v|/m)$ and $\chi(|v|/m) f_0(x,v) = \chi(|v|/m) g(0,x,v)$ for $(x,v) \in \gamma_-$. Apply previous result to compute the traces of the derivatives of f^m. It is standard (using Green's identity) to show that $\partial_t f^m, \nabla_x f^m$ and $\nabla_v f^m$ are Cauchy and we can pass a limit. $\square$

We now study weighted $W^{1,p}$ estimate. Recall (13). We first define an effective collision frequency:

$$\begin{aligned} \nu_{l,\lambda}(t,x,v) &= \nu(v) - \lambda \mathbf{d}^{-1}(t,x,v) \{\partial_t + v \cdot \nabla_x\} \mathbf{d}(t,x,v) \\ &= \nu(v) - \frac{\lambda}{2\mathbf{d}^2} \Big(-l \langle v \rangle e^{-\langle v \rangle t} \alpha - e^{-l\langle v \rangle t} 2v \{v \cdot \nabla^3 \xi(x) \cdot v\} \xi(x) \Big) \\ &\qquad \times \Big(\chi(e^{-l\langle v \rangle t}\alpha/\varepsilon) + \varepsilon^{-1} \chi'(e^{-l\langle v \rangle t}\alpha/\varepsilon)[e^{-l\langle v \rangle t}\alpha - 1] \Big). \end{aligned} \quad (54)$$

Clearly $\nu_{l,\lambda}(t,x,v) \sim_\varepsilon \nu(v) + \lambda(l - O_\xi(1)) \langle v \rangle \sim \lambda \langle v \rangle$ from our choice (14), and from the definition, it is easy to verify that

$$\{\partial_t + v \cdot \nabla_x\} \mathbf{d}^\lambda = [-\nu_{l,\lambda}(v) + \nu(v)] \mathbf{d}^\lambda, \quad (55)$$

so that

$$\begin{aligned} \mathbf{d}^\lambda \{\partial_t + v \cdot \nabla_x + \nu(v)\} \partial f &= \{\partial_t + v \cdot \nabla_x + \nu(v)\}[\mathbf{d}^\lambda \partial f] - \partial f \{\partial_t + v \cdot \nabla_x\} \mathbf{d}^\lambda \\ &= \{\partial_t + v \cdot \nabla_x + \nu_{l,\lambda}\}[\mathbf{d}^\lambda \partial f]. \end{aligned} \quad (56)$$

Proposition 2. *Let f be a solution of (32). Assume (35) and $\langle v\rangle g \in L^\infty([0,T]\times\gamma_-)$, $\langle v\rangle H \in L^\infty([0,T]\times\Omega\times\mathbb{R}^3)$. For any fixed $p\in[2,\infty]$, assume*

$$\begin{aligned} \mathbf{d}^\lambda\partial_t g,\ \mathbf{d}^\lambda\nabla_\tau g &\in L^\infty([0,T];L^p(\gamma_-)),\\ \mathbf{d}^\lambda\big\{|\nabla_\tau g| + \frac{1}{n(x)\cdot v}\big(|\partial_t g| + \langle v\rangle|\nabla_\tau g| + |H|\big)\big\} &\in L^\infty([0,T];L^p(\gamma_-)),\\ \mathbf{d}^\lambda\big| - v\cdot\nabla_x f_0 - \nu(v)f_0 + H_0\big| &\in L^p(\Omega\times\mathbb{R}^3), \end{aligned}$$

and assume $1/p+1/q=1$ there exist $TC_T = O(T)$ and $\varepsilon \ll 1$ such that for all $t\in[0,T]$

$$\left|\iint_{\Omega\times\mathbb{R}^3}\mathbf{d}^\lambda\partial H(t)h(t)\right| \leq C_T\big\{||h(t)||_q + \varepsilon||\nu_{l,\lambda}^{1/q}h(t)||_q\big\}.$$

Then $f(t,x,v)$ satisfies

$$||f(t)||_\infty \leq ||f_0||_\infty + \sup_{0\leq s\leq t}||g(s)||_\infty + \Big|\Big|\int_0^t H(s)ds\Big|\Big|_\infty.$$

Recall $\partial = [\partial_t,\nabla_x,\nabla_v]$, then

$$\begin{aligned} \{\partial_t + v\cdot\nabla_x + \nu_{l,\lambda}\}[\mathbf{d}^\lambda\partial f] &= \mathbf{d}^\lambda\partial H - \partial v\cdot\mathbf{d}^\lambda\nabla_x f - \partial\nu(v)\mathbf{d}^\lambda f,\\ \mathbf{d}^\lambda\partial f|_{t=0} &= \mathbf{d}^\lambda\partial f_0, \quad \mathbf{d}^\lambda\partial f|_{\gamma_-} = \mathbf{d}^\lambda[\partial g|_{\gamma_-}], \end{aligned}$$

where $[\partial g|_{\gamma_-}]$ is given in (36). Moreover, recalling (33) and (34), we have for $2\leq p<\infty$,

$$\begin{aligned} &\int_{\Omega\times\mathbb{R}^3}|\mathbf{d}^\lambda\partial f(t)|^p + \int_0^t\int_{\Omega\times\mathbb{R}^3}\nu_{l,\lambda}|\mathbf{d}^\lambda\partial f|^p + \int_0^t\int_{\gamma_+}|\mathbf{d}^\lambda\partial f|^p \\ &\quad\lesssim \int_{\Omega\times\mathbb{R}^3}|\mathbf{d}^\lambda\partial f_0|^p + \int_0^t\int_{\gamma_-}|\mathbf{d}^\lambda\partial g|^p + \int_0^t\int_{\Omega\times\mathbb{R}^3}|\mathbf{d}^\lambda\partial H - \partial v\cdot\mathbf{d}^\lambda\nabla_x f - \partial\nu\mathbf{d}^\lambda f||\mathbf{d}^\lambda\partial f|^{p-1},\\ &||\mathbf{d}^\lambda\partial f(t)||_\infty \lesssim ||\mathbf{d}^\lambda\partial f_0||_\infty + ||\mathbf{d}^\lambda\partial g||_\infty + \int_0^t||\mathbf{d}^\lambda\partial H - \partial v\cdot\mathbf{d}^\lambda\nabla_x f - \partial\nu\mathbf{d}^\lambda f||_\infty, \qquad \text{for } p=\infty. \end{aligned} \tag{57}$$

Proof. First we assume f_0, g and H have compact supports in $\{v\in\mathbb{R}^3 : |v|<m\}$. We estimate ∂f in the bulk. From the velocity lemma (Lemma 6), we have

$$\sup_{t\leq t_{\mathbf{b}}}\frac{\mathbf{d}(t,x,v)^\lambda}{\mathbf{d}(0,x-tv,v)^\lambda} \leq e^{C_{m,\lambda}t}, \quad \sup_{t\geq t_{\mathbf{b}}}\frac{\mathbf{d}(t,x,v)^\lambda}{\mathbf{d}(t-t_{\mathbf{b}},x_{\mathbf{b}},v)^\lambda} \leq e^{C_{m,\lambda}t_{\mathbf{b}}}, \quad \sup_{\max\{t-t_{\mathbf{b}},0\}\leq s\leq t}\frac{\mathbf{d}(t,x,v)^\lambda}{\mathbf{d}(t-s,x-sv,v)^\lambda} \leq e^{C_{m,\lambda}s}.$$

Multiply $\mathbf{d}(t,x,v)^\lambda$ by (42), (43) and (44) and use the above inequalities to get

$$\begin{aligned} &\mathbf{d}^\lambda|\partial_t f(t,x,v)|\\ &\lesssim e^{C_{m,\lambda}t}e^{-t\nu(v)}\mathbf{d}^\lambda\big|[\nu f_0 + v\cdot\nabla_x f_0 - H_{|t=0}](x-tv,v)\big|\mathbf{1}_{\{t<t_{\mathbf{b}}\}}\\ &\quad + e^{C_{m,\lambda}t_{\mathbf{b}}}e^{-t_{\mathbf{b}}\nu}\mathbf{d}^\lambda\partial_t\big|g(t-t_{\mathbf{b}},x_{\mathbf{b}},v)\big|\mathbf{1}_{\{t>t_{\mathbf{b}}\}}\\ &\quad + \int_0^{\min(t,t_{\mathbf{b}})}e^{C_{m,\lambda}s}e^{-s\nu}\mathbf{d}^\lambda\big|\partial_t H(t-s,x-vs,v)\big|\mathrm{d}s,\\ &\mathbf{d}^\lambda|\nabla_x f(t,x,v)|\\ &\lesssim e^{C_{m,\lambda}t}e^{-t\nu}\mathbf{d}^\lambda\big|\nabla_x f_0(x-tv,v)\big|\mathbf{1}_{\{t<t_{\mathbf{b}}\}} + e^{C_{m,\lambda}t_{\mathbf{b}}}e^{-t_{\mathbf{b}}\nu}\sum_{i=1}^2\tau_i\mathbf{d}^\lambda|\partial_{\tau_i}g(t-t_{\mathbf{b}},x_{\mathbf{b}},v)|\mathbf{1}_{\{t>t_{\mathbf{b}}\}}\\ &\quad + e^{C_{m,\lambda}t_{\mathbf{b}}}e^{-t_{\mathbf{b}}\nu}n(x_{\mathbf{b}})\frac{\mathbf{d}^\lambda(t-t_{\mathbf{b}},x_{\mathbf{b}},v)}{|v\cdot n(x_{\mathbf{b}})|}\Big|\Big\{\partial_t g + \sum_{i=1}^2(v\cdot\tau_i)\partial_{\tau_i}g + \nu g - H\Big\}(t-t_{\mathbf{b}},x_{\mathbf{b}},v)\Big|\mathbf{1}_{\{t>t_{\mathbf{b}}\}}\\ &\quad + \int_0^{\min(t,t_{\mathbf{b}})}e^{C_{m,\lambda}s}e^{-s\nu}\mathbf{d}^\lambda\big|\nabla_x H(t-s,x-vs,v)\big|\mathrm{d}s, \end{aligned} \tag{58}$$

$$
\begin{aligned}
&\mathbf{d}^{\lambda}|\nabla_v f(t,x,v)| \\
&\lesssim\; e^{C_{m,\lambda}t}e^{-t\nu}\mathbf{d}^{\lambda}\big|[-t\nabla_x f_0+\nabla_v f_0-t\nabla_v\nu(v)f_0](x-tv,v)\big|\mathbf{1}_{\{t<t_{\mathbf{b}}\}} \\
&\quad+e^{C_{m,\lambda}t_{\mathbf{b}}}e^{-t_{\mathbf{b}}\nu}\sum_{i=1}^{2}\tau_i\mathbf{d}^{\lambda}|\partial_{\tau_i}g(t-t_{\mathbf{b}},x_{\mathbf{b}},v)|\mathbf{1}_{\{t>t_{\mathbf{b}}\}} \\
&\quad+e^{C_{m,\lambda}t_{\mathbf{b}}}e^{-t_{\mathbf{b}}\nu}n(x_{\mathbf{b}})\frac{\mathbf{d}(t-t_{\mathbf{b}},x_{\mathbf{b}},v)^{\lambda}}{|v\cdot n(x_{\mathbf{b}})|}\Big|\Big\{\partial_t g+\sum_{i=1}^{2}(v\cdot\tau_i)\partial_{\tau_i}g+\nu g-H\Big\}(t-t_{\mathbf{b}},x_{\mathbf{b}},v)\Big|\mathbf{1}_{\{t>t_{\mathbf{b}}\}} \\
&\quad+e^{C_{m,\lambda}t_{\mathbf{b}}}e^{-t_{\mathbf{b}}\nu}\mathbf{d}^{\lambda}\{|\nabla_v g(t-t_{\mathbf{b}},x_{\mathbf{b}},v)|+|t_{\mathbf{b}}\nabla_v\nu(v)||g(t-t_{\mathbf{b}},x_{\mathbf{b}},v)|\}\mathbf{1}_{\{t>t_{\mathbf{b}}\}} \\
&\quad+\int_0^{\min(t,t_{\mathbf{b}})}e^{C_{m,\lambda}s}e^{-s\nu}\mathbf{d}^{\lambda}\big|\{\nabla_v H-s\nabla_x H-s\nabla\nu H\}(t-s,x-vs,v)\big|\mathrm{d}s.
\end{aligned}
$$

Following (45) and (46) of Proposition 1 and using the condition of Proposition 2, we deduce

$$
\begin{aligned}
||\mathbf{d}^{\lambda}\partial_t f(t)||_p &\lesssim_{t,m,\lambda} ||\mathbf{d}^{\lambda}[v\cdot\nabla_x f_0+\nu f_0-H(0,\cdot,\cdot)]||_p+\left[\int_0^t||\mathbf{d}^{\lambda}\partial_t g(s)||^p_{\gamma,p}\mathrm{d}s\right]^{1/p}+\left[\int_0^t||\mathbf{d}^{\lambda}\partial_t H(s)||^p_p\mathrm{d}s\right]^{1/p}, \\
||\mathbf{d}^{\lambda}\nabla_x f(t)||_p &\lesssim_{t,m,\lambda} ||\mathbf{d}^{\lambda}\nabla_x f_0||_p+\sum_{i=1}^{2}\left[\int_0^t||\mathbf{d}^{\lambda}\partial_{\tau_i}g(s)||^p_{\gamma,p}\mathrm{d}s\right]^{1/p} \\
&\quad+\left[\int_0^t\Big|\Big|\frac{\mathbf{d}^{\lambda}}{v\cdot n}\{\partial_t g+\sum(v\cdot\tau_i)\partial_{\tau_i}g+\nu g-H\}\Big|\Big|^p_{\gamma,p}\mathrm{d}s\right]^{1/p}+\left[\int_0^t||\mathbf{d}^{\lambda}\nabla_x H(s)||^p_p\mathrm{d}s\right]^{1/p}, \\
||\mathbf{d}^{\lambda}\nabla_v f(t)||_p &\lesssim_{t,m,\lambda} ||\mathbf{d}^{\lambda}\nabla_v f_0||_p+\sum_{i=1}^{2}\left[\int_0^t||\mathbf{d}^{\lambda}\partial_{\tau_i}g(s)||^p_{\gamma,p}\mathrm{d}s\right]^{1/p}+\sup_{0\le s\le t}||\langle v\rangle g(s)||_\infty \\
&\quad+\left[\int_0^t\Big|\Big|\frac{\mathbf{d}^{\lambda}}{v\cdot n}\{\partial_t g+\sum(v\cdot\tau_i)\partial_{\tau_i}g+\nu g-H\}\Big|\Big|^p_{\gamma,p}\mathrm{d}s\right]^{1/p}+\left[\int_0^t||\mathbf{d}^{\lambda}\nabla_v g(s)||^p_p\mathrm{d}s\right]^{1/p} \\
&\quad+\left[\int_0^t||\mathbf{d}^{\lambda}\nabla_v H(s)||^p_p+||\mathbf{d}^{\lambda}\nabla_x H(s)||^p_p\mathrm{d}s\right]^{1/p}+\sup_{0\le s\le t}||\langle v\rangle H(s)||_\infty.
\end{aligned}
$$

By the hypothesis of Proposition 2 and assumption on f_0, g and H to have compact support, the right hand sides are bounded and hence $\mathbf{d}^{\lambda}\partial_t f, \mathbf{d}^{\lambda}\nabla_x f$, and $\mathbf{d}^{\lambda}\nabla_v f$ are in $L^\infty([0,T];L^p(\Omega\times\mathbb{R}^3))$.

Since f_0, g and H are compactly supported on $\{v\in\mathbb{R}^3:|v|\le m\}$, the derivatives $\mathbf{d}^{\lambda}\partial_t f, \mathbf{d}^{\lambda}\nabla_x f$ and $\mathbf{d}^{\lambda}\nabla_v f$ are compactly supported on $\{v\in\mathbb{R}^3:|v|\le m\}$ and hence from (56) and (51)

$$
\{\partial_t+v\cdot\nabla_x+\nu_{l,\lambda}\}[\mathbf{d}^{\lambda}\partial f]=\mathbf{d}^{\lambda}\partial H-\partial v\cdot\mathbf{d}^{\lambda}\nabla_x f-\partial\nu(v)\mathbf{d}^{\lambda}f. \tag{59}
$$

Moreover, from the general definition of traces, by choosing a test function multiplied by $\mathbf{d}^{\lambda}$, we deduce $\mathbf{d}^{\lambda}\partial f$ has the same trace as $\mathbf{d}^{\lambda}[\partial f|_{\gamma}]$.

Now we can apply Lemma 5 to have (57) which does not depend on the velocity cut-off. Therefore for the general case, we use (52) and (53) and pass a limit to conclude the proof. □

Proposition 3. *Let f be a solution of (32). Assume the compatibility condition (35) and compatibility condition*

$$
\partial_t g(0,x,v)=-v\cdot\nabla_x f_0-\nu(v)f_0+H(0,x,v),\qquad \text{for } (x,v)\in\gamma_-\cup\gamma_0. \tag{60}
$$

Assume

$$
\begin{aligned}
\mathbf{d}\partial_t g,\ \mathbf{d}\nabla_\tau g &\in C^0([0,T]\times\gamma_-\cup\gamma_0), \\
\frac{\mathbf{d}}{n(x)\cdot v}\Big\{\partial_t g+\sum_{i=1}^{2}(v\cdot\tau_i)\partial_{\tau_i}g+\nu g-H\Big\} &\in C^0([0,T]\times\gamma_-\cup\gamma_0), \\
H,\ \mathbf{d}\partial H &\in C^0([0,T]\times\Omega\times\mathbb{R}^3), \\
\mathbf{d}\big\{-v\cdot\nabla_x f_0-\nu(v)f_0+H_0\big\},\ \mathbf{d}\nabla_x f_0,\ \mathbf{d}\nabla_v f_0 &\in C^0(\bar\Omega\times\mathbb{R}^3).
\end{aligned}
$$

Then $\mathbf{d}\partial f\in C^0([0,T]\times\bar\Omega\times\mathbb{R}^3)$.

Proof. We only need to prove that $\mathbf{d}\partial f \in C^0([0,T]\times\bar\Omega\times\mathbb{R}^3)$. Recall that we have compatibility conditions (35) and (60). We multiply by $\mathbf{d}$ the formulas (42), (43) and (44). Clearly $\mathbf{d}\partial f(t,x,v)$ is continuous for $(x,v)\notin\gamma_0$ and $t\neq t_{\mathbf{b}}(x,v)$. We first show that $\mathbf{d}\partial f(t,x,v)$ is continuous at $(x,v)\in\gamma_0$ with $t\neq t_{\mathbf{b}}(x,v)$ and then prove that $\mathbf{d}\partial f(t,x,v)$ is continuous at $t=t_{\mathbf{b}}(x,v)$ for $(x,v)\in\bar\Omega\times\mathbb{R}^3$.

We first consider the time-derivative $\mathbf{d}\partial_t f$ at the grazing set γ_0. We define

$$\begin{aligned}\mathbf{d}\partial_t f(t,x,v)&\equiv \mathbf{d}\partial_t g(t,x,v)\quad \text{for}\quad t>0,\ (x,v)\in\gamma_0,\\ \mathbf{d}\partial_t f(0,x,v)&\equiv \mathbf{d}\{-v\cdot\nabla_x f_0(x,v)-\nu(v)f_0(x,v)+H(0,x,v)\}\quad\text{for}\quad (x,v)\in\gamma_0.\end{aligned}$$

Due to (60) we have $\mathbf{d}\partial_t f(t,x,v)\to\mathbf{d}\partial_t f(0,x,v)$ as $t\downarrow 0$. In order to show that $\mathbf{d}\partial_t f(t,x,v)$ is continuous at $(x,v)\in\gamma_0$ we choose a sequence $(y_j,u_j)\in(\bar\Omega\times\mathbb{R}^3)\backslash\gamma_0$ such that $(y_j,u_j)\to(x,v)$. By the velocity lemma (Lemma 6) and $(y_j,u_j)\notin\gamma_0$, we have $(x_{\mathbf{b}}(y_j,u_j),u_j)<0$. Since $t_{\mathbf{b}}(y_j,u_j)\to t_{\mathbf{b}}(x,v)=0$ we may assume $t>t_{\mathbf{b}}(y_j,u_j)$ for sufficiently large $j\gg 1$. Now we multiply (42) by $\mathbf{d}(t,x,v)$ to get

$$\begin{aligned}\mathbf{d}\partial_t f(t,y_j,u_j)=&\frac{\mathbf{d}(t,y_j,u_j)}{\mathbf{d}(t-t_{\mathbf{b}}(y_j,u_j),x_{\mathbf{b}}(y_j,u_j),u_j)}e^{-t_{\mathbf{b}}(y_j,u_j)\nu(u_j)}\mathbf{d}\partial_t g(t-t_{\mathbf{b}}(y_j,u_j),x_{\mathbf{b}}(y_j,u_j),u_j)\\&+\int_0^{t_{\mathbf{b}}(y_j,u_j)}\frac{\mathbf{d}(t,y_j,u_j)}{\mathbf{d}(t-s,y_j-su_j,u_j)}e^{-s\nu(u_j)}\mathbf{d}\partial_t H(t-s,y_j-su_j,u_j)\mathrm{d}s.\end{aligned}$$

From Lemma 6 we have, for $j\gg1$,

$$\begin{aligned}\frac{\mathbf{d}(t,y_j,u_j)}{\mathbf{d}(t-t_{\mathbf{b}}(y_j,u_j),x_{\mathbf{b}}(y_j,u_j),u_j)}&=e^{-\frac{l\langle u_j\rangle}{2}t_{\mathbf{b}}(y_j,u_j)}\frac{\alpha(t;t,y_j,u_j)}{\alpha(t-t_{\mathbf{b}}(y_j,u_j);t,y_j,u_j)}\le e^{C_{\Omega,l}\langle u_j\rangle t_{\mathbf{b}}(y_j,u_j)}\lesssim_{\Omega,l}1,\\ \frac{\mathbf{d}(t,y_j,u_j)}{\mathbf{d}(t-s,y_j-su_j,u_j)}&=e^{-\frac{l\langle u_j\rangle}{2}s}\frac{\alpha(t;t,y_j,u_j)}{\alpha(t-s;t,y_j,u_j)}\le e^{C_{\Omega,l}\langle u_j\rangle s}\lesssim_{\Omega,l}1,\quad\text{for}\quad 0\le s\le\min(t,t_{\mathbf{b}}),\end{aligned}$$

and hence

$$\frac{\mathbf{d}(t,y_j,u_j)}{\mathbf{d}(t-t_{\mathbf{b}}(y_j,u_j),x_{\mathbf{b}}(y_j,u_j),u_j)}\to\frac{\mathbf{d}(t,x,v)}{\mathbf{d}(t,x,v)}=1.$$

Since the integrand of $\int_0^{t_{\mathbf{b}}(j_j,u_j)}(\cdots)\mathrm{d}s$ is finite due to the hypothesis of the proposition and the above inequality we conclude from $t_{\mathbf{b}}(y_j,u_j)\to t_{\mathbf{b}}(x,v)=0$

$$\left|\int_0^{t_{\mathbf{b}}(y_j,u_j)}\frac{\mathbf{d}(t,y_j,u_j)}{\mathbf{d}(t-s,y_j-su_j,u_j)}e^{-s\nu(u_j)}\mathbf{d}\partial_t H(t-s,y_j-su_j,u_j)\mathrm{d}s\right|\to 0,$$

and $\mathbf{d}\partial_t f(t,y_j,u_j)\to\mathbf{d}\partial_t g(t,x,v)$.

Therefore $\mathbf{d}\partial_t f(t,x,v)$ is continuous at γ_0.

In order to show that $\mathbf{d}\partial_t f(t,x,v)$ is continuous at $t=t_{\mathbf{b}}(x,v)$ we separate two cases: If $t<t_{\mathbf{b}}$ we use (42) to get

$$\begin{aligned}\lim_{t\uparrow t_{\mathbf{b}}}\mathbf{d}\partial_t f(t,x,v)=&\frac{\mathbf{d}(t,x,v)}{\mathbf{d}(0,x-tv,v)}e^{-t\nu(v)}\mathbf{d}\{-v\cdot\nabla_x f_0-\nu(v)f_0+H|_{t=0}\}(x-tv,v)\\&+\int_0^t\frac{\mathbf{d}(t,x,v)}{\mathbf{d}(t-s,x-sv,v)}e^{-s\nu(v)}\mathbf{d}\partial_t H(t-s,x-sv,v)\mathrm{d}s.\end{aligned}$$

Using Lemma 6, we deduce, for all $0\le s\le t<t_{\mathbf{b}}$,

$$\frac{\mathbf{d}(t,x,v)}{\mathbf{d}(s,x-sv,v)}=e^{-\frac{l\langle v\rangle}{2}s}\frac{\alpha(t;t,x,v)}{\alpha(s;t,x,v)}\le e^{-\frac{l\langle v\rangle}{2}s}e^{C_\Omega\langle v\rangle s}<\infty.$$

Hence,

$$\begin{aligned}\lim_{t\uparrow t_{\mathbf{b}}}\mathbf{d}\partial_t f(t,x,v)=&\frac{\mathbf{d}(t_{\mathbf{b}},x,v)}{\mathbf{d}(0,x_{\mathbf{b}},v)}e^{-t_{\mathbf{b}}\nu(v)}\mathbf{d}\{-v\cdot\nabla_x f_0-\nu f_0+H|_{t=0}\}(x_{\mathbf{b}},v)\\&+\int_0^{t_{\mathbf{b}}}\frac{\mathbf{d}(t_{\mathbf{b}},x,v)}{\mathbf{d}(t_{\mathbf{b}}-s,x-sv,v)}e^{-s\nu(v)}\mathbf{d}\partial_t H(t_{\mathbf{b}}-s,x-sv,v)\mathrm{d}s.\end{aligned}$$

Similarly for $t > t_{\mathbf{b}}$ we use (42) again to obtain

$$\lim_{t\downarrow t_{\mathbf{b}}} \mathbf{d}\partial_t f(t,x,v) = \frac{\mathbf{d}(t_{\mathbf{b}},x,v)}{\mathbf{d}(0,x_{\mathbf{b}},v)} e^{-t_{\mathbf{b}}\nu(v)} \mathbf{d}\partial_t g(0,x_{\mathbf{b}},v) + \int_0^{t_{\mathbf{b}}} \frac{\mathbf{d}(t_{\mathbf{b}},x,v)}{\mathbf{d}(t_{\mathbf{b}}-s,x-sv,v)} e^{-s\nu(v)} \mathbf{d}\partial_t H(t_{\mathbf{b}},-s,x-sv,v)\mathrm{d}s.$$

Finally we use the compatibility condition (60) and (36) for $\lambda = 1$ to conclude $\lim_{t\downarrow t_{\mathbf{b}}} \mathbf{d}\partial_t f(t,x,v) = \lim_{t\uparrow t_{\mathbf{b}}} \mathbf{d}\partial_t f(t,x,v)$.

We next consider the spatial derivative $\mathbf{d}\nabla_x f$. We define

$$\mathbf{d}\nabla_x f(t,x,v) = \sum_{i=1}^{2} \tau_i \mathbf{d}\partial_{\tau_i} g(t,x,v) + n e^{-\frac{l\langle v\rangle}{2}t} |\nabla\xi(x)| \Big\{ \partial_t g + \sum_{i=1}^{2} (v\cdot\tau_i)\partial_{\tau_i} g + \nu g - H \Big\}(t,x,v), \quad \text{for } t>0,\ (x,v)\in\gamma_0,$$

$$\mathbf{d}\nabla_x f(0,x,v) = \mathbf{d}\nabla_x f_0(x,v), \quad \text{for } (x,v)\in\gamma_0.$$

Due to (60) and (36) for $\lambda = 1$, we have, for $(x,v)\in\gamma_0$,

$$\begin{aligned}\lim_{t\downarrow 0} \mathbf{d}\nabla_x f(t,x,v) &= \sum_{i=1}^{2} \tau_i \mathbf{d}\partial_{\tau_i} f_0(x,v) + n|\nabla\xi(x)| \Big\{ [-v\cdot\nabla_x f_0 - \nu f_0 + H|_{t=0}] + \sum_{i=1}^{2} (v\cdot\tau_i)\partial_{\tau_i} f_0 + \nu f_0 - H|_{t=0} \Big\}(x,v)\\ &= \sum_{i=1}^{2} \tau_i \mathbf{d}\partial_{\tau_i} f_0(x,v) + n|\nabla\xi(x)|(-v\cdot n(x))\partial_n f_0(x,v) = \mathbf{d}\nabla_x f_0(x,v),\end{aligned}$$

where we have used $\mathbf{d}(0,x,v) = |\nabla\xi(x)\cdot v|$ or $\mathbf{d}(0,x,v) = -|\nabla\xi(x)\cdot v|$ for $(x,v)\in\gamma_0$. In order to show that $\mathbf{d}\nabla_x f(t,x,v)$ is continuous at $(x,v)\in\gamma_0$ and $t>0$, we choose a sequence $(y_j,u_j)\to(x,v)\in\gamma_0$ as in the proof for the time-derivative $\mathbf{d}\partial_t f$. Then using (43) for sufficiently large $j\gg 1$, we deduce

$$\begin{aligned}&\mathbf{d}\nabla_x f(t,y_j,u_j) = \frac{\mathbf{d}(t,y_j,u_j)}{\mathbf{d}(t-t_{\mathbf{b}}(y_j,u_j),x_{\mathbf{b}}(y_j,u_j),u_j)} \Big\{ e^{-t_{\mathbf{b}}(y_j,u_j)\nu(u_j)} \sum_{i=1}^{2} \tau_i \mathbf{d}\partial_{\tau_i} g(t-t_{\mathbf{b}}(y_j,u_j),x_{\mathbf{b}}(y_j,u_j),u_j)\\ &+ n(x_{\mathbf{b}}(y_j,u_j)) \frac{\mathbf{d}(t-t_{\mathbf{b}}(y_j,u_j),x_{\mathbf{b}}(y_j,u_j),u_j)}{|v\cdot\nabla\xi(x_{\mathbf{b}}(y_j,u_i))||\nabla\xi(x_{\mathbf{b}}(y_j,u_j))|^{-1}} \big\{ \partial_t g + \sum_{i=1}^{2} (v\cdot\tau_i)\partial_{\tau_i} g + \nu g - H \big\}(t-t_{\mathbf{b}}(y_j,u_j),x_{\mathbf{b}}(y_j,u_j),u_j) \Big\}\\ &\to \sum_{i=1}^{2} \tau_i(x) \mathbf{d}\partial_{\tau_i} g(t,x,v) + n(x) e^{-\frac{l\langle v\rangle}{2}t} |\nabla\xi(x)| \Big\{ \partial_t g + \sum_{i=1}^{2} (v\cdot\tau_i)\partial_{\tau_i} g + \nu g - H \Big\}(t,x,v).\end{aligned}$$

In order to show that $\mathbf{d}\nabla_x f(t,x,v)$ is continuous at $t = t_{\mathbf{b}}(x,v)$, we take limits $t\to t_{\mathbf{b}}$ for both $t < t_{\mathbf{b}}$ and $t > t_{\mathbf{b}}$ in (43). Due to the common H integral, it suffices to observe that the part of $\lim_{t\uparrow t_{\mathbf{b}}}$

$$\frac{\mathbf{d}(t_{\mathbf{b}},x,v)}{\mathbf{d}(0,x_{\mathbf{b}},v)} e^{-t_{\mathbf{b}}\nu(v)} \mathbf{d}(0,x_{\mathbf{b}},v)\nabla_x f_0(x_{\mathbf{b}},v),$$

coincides with the part of $\lim_{t\downarrow t_{\mathbf{b}}}$

$$\begin{aligned}&\frac{\mathbf{d}(t_{\mathbf{b}},x,v)}{\mathbf{d}(0,x_{\mathbf{b}},v)} e^{-t_{\mathbf{b}}(x,v)\nu(v)} \Big\{ \sum_{i=1}^{2} \tau_i \mathbf{d}\partial_{\tau_i} g - n(x_{\mathbf{b}}) \frac{\mathbf{d}(0,x_{\mathbf{b}},v)}{v\cdot n(x_{\mathbf{b}})} \big\{ \partial_t g + \sum_{i=1}^{2} (v\cdot\tau_i)\partial_{\tau_i} g + \nu g - H \big\} \Big\}(0,x_{\mathbf{b}},v)\\ &= \frac{\mathbf{d}(t_{\mathbf{b}},x,v)}{\mathbf{d}(0,x_{\mathbf{b}},v)} e^{-t_{\mathbf{b}}(x,v)\nu(v)} \Big\{ \sum_{i=1}^{2} \tau_i \mathbf{d}\partial_{\tau_i} f_0(x_{\mathbf{b}},v) + n(x_{\mathbf{b}}) \frac{\mathbf{d}(0,x_{\mathbf{b}},v)}{v\cdot n(x_{\mathbf{b}})} \big\{ (v\cdot n(x_{\mathbf{b}}))\partial_n f_0(x_{\mathbf{b}},v) \big\} \Big\}\\ &= \frac{\mathbf{d}(t_{\mathbf{b}},x,v)}{\mathbf{d}(0,x_{\mathbf{b}},v)} e^{-t_{\mathbf{b}}\nu(v)} \mathbf{d}(0,x_{\mathbf{b}},v)\nabla_x f_0(x_{\mathbf{b}},v),\end{aligned}$$

where we used (60) and (36). This completes the proof of continuity of $\mathbf{d}\nabla_x f$.

Finally we consider $\nabla_v f$. Because of the proof of continuity of $\mathbf{d}\nabla_x f$, we only need to show

$$\lim_{t\downarrow t_{\mathbf{b}}} \mathbf{d}(t,x,v) e^{-t\nu(v)} [\nabla_v f_0 - t\nabla_v \nu(v) f_0](x-tv,v) = \lim_{t\uparrow t_{\mathbf{b}}} \mathbf{d}(t,x,v) e^{-t_{\mathbf{b}}\nu(v)} [\nabla_v g - t_{\mathbf{b}}\nabla_v\nu(v) g](t-t_{\mathbf{b}},x_{\mathbf{b}},v),$$

which is clear from $f|_{\gamma_-} = g$ and $\mathbf{d}\nabla_v f|_{\gamma_-} = \mathbf{d}\nabla_v g$ and $\mathbf{d}\nabla_v f|_{t=0} = \mathbf{d}\nabla_v f$ in (36). $\square$

4 $W^{1,p}$ $(1<p<2)$ Estimate

Consider the following iteration :

$$\begin{aligned}\{\partial_t + v\cdot\nabla_x + \nu(v)\} f^{m+1} - Kf^m = \Gamma(f^m, f^m), \quad f^{m+1}(0,x,v) = f_0(x,v),\\ f^{m+1}(t,x,v)|_{\gamma_-} = c_\mu\sqrt{\mu(v)}\int_{n\cdot u>0} f^m(t,x,u)\sqrt{\mu(u)}\{n\cdot u\}\mathrm{d}u,\end{aligned} \tag{61}$$

with $f^0 \equiv f_0$, and with the compatibility condition for the initial datum (4). Remark that the normalized Maxwellian is $\mu(v) = e^{-\frac{|v|^2}{2}}$. From [2, 7], we have a uniform bound of

$$\sup_m \sup_{0\le t<\infty} ||\langle v\rangle^\beta f^m(t)||_\infty \ \lesssim_\Omega \ ||\langle v\rangle^\beta f_0||_\infty \ll 1, \tag{62}$$

for $\beta > 4$. We apply Proposition 1 for $m = 1, 2, ...$ with

$$H = -Kf^m + \Gamma(f^m, f^m), \quad g = c_\mu\sqrt{\mu(v)}\int_{n\cdot u>0} f^m(t,x,u)\sqrt{\mu(u)}\{n(x)\cdot u\}\mathrm{d}u. \tag{63}$$

Recall $\partial = [\partial_t, \nabla_x, \nabla_v]$. Then ∂f^m satisfies

$$\{\partial_t + v\cdot\nabla_x + \nu(v)\}\partial f^{m+1} = \mathcal{G}^m, \quad \partial f^{m+1}(0,x,v) = \partial f_0(x,v), \tag{64}$$

where

$$\begin{aligned}&\mathcal{G}^m = -[\partial v]\cdot\nabla_x f^{m+1} - \partial\nu(v) f^{m+1} - \partial[Kf^m - \Gamma(f^m,f^m)], \qquad (65)\\ &|\mathcal{G}^m| \lesssim |\nabla_x f^{m+1}| + K|\partial f^m| + |\Gamma(\partial f^m, f^m)| + |\Gamma(f^m, \partial f^m)| + |\nabla_v\nu(v)||f^{m+1}| + |K_v f^m| + |\Gamma_v(f^m,f^m)|,\end{aligned}$$

with $K_v f^m$ and $\Gamma_v(f^m, f^m)$ defined in (22) and (29). Furthermore using (19) and Lemma 1, Lemma 2 and Lemma 3 yields

$$\begin{aligned}&|\nabla_v\nu(v)||f^{m+1}| + |K_v f^m| + |\Gamma_v(f^m,f^m)|\\ &\lesssim \left\{|\nabla_v\nu(v)|\langle v\rangle^{-\beta} + |K_v\langle v\rangle^{-\beta}| + |\Gamma_v(\langle v\rangle^{-\beta},\langle v\rangle^{-\beta})|\right\}||\langle v\rangle^\beta f_0||_\infty \lesssim \langle v\rangle^{-\beta+\gamma}||\langle v\rangle^\beta f_0||_\infty.\end{aligned} \tag{66}$$

For $(x,v)\in\gamma_-$, from (63) and (34), the boundary condition is bounded by

$$\begin{aligned}|\partial f^{m+1}(t,x,v)| \ &\lesssim \ c_\mu\sqrt{\mu(v)}\left(1+\frac{\langle v\rangle}{|n(x)\cdot v|}\right)\int_{n(x)\cdot u>0}|\partial f^m(t,x,u)|\langle u\rangle\sqrt{\mu}\{n(x)\cdot u\}\mathrm{d}u\\ &\quad + \frac{1}{|n(x)\cdot v|}\Big\{\nu(v)|f^{m+1}| + |Kf^m| + |\Gamma(f^m,f^m)|\Big\}\\ &\lesssim \ \sqrt{\mu(v)}\left(1+\frac{\langle v\rangle}{|n(x)\cdot v|}\right)\int_{n(x)\cdot u>0}|\partial f^m(t,x,u)|\mu^{1/4}\{n(x)\cdot u\}\mathrm{d}u + \frac{\langle v\rangle^{\gamma-\beta}}{|n(x)\cdot v|}||\langle v\rangle^\beta f_0||_\infty,\end{aligned} \tag{67}$$

where we used the boundary condition for $f^{m+1}|_{\gamma_-}$ again, and (62), (19) and Lemma 2 and $\langle u\rangle\sqrt{\mu(u)} \lesssim \mu(u)^{\frac14}$.

Set $\partial f^0 = [\partial_t f^0, \nabla_x f^0, \nabla_v f^0] = [0,0,0]$. The main estimate is the following :

Lemma 7. *For $1\le p<2$, if $0<T_*\ll 1$ and $||\langle v\rangle^\beta f_0||_\infty \ll 1$ for $\beta>4$, and the compatibility condition (4) then uniformly-in-m,*

$$\sup_{0\le t\le T_*}||\partial f^m||_p^p + \int_0^{T_*}|\partial f^m|_{\gamma,p}^p + \int_0^{T_*}||\nu^{1/p}\partial f^m||_p^p \lesssim_{\Omega,T_*} ||\partial f_0||_p^p + ||\langle v\rangle^\beta f_0||_\infty^p. \tag{68}$$

Recall that the time derivative of the initial datum is defined as $\partial_t f_0 \equiv -v\cdot\nabla_x f_0 - Lf_0 + \Gamma(f_0,f_0)$. We remark that the sequence (61) is the one used in [7, 2] and shown to be Cauchy in L^∞. Therefore the limit function f is a solution of the Boltzmann equation with the diffuse boundary condition (1). On the other hand, due to the weak lower semi-continuity for L^p in the case of $p>1$, once we have Lemma 7 then we pass a limit $\partial f^m \rightharpoonup \partial f$ weakly in $\sup_{t\in[0,T_*]}||\cdot||_p^p$ and $\int_0^{T_*}||\nu^{1/p}\cdot||_p^p$ and $\partial f^m|_\gamma \rightharpoonup \partial f|_\gamma$ in $\int_0^{T_*}|\cdot|_{\gamma,p}^p$ to conclude that ∂f satisfies the same estimate of (68). Repeat the same procedure for $[T_*, 2T_*], [2T_*, 3T_*], \cdots$, to conclude Theorem 1.

Proof of Lemma 7. We prove the lemma by induction. From Proposition 1, ∂f^1 exists. Because of our choice ∂f^0 the estimate (68) is valid for $m = 1$. Now assume that ∂f^i exists and (68) is valid for all $i = 1, 2, \cdots, m$. Applying Proposition 1 to show that ∂f^{m+1} exists and to get (37) − (39), we have

$$\begin{aligned}&\sup_{0\le s\le t} ||\partial f^{m+1}(s)||_p^p + \int_0^t |\partial f^{m+1}|_{\gamma_+,p}^p + \int_0^t ||\nu^{1/p}\partial f^{m+1}||_p^p \\ \lesssim\ & ||\partial f_0||_p^p + \int_0^t |\partial f^{m+1}|_{\gamma_-,p}^p + \int_0^t \iint_{\Omega\times\mathbb{R}^3} \Big\{|K\partial f^m| + |\Gamma(f^m, \partial f^m)| + |\Gamma(\partial f^m, f^m)|\Big\}|\partial f^{m+1}|^{p-1} \\ &+ t \sup_{0\le s\le t} ||\partial f^{m+1}(s)||_p^p + ||\langle v\rangle^\beta f_0||_\infty + ||\langle v\rangle^\beta f_0||_\infty \int_{\mathbb{R}^3} |\partial f^{m+1}|^p \mathrm{d}v,\end{aligned}$$

where we have used (65) and (66) and

$$\begin{aligned}\iint_{\Omega\times\mathbb{R}^3} \langle v\rangle^{-\beta+\gamma}||\langle v\rangle^\beta f_0||_\infty|\partial f^{m+1}|^{p-1}\mathrm{d}v \le & ||\langle v\rangle^\beta f_0||_\infty\{\iint \langle v\rangle^{-\beta+\gamma}\mathrm{d}v + \iint \langle v\rangle^{-\beta+\gamma}|\partial f^{m+1}|^p\} \\ \lesssim & ||\langle v\rangle^\beta f_0||_\infty\{1 + \iint_{\Omega\times\mathbb{R}^3} |\partial f^{m+1}|^p \mathrm{d}v\}.\end{aligned} \tag{69}$$

Estimate for K and Γ : Use Lemma 1 and Lemma 2 to have

$$\begin{aligned}&\int_0^t \iint_{\Omega\times\mathbb{R}^3} \Big\{|K\partial f^m| + |\Gamma(f^m, \partial f^m)| + |\Gamma(\partial f^m, f^m)|\Big\}|\partial f^{m+1}|^{p-1} \\ \lesssim & \int_0^t ||K\partial f^m||_p^p + \int_0^t ||\partial f^{m+1}||_p^p + \sup_x\Big\{\int_{\mathbb{R}^3} \nu|f^m|^p\mathrm{d}v\Big\}^{1/p}\Big\{\int_0^t ||\nu^{1/p}\partial f^m||_p^p + \int_0^t ||\nu^{1/p}\partial f^{m+1}||_p^p\Big\} \\ \lesssim & \int_0^t ||\partial f^m||_p^p + \int_0^t ||\partial f^{m+1}||_p^p + C_{\gamma,p,\beta}||\langle v\rangle^\beta f_0||_\infty\Big\{\int_0^t ||\nu^{1/p}\partial f^m||_p^p + \int_0^t ||\nu^{1/p}\partial f^{m+1}||_p^p\Big\},\end{aligned} \tag{70}$$

where we used $\int_{\mathbb{R}^3} \frac{\nu(v)}{\langle v\rangle^{p\beta}}\mathrm{d}v \lesssim \int_{\mathbb{R}^3}\langle v\rangle^{\gamma-p\beta}\mathrm{d}v = C_{\gamma,p,\beta} < \infty$ for $\beta > 4$.

Boundary Estimate : Recall (11). We use (67) to obtain

$$\begin{aligned}&\int_0^t \int_{\gamma_-} |\partial f^{m+1}(s)|^p \\ \lesssim_p & \sup_{x\in\partial\Omega}\left(\int_{\gamma_-} \sqrt{\mu(v)}^p\left(|n\cdot v| + \frac{\langle v\rangle^p}{|n\cdot v|^{p-1}}\right)\mathrm{d}v\right)\int_0^t\int_{\partial\Omega}\left[\int_{u\cdot n(x)>0}|\partial f^m(s,x,u)|\mu^{1/4}(u)\{n\cdot u\}\mathrm{d}u\right]^p \mathrm{d}S_x\mathrm{d}s \\ &+ \sup_{x\in\partial\Omega}\left(\int_{\gamma_-}\langle v\rangle^{-p(\beta-\gamma)}|n\cdot v|^{1-p}\mathrm{d}v\right)\times t||\langle v\rangle^\beta f_0||_\infty^p \\ \lesssim_p & \int_0^t\int_{\partial\Omega}\left[\int_{u\cdot n(x)>0}|\partial f^m(s,x,u)|\mu^{1/4}(u)\{n\cdot u\}\mathrm{d}u\right]^p \mathrm{d}S_x\mathrm{d}s + t||\langle v\rangle^\beta f_0||_\infty^p.\end{aligned} \tag{71}$$

It suffices to estimate $\int_0^t\int_{\partial\Omega}\left[\int_{u\cdot n(x)>0}|\partial f^m(s,x,u)|\mu^{1/4}(u)\{n\cdot u\}\mathrm{d}u\right]^p \mathrm{d}S_x\mathrm{d}s$. We split the $\{u\in\mathbb{R}^3 : n(x)\cdot u > 0\}$ as

$$\int_0^t\int_{\partial\Omega}\left[\int_{n\cdot u>0}|\partial f^m|\mu^{1/4}\{n\cdot u\}\mathrm{d}u\right]^p \lesssim_p \int_0^t\int_{\partial\Omega}\left[\int_{(x,u)\in\gamma_+\setminus\gamma_+^\varepsilon}\mathrm{d}u\right]^p + \int_0^t\int_{\partial\Omega}\left[\int_{(x,u)\in\gamma_+^\varepsilon}\mathrm{d}u\right]^p. \tag{72}$$

We use Hölder's inequality to bound

$$\left[\int_{(x,u)\in\gamma_+^\varepsilon}\mathrm{d}u\right]^p \le \left[\int_{(x,u)\in\gamma_+^\varepsilon}\mu^{\frac{p}{4(p-1)}}\{n\cdot u\}\mathrm{d}u\right]^{p-1}\left[\int_{(x,u)\in\gamma_+^\varepsilon}|\partial f^m(s,x,u)|^p\{n(x)\cdot u\}\mathrm{d}u\right],$$

to bound the second term of (72)

$$\int_0^t \int_{\partial\Omega} \left[\int_{(x,u)\in\gamma_+^\varepsilon} \mathrm{d}u \right]^p \lesssim_p \varepsilon \int_0^t |\partial f^m(s)|^p_{\gamma_+,p} \mathrm{d}s. \tag{73}$$

For the first term (non-grazing part) of (72) we use Hölder's inequality and Lemma 4 and Lemma 1 and Lemma 2 for f^m to estimate

$$\begin{aligned}
& \int_0^t \int_{\partial\Omega} \left[\int_{(x,u)\in\gamma_+\setminus\gamma_+^\varepsilon} \mathrm{d}u \right]^p \\
\lesssim_\varepsilon \quad & ||\partial f_0||_p^p + \int_0^t ||\partial f^m(s)||_p^p \mathrm{d}s + \int_0^t \iint_{\Omega\times\mathbb{R}^3} \big|[\partial_t + v\cdot\nabla_x + \nu(v)]\partial f^m\big| |\partial f^m|^{p-1} \mathrm{d}x\mathrm{d}v\mathrm{d}s \\
\lesssim_\varepsilon \quad & ||\partial f_0||_p^p + \int_0^t ||\partial f^m(s)||_p^p \qquad (74)\\
& + \int_0^t \iint_{\Omega\times\mathbb{R}^3} \big\{ \langle v\rangle^{-\beta+\gamma} ||\langle v\rangle^\beta f_0||_\infty + |\partial f^m| + |K\partial f^{m-1}| + |\Gamma(f^{m-1},\partial f^{m-1})| + |\Gamma(\partial f^{m-1}, f^{m-1})| \big\} |\partial f^m|^{p-1} \\
\lesssim_\varepsilon \quad & ||\partial f_0||_p^p + (1 + ||\langle v\rangle^\beta f_0||_\infty) \int_0^t ||\partial f^m(s)||_p^p + (1 + ||\langle v\rangle^\beta f_0||_\infty) \int_0^t ||\partial f^{m-1}(s)||_p^p + t ||\langle v\rangle^\beta f_0||_\infty^p.
\end{aligned}$$

Putting together the estimates (69), (73), (74) and (70), and choosing sufficiently small $\varepsilon \ll 1, T_* \ll 1, ||\langle v\rangle^\beta f_0||_\infty \ll 1$, we deduce that

$$\begin{aligned}
& \sup_{0\le t\le T_*} ||\partial f^{m+1}(t)||_p^p + \int_0^{T_*} |\partial f^{m+1}|^p_{\gamma_+,p} + \int_0^{T_*} ||\nu^{1/p}\partial f^{m+1}||_p^p \\
\lesssim & C_{T_*,\Omega} \big\{ ||\partial f_0||_p^p + ||\langle v\rangle^\beta f_0||_\infty \big\} + \frac{1}{8} \max_{i=m,m-1} \Big\{ \sup_{0\le t\le T_*} ||\partial f^i(t)||_p^p + \int_0^{T_*} |\partial f^i|^p_{\gamma_+,p} + \int_0^{T_*} ||\nu^{1/p}\partial f^i||_p^p \Big\}.
\end{aligned} \tag{75}$$

To conclude the proof we use the following fact from [2] : Suppose $a_i \ge 0, D \ge 0$ and $A_i = \max\{a_i, a_{i-1}, \cdots, a_{i-(k-1)}\}$ for fixed $k \in \mathbb{N}$.

$$If \quad a_{m+1} \le \frac{1}{8} A_m + D \quad \text{then} \quad A_m \ \le\ \frac{1}{8} A_0 + \left(\frac{8}{7}\right)^2 D, \quad \text{for } \frac{m}{k} \gg 1. \tag{76}$$

Proof of (76): In fact, we can iterate for $m, m-1, ...$ to get

$$\begin{aligned}
a_m \ &\le\ \frac{1}{8} \max\{\frac{1}{8}A_{m-2} + D, A_{m-2}\} + D \le \frac{1}{8} A_{m-2} + (1 + \frac{1}{8}) D \\
&\le\ \frac{1}{8} \max\{\frac{1}{8}A_{m-3} + D, A_{m-3}\} + (1 + \frac{1}{8}) D \le \frac{1}{8} A_{m-3} + (1 + \frac{1}{8} + \frac{1}{8^2}) D \\
&\le\ \frac{1}{8} A_{m-k} + \frac{8}{7} D.
\end{aligned}$$

Similarly $a_{m-i} \le \frac{1}{8} A_{m-k} + \frac{8}{7} D$ for all $i = 0, 1, \cdots, k-1$. Therefore if $1 \ll m/k \in \mathbb{N}$,

$$\begin{aligned}
A_m \ &=\ \max\{a_m, a_{m-1}, \cdots, a_{m-(k-1)}\} \le \frac{1}{8} A_{m-k} + \frac{8}{7} D \\
&\le\ \frac{1}{8^2} A_{m-2k} + \frac{8}{7}(1 + \frac{1}{8}) D \le \frac{1}{8^3} A_{m-3k} + \frac{8}{7}(1 + \frac{1}{8} + \frac{1}{8^2}) D \\
&\le\ \left(\frac{1}{8}\right)^{[\frac{m}{k}]} A_{m-[\frac{m}{k}]k} + \left(\frac{8}{7}\right)^2 D \le \left(\frac{1}{8}\right)^{\frac{m}{k}} A_0 + \left(\frac{8}{7}\right)^2 D \ \le\ \frac{1}{8} A_0 + \left(\frac{8}{7}\right)^2 D.
\end{aligned}$$

This completes the proof of (76).

In (75), setting $k = 2$ and

$$a_i = \sup_{0\le t\le T_*} ||\partial f^i(t)||_p^p + \int_0^t |\partial f^i|^p_{\gamma_+,p} + \int_0^t ||\nu^{1/p}\partial f^i||_p^p, \qquad D = C_{T_*,\Omega}\big\{ ||\partial f_0||_p^p + ||\langle v\rangle^\beta f_0||_\infty \big\},$$

and applying (76), we complete the proof of the lemma. $\square$

The following result indicates that Theorem 1 is optimal :

Lemma 8. *Let $\Omega = B(0;1)$ with $B(0;1) = \{x \in \mathbb{R}^3 : |x| < 1\}$. There exists an initial datum $f_0(x,v) \in C^\infty$ with $f_0 \subset\subset B(0;1) \times B(0;1)$ so that the solution f to*

$$\partial_t f + v \cdot \nabla_x f = 0, \quad f|_{t=0} = f_0, \quad f(t,x,v)|_{\gamma_-} = c_\mu \sqrt{\mu(v)} \int_{n(x)\cdot u>0} f(t,x,u)\sqrt{\mu(u)}\{n(x)\cdot u\}\mathrm{d}u, \tag{77}$$

satisfies

$$\int_0^1 \int_{\gamma_-} |\nabla_x f(s,x,v)|^2 \mathrm{d}\gamma \mathrm{d}s \;=\; +\infty,$$

so that the estimate (5) of Theorem 1 fails for $p=2$.

Proof. We prove by contradiction. Suppose $\int_0^1 \int_{\gamma_-} |\partial f(s,x,v)|^2 \mathrm{d}\gamma \mathrm{d}s < +\infty$. Then

$$\partial_n f(t,x,v) = \frac{1}{n\cdot v}\Big\{ -\partial_t f - (\tau_1\cdot v)\partial_{\tau_1} f - (\tau_2 \cdot v)\partial_{\tau_2} f\Big\}, \quad for\ (x,v)\in\gamma_-,$$

We use the boundary condition to define :

$$\begin{aligned}
\partial_t f(t,x,v)|_{\gamma_-} &= c_\mu \sqrt{\mu(v)} A(t,x) \equiv c_\mu\sqrt{\mu}\int_{n\cdot u>0} \partial_t f \sqrt{\mu}\{n\cdot u\}\mathrm{d}u,\\
\partial_{\tau_i} f(t,x,v)|_{\gamma_-} &= c_\mu \sqrt{\mu(v)} B_i(t,x)\\
&\equiv c_\mu\sqrt{\mu}\int_{n\cdot u>0}\partial_{\tau_i} f\sqrt{\mu}\{n\cdot u\}\mathrm{d}u + c_\mu\sqrt{\mu}\int_{n\cdot u>0}\nabla_v f \frac{\partial \mathcal{T}}{\partial \tau_i}\mathcal{T}^{-1} u \sqrt{\mu}\{n\cdot u\}\mathrm{d}u.
\end{aligned}$$

We make a change of variables $v_\perp = v\cdot n(x),\ v_{\tau_1} = v\cdot \tau_1(x),\ v_{\tau_2} = v\cdot\tau_2(x)$ to compute

$$\begin{aligned}
&\int_{\partial\Omega}\mathrm{d}S_x \int_0^\infty \mathrm{d}v_\perp \iint_{\mathbb{R}^2} \mathrm{d}v_{\tau_1}\mathrm{d}v_{\tau_2}\,\frac{\mu(v)}{v_\perp}\Big\{(A)^2 + (v_{\tau_1})^2(B_1)^2 + (v_{\tau_2})^2(B_2)^2 + 2v_{\tau_1}AB_1 + 2v_{\tau_2}AB_2 + 2v_{\tau_1}v_{\tau_2}B_1B_2\Big\}\\
&= \int_0^\infty \mathrm{d}v_\perp \frac{e^{-\frac{|v_\perp|^2}{2}}}{v_\perp}\int_{\partial\Omega}\mathrm{d}S_x\big\{(A)^2 + 2\pi(B_1)^2 + 2\pi(B_2)^2\big\}.
\end{aligned}$$

Note that the integration over $\partial\Omega$ is a function of t only (independent of v). Since $\int_0^\infty \frac{\mathrm{d}v_\perp}{v_\perp} = \infty$, we conclude that $A = B_1 = B_2 \equiv 0$ for $(t,x)\in[0,\infty)\times\partial\Omega$. In particular from $A(t,x)=0$ we have for all $t\geq 0$

$$\int_{n(x)\cdot u>0} f(t,x,u)\sqrt{\mu(u)}\{n(x)\cdot u\}\mathrm{d}u = \int_{n(x)\cdot u>0} f(0,x,u)\sqrt{\mu(u)}\{n(x)\cdot u\}\mathrm{d}u. \tag{78}$$

We now choose the initial datum to vanish near $\partial\Omega$:

$$f_0(x,v) = \phi(|x|)\phi(|v|),$$

where $\phi\in C^\infty([0,\infty))$ and $\phi\geq 0$ and $\mathrm{supp}\phi \subset\subset [0,1)$ and $\phi\equiv 1$ on $[0,\frac{1}{2}]$. Clearly

$$c_\mu\sqrt{\mu(v)}\int_{n(x)\cdot u>0} f_0(x,u)\sqrt{\mu(u)}\{n(x)\cdot u\}\mathrm{d}u = 0.$$

Hence $f(t,x,v)\geq 0$ from $f_0\geq 0$ and the zero inflow boundary condition from (78) and the above equality. Moreover following the backward trajectory to the initial plane for $t\in[\frac{1}{8},\frac{1}{4}]$ and $(x,v)\in\gamma_+$ and $|v-\frac{x}{|x|}|<\frac{1}{64}$, and $|v|\in[\frac{1}{8},\frac{1}{2}]$,

$$f(t,x,v) = f_0(x-tv,v) = 1,$$

which contradicts to $c_\mu\sqrt{\mu(v)}\int_{n\cdot u>0} f(t,x,u)\sqrt{\mu(u)}\{n(x)\cdot u\}\mathrm{d}u = 0$ for $(t,x,v)\in[0,\infty)\times\gamma_-$ from (78). $\square$

5 Weighted $W^{1,p}$ $(2 \le p < \infty)$ Estimate

We now establish the weighted $W^{1,p}$ estimate for $2 \le p < \infty$ with the same iteration (61). From [2, 7] for $0 < \zeta < \frac{1}{4}$, we have a uniform bound

$$\sup_m \sup_{0 \le t < \infty} ||e^{\zeta |v|^2} f^m(t)||_\infty \lesssim_\Omega ||e^{\zeta|v|^2} f_0||_\infty. \tag{79}$$

Recall $\partial = [\partial_t, \nabla_x, \nabla_v]$ and $\mathbf{d}^\lambda \partial f^m$ satisfies

$$\{\partial_t + v \cdot \nabla_x + \nu_{l,\lambda}\} \mathbf{d}^\lambda \partial f^{m+1} = \mathcal{N}^m, \quad \mathbf{d}^\lambda \partial f^{m+1}(0,x,v) = \mathbf{d}^\lambda \partial f_0(x,v), \tag{80}$$

where $\mathcal{N}^m = \mathbf{d}^\lambda \mathcal{G}^m$ in (65). Using (19) and (20), (23), (24) and (27), (29) and Lemma 2 and Lemma 3 to have

$$\begin{aligned} |\mathcal{N}^m| \lesssim & \mathbf{d}^\lambda \big\{ |K\partial f^m| + |\Gamma(f^m, \partial f^m)| + |\Gamma(\partial f^m, f^m)| \big\} \\ & + \mathbf{d}^\lambda \big\{ |\partial f^{m+1}| + |\nabla_v \nu(v)||f^{m+1}| + e^{-\frac{\zeta}{4}|v|^2} ||e^{\zeta|v|^2} f_0||_\infty (1 + ||e^{\zeta|v|^2} f_0||_\infty) \big\}. \end{aligned} \tag{81}$$

For $(x,v) \in \gamma$, $\mathbf{d}(t,x,v) \backsim e^{-\frac{l\langle v\rangle}{2}t} |n(x)\cdot v|$, and from (34) and (67), the boundary condition is bounded for $\lambda < \frac{p-1}{p}$ by

$$\begin{aligned} \mathbf{d}^\lambda |\partial f^{m+1}(t,x,v)| \lesssim & \ \mathbf{d}^\lambda \sqrt{\mu} \Big(1 + \frac{\langle v \rangle}{|n(x)\cdot v|} \Big) \int_{n\cdot u > 0} |\partial f^m(t,x,u)| \langle u \rangle \sqrt{\mu} \{n \cdot u\} \mathrm{d}u \\ & + \frac{\mathbf{d}^\lambda(t,x,v)}{|n(x)\cdot v|} e^{-\frac{\zeta}{4}|v|^2} ||e^{\zeta|v|^2} f_0||_\infty (1 + ||e^{\zeta|v|^2} f_0||_\infty). \end{aligned} \tag{82}$$

Set $f^0 = f_0$ and $\partial f^0 = [\partial_t f^0, \nabla_x f^0, \nabla_v f^0] = [0,0,0]$. The main estimate is the following :

Lemma 9. *For $p \ge 2$, $\frac{p-2}{p} < \lambda < \frac{p-1}{p}$, if $0 < T_* \ll 1$ and $||e^{\zeta|v|^2} f_0||_\infty \ll 1$ for $0 < \zeta < \frac{1}{4}$, then uniformly-in-m,*

$$\sup_{0 \le t \le T_*} ||\mathbf{d}^\lambda \partial f^m(t)||_p^p + \int_0^{T_*} |\mathbf{d}^\lambda \partial f^m|_{\gamma,p}^p + \int_0^{T_*} ||\nu_{l,\lambda}^{1/p} \mathbf{d}^\lambda \partial f^m||_p^p \lesssim_{\Omega, T_*} ||\mathbf{d}^\lambda \partial f_0||_p^p + ||e^{\zeta|v|^2} f_0||_\infty^p. \tag{83}$$

Remark that $\nu_{l,\lambda}(t,x,v) \sim \langle v \rangle$. Once we have Lemma 9 then we pass to the limit, $\mathbf{d}^\lambda \partial f^m \rightharpoonup \mathbf{d}^\lambda \partial f$ weakly with norms $\sup_{t\in[0,T_*]} || \cdot ||_p^p$ and $\int_0^{T_*} ||\nu_{l,\lambda}^{1/p} \cdot ||_p^p$ and $\mathbf{d}^\lambda \partial f^m|_\gamma \rightharpoonup \mathbf{d}^\lambda \partial f|_\gamma$ in $\int_0^{T_*} |\cdot|_{\gamma,p}^p$ and $\mathbf{d}^\lambda \partial f$ satisfies (83). Repeat the same procedure for $[T_*, 2T_*], [2T_*, 3T_*], \cdots$, to conclude Theorem 2.

Proof of Lemma 9. We prove the Lemma by induction. From Proposition 2 ∂f^1 exits. More precisely we construct $\partial_t f^1, \nabla_x f^1$ first and then $\nabla_v f^1$. Because of our choice of ∂f^0 the estimate (83) is valid for $m = 1$. Now assume that ∂f^i exists and (83) is valid for all $i = 1, 2, \cdots, m$. Applying Proposition 2 we deduce that ∂f^{m+1} exists. From Lemma 5 we have

$$\begin{aligned} & \sup_{0 \le s \le t} ||\mathbf{d}^\lambda \partial f^{m+1}(s)||_p^p + \int_0^t |\mathbf{d}^\lambda \partial f^{m+1}|_{\gamma_+,p}^p + \int_0^t ||\nu_{l,\lambda}^{1/p} \mathbf{d}^\lambda \partial f^{m+1}||_p^p \\ \lesssim & ||\mathbf{d}^\lambda \partial f_0||_p^p + \int_0^t |\mathbf{d}^\lambda \partial f^{m+1}|_{\gamma_-,p}^p + \{t + \varepsilon\} \sup_{0 \le s \le t} ||\mathbf{d}^\lambda \partial f^{m+1}(s)||_p^p + t ||e^{\zeta|v|^2} f^m||_\infty^p (1 + ||e^{\zeta|v|^2} f^m||_\infty^p) \\ & + t ||e^{\zeta|v|^2} f^{m+1}||_\infty^p + \int_0^t \iint_{\Omega\times\mathbb{R}^3} \mathbf{d}^{\lambda p} \Big\{ |K\partial f^m| + |\Gamma(f^m, \partial f^m)| + |\Gamma(\partial f^m, f^m)| \Big\} |\partial f^{m+1}|^{p-1}. \end{aligned} \tag{84}$$

Estimate for $K\partial f^m$: The key estimate is the following : For $0 < \lambda < \frac{p-1}{p}$, $0 < \zeta \le \frac{1}{4}$, and some $C_{l,\lambda,p} > 0$,

$$\sup_{x\in\Omega} \int_{\mathbb{R}^3} \mathbf{k}_\zeta(v,u) \frac{\mathbf{d}(t,x,v)^{\frac{\lambda p}{p-1}}}{\mathbf{d}(t,x,u)^{\frac{\lambda p}{p-1}}} \mathrm{d}u \ \lesssim_{\Omega,\zeta} \ \langle v \rangle^{\frac{\lambda p}{p-1}} e^{C_{l,\lambda,p} t^2}. \tag{85}$$

Recall the function $\mathbf{k}_\zeta(v,u)$ in (26) and remark that if $\zeta = \frac{1}{4}$ then $\mathbf{k}_\zeta(v,u) \equiv \mathbf{k}(v,u)$. We split

$$\int_{\mathbb{R}^3} \mathbf{k}_\zeta(v,u) \frac{\mathbf{d}(t,x,v)^{\frac{\lambda p}{p-1}}}{\mathbf{d}(t,x,u)^{\frac{\lambda p}{p-1}}} \mathrm{d}u = \int_{\mathbf{d}(t,x,u)>\varepsilon} + \int_{\mathbf{d}(t,x,u)\le\varepsilon} .$$

The first term is bounded by

$$\int_{\mathbb{R}^3} \mathbf{k}_\zeta(v,u) \mathrm{d}u \lesssim_{\varepsilon,\gamma} 1,$$

from the lower bound of $\mathbf{d}(t,x,u)$. Denote $u_\perp = u\cdot n(x) = u \cdot \frac{\nabla\xi(x)}{|\nabla\xi(x)|}$ and $u_\parallel = u - u_\perp n(x)$. The second term for $0 \le \gamma \le 1$ is bounded by

$$\begin{aligned}
& |v|^{\frac{\lambda p}{p-1}} \int_{\mathbb{R}^3} \{|v-u|^\gamma + \frac{1}{|v-u|^{2-\gamma}}\} e^{-C_\zeta |v-u|^2} \frac{e^{-\frac{l\lambda p}{2(p-1)}\langle v\rangle t}}{e^{-\frac{l\lambda p}{2(p-1)}\langle u\rangle t}} \frac{1}{|u\cdot\nabla\xi(x)|^{\frac{\lambda p}{p-1}}} \mathrm{d}u \\
\lesssim_\Omega \ & |v|^{\frac{\lambda p}{p-1}} \int_{\mathbb{R}^3} \{1 + |v-u|^{-2+\gamma}\} e^{-\frac{C_\zeta |v-u|^2}{2}} e^{\frac{l\lambda p}{2(p-1)} t|v-u|} |u_\perp|^{\frac{-\lambda p}{p-1}} \mathrm{d}u \\
\lesssim_\Omega \ & |v|^{\frac{\lambda p}{p-1}} e^{C_{l,\lambda,p} t^2} \int_{\mathbb{R}^2} \mathrm{d}u_\parallel \int_{\mathbb{R}} \mathrm{d}u_\perp \{1 + |v-u|^{-2+\gamma}\} e^{-\frac{C_\zeta|v-u|^2}{4}} |u_\perp|^{-\frac{\lambda p}{p-1}} \\
\lesssim_\Omega \ & |v|^{\frac{\lambda p}{p-1}} e^{C_{l,\lambda,p} t^2} \int_{\mathbb{R}^2} \{1 + |v_\parallel - u_\parallel|^{-2+\gamma+\delta}\} e^{-\frac{C_\zeta|v-u|^2}{16}} \mathrm{d}u_\parallel \int_{\mathbb{R}} \frac{e^{-\frac{C_\zeta|v_\perp - u_\perp|^2}{8}}}{|u_\perp|^{\frac{\lambda p}{p-1}} |v_\perp - u_\perp|^\delta} \mathrm{d}u_\perp \\
\lesssim_\Omega \ & C_\gamma |v|^{\frac{\lambda p}{p-1}} e^{C_{l,\lambda,p} t^2} \int_{\mathbb{R}} \frac{e^{-C_\zeta \frac{|v_\perp - u_\perp|^2}{8}}}{|u_\perp|^{\frac{\lambda p}{p-1}} |v_\perp - u_\perp|^\delta} \mathrm{d}u_\perp .
\end{aligned}$$

where $0 < \lambda < \frac{p-1}{p}$ and $\delta > 0$ is chosen such that $\frac{\lambda p}{p-1} + \delta < 1$ and $-2+\gamma+\delta > -2$, and

$$e^{\frac{l\lambda p}{2(p-1)} t|v-u|} \lesssim e^{C_{l,\lambda,p} t^2} \times e^{-\frac{C_\zeta |v-u|^2}{4}}, \tag{86}$$

for some $C_{l,\lambda,p} > 0$. Furthermore we split the last integration as $\int_{|u_\perp|/2 \le |v_\perp - u_\perp|} + \int_{|u_\perp|/2 \ge |v_\perp - u_\perp|}$. Both of them are bounded by

$$C\left[\int \frac{e^{-\frac{C_\zeta|v_\perp - u_\perp|^2}{8}}}{|u_\perp|^{\frac{\lambda p}{p-1}+\delta}} \mathrm{d}u_\perp + \int \frac{e^{-\frac{C_\zeta|v_\perp - u_\perp|^2}{8}}}{|v_\perp - u_\perp|^{\frac{\lambda p}{p-1}+\delta}} \mathrm{d}u_\perp \right] \lesssim \langle v_\perp\rangle^{-\frac{\lambda p}{p-1}-\delta} + 1.$$

Therefore we conclude (85).

In order to estimate $K\partial f^m$ contribution in (84) recall, for $1/p + 1/q = 1$,

$$\begin{aligned}
\mathbf{d}^\lambda |K\partial f^m| \ &\le \ \mathbf{d}^\lambda \int |\mathbf{k}(v,u)||\partial f^m(u)| \mathrm{d}u = \mathbf{d}^\lambda(v) \int |\mathbf{k}(v,u)| \frac{\mathbf{d}(u)^\lambda |\partial f^m(u)|}{\mathbf{d}(u)^\lambda} \mathrm{d}u \\
&\le \ \left(\int |\mathbf{k}(v,u)| \frac{\mathbf{d}(v)^{\lambda q}}{\mathbf{d}(u)^{\lambda q}} \mathrm{d}u\right)^{\frac{1}{q}} \left(\int |\mathbf{k}(v,u)| \left|\mathbf{d}^\lambda \partial f^m(u)\right|^p \mathrm{d}u\right)^{\frac{1}{p}} \\
&\le \ \langle v\rangle^\lambda e^{C_{l,\lambda,p} t^2} \left(\int |\mathbf{k}(v,u)| \left|\mathbf{d}^\lambda \partial f^m(u)\right|^p \mathrm{d}u\right)^{\frac{1}{p}} .
\end{aligned}$$

The $K\partial f^m$ contribution in (84) is therefore bounded by

$$\begin{aligned}
&\int_0^t \iint_{\Omega\times\mathbb{R}^3} \mathbf{d}^{\lambda p}|K\partial f^m||\partial f^{m+1}|^{p-1}\mathrm{d}v\mathrm{d}x\mathrm{d}s\\
\leq\ & \int_0^t \iint_{\Omega\times\mathbb{R}^3} \langle v\rangle^{\frac{\lambda}{p}} e^{C_{l,\lambda,p}s^2}\Big(\int |\mathbf{k}(v,u)||\mathbf{d}^\lambda\partial f^m(u)|^p\mathrm{d}u\Big)^{\frac{1}{p}}\langle v\rangle^{\frac{\lambda}{q}}\mathbf{d}^{\lambda(p-1)}|\partial f^{m+1}(v)|^{p-1}\\
\leq\ & C_\varepsilon \int_0^t \iint_{\Omega\times\mathbb{R}^3} e^{pC_{l,\lambda,p}s^2}\langle u\rangle^\lambda|\mathbf{d}^\lambda\partial f^m(u)|^p\Big(\int |\mathbf{k}(v,u)|\frac{\langle v\rangle^\lambda}{\langle u\rangle^\lambda}\mathrm{d}v\Big)\mathrm{d}x\mathrm{d}u\mathrm{d}s\\
&+\varepsilon\int_0^t \iint_{\Omega\times\mathbb{R}^3}\langle v\rangle^\lambda\mathbf{d}^{\lambda p}|\partial f^{m+1}|^p\mathrm{d}x\mathrm{d}v\mathrm{d}s\\
\leq\ & C\int_0^t \iint_{\Omega\times\mathbb{R}^3} e^{C_{l,\lambda,p}s^2}\langle v\rangle^\lambda|\mathbf{d}^\lambda\partial f^m|^p+\varepsilon\int_0^t \iint_{\Omega\times\mathbb{R}^3}\langle v\rangle^\lambda|\mathbf{d}^\lambda\partial f^{m+1}|^p\\
\leq\ & Cte^{C_{l,\lambda,p}t^2}\sup_{0\leq s\leq t}\iint_{\Omega\times\mathbb{R}^3}|\mathbf{d}^\lambda\partial f^m|^p+(\delta+\varepsilon)\max_{i=m,m+1}\int_0^t \iint_{\Omega\times\mathbb{R}^3}\langle v\rangle|\mathbf{d}^\lambda\partial f^i|^p,
\end{aligned} \tag{87}$$

where we used $\langle v\rangle^\lambda \leq C_\delta+\delta\langle v\rangle$ and (19) and (20) in Lemma 1 and $e^{C_{l,\lambda,p}s^2}$ factor comes from (85).

Estimate of nonlinear terms : Recall $\mathbf{k}_\zeta(v,u)$ in (26), (25) and (85). In order to estimate the nonlinear terms in (84) we apply (85) to have

$$\begin{aligned}
&\mathbf{d}^\lambda\big\{|\Gamma(f^m,\partial f^m)|+|\Gamma(\partial f^m,f^m)|\big\}(s,x,v)\\
\lesssim_\zeta\ & ||e^{\zeta|v|^2}f_0||_\infty\Big\{\nu_\zeta(v)\mathbf{d}^\lambda|\partial f^m|+\mathbf{d}^\lambda\Big|\int_{\mathbb{R}^3}\mathbf{k}_\zeta(v,u)\partial f^m(u)\mathrm{d}u\Big|\Big\}\\
\lesssim_\zeta\ & ||e^{\zeta|v|^2}f_0||_\infty\langle v\rangle\mathbf{d}^\lambda|\partial f^m|\\
&+||e^{\zeta|v|^2}f_0||_\infty\Big(\int_{\mathbb{R}^3}\mathbf{k}_\zeta(v,u)\frac{\mathbf{d}(s,x,v)^{\lambda q}}{\mathbf{d}(s,x,u)^{\lambda q}}\mathrm{d}u\Big)^{\frac{1}{q}}\Big(\int_{\mathbb{R}^3}\mathbf{k}_\zeta(v,u)|\mathbf{d}^\lambda\partial f^m(u)|^p\mathrm{d}u\Big)^{\frac{1}{p}}\\
\lesssim_\zeta\ & ||e^{\zeta|v|^2}f_0||_\infty\Big\{\langle v\rangle\mathbf{d}^\lambda|\partial f^m|+\langle v\rangle e^{Cs^2}\Big(\int_{\mathbb{R}^3}\mathbf{k}_\zeta(v,v')|\mathbf{d}^\lambda\partial f^m(u)|^p\mathrm{d}u\Big)^{\frac{1}{p}}\Big\},
\end{aligned}$$

where at the last line we used $\frac{p-2}{p}<\lambda<\frac{p-1}{p}$ so that $\langle v\rangle^\lambda\leq\langle v\rangle$.

Therefore the nonlinear contributions in (84) are bounded by

$$\begin{aligned}
&\int_0^t\iint \mathbf{d}^{\lambda p}|\Gamma(f^m,\partial f^m)+\Gamma(\partial f^m,f^m)||\partial f^{m+1}|^{p-1}\mathrm{d}v\mathrm{d}x\mathrm{d}s \qquad (88)\\
&\lesssim_\zeta ||e^{\zeta|v|^2}f_0||_\infty\int_0^t\iint\langle v\rangle^{\frac{1}{q}}\mathbf{d}^{\frac{\lambda p}{q}}|\partial f^{m+1}|^{p-1}\\
&\quad\times\Big\{\langle v\rangle^{\frac{1}{p}}\mathbf{d}^\lambda|\partial f^m|+e^{Cs^2}\langle v\rangle^{\frac{1}{p}}\Big(\int_{\mathbb{R}^3}\mathbf{k}_\zeta(v,u)\mathbf{d}^{\lambda p}|\partial f^m(s,x,u)|\Big)^{\frac{1}{p}}\Big\}\\
&\lesssim_\zeta ||e^{\zeta|v|^2}f_0||_\infty\Big\{\int_0^t\iint\langle v\rangle|\mathbf{d}^\lambda\partial f^m|^p+\int_0^t\iint\langle v\rangle|\mathbf{d}^\lambda\partial f^{m+1}|^p\\
&\qquad+\int_0^t e^{C_ps^2}\iiint\mathbf{k}_\zeta(v,u)\frac{\langle v\rangle^\lambda}{\langle u\rangle^\lambda}\langle u\rangle^\lambda|\mathbf{d}^\lambda\partial f^m(u)|^p\mathrm{d}u\mathrm{d}v\mathrm{d}x\Big\}\\
&\lesssim_\zeta ||e^{\zeta|v|^2}f_0||_\infty\Big\{\int_0^t\iint e^{C_{l,\lambda,p}s^2}\langle v\rangle|\mathbf{d}^\lambda\partial f^m|^p+\int_0^t\iint\langle v\rangle|\mathbf{d}^\lambda\partial f^{m+1}|^p\Big\},
\end{aligned}$$

where we have used Lemma 1.

Boundary Estimate: Recall (11). We use (82) to estimate the contribution of γ_-

$$\begin{aligned}
&\int_0^t \int_{\gamma_-} |\mathbf{d}^\lambda \partial f^{m+1}(s,x,v)|^p \\
&\lesssim_p \int_0^t \int_{\gamma_-} \mathbf{d}^{\lambda p} \sqrt{\mu}^p \Big(1+\frac{\langle v\rangle}{|n(x)\cdot v|}\Big)^p \left[\int_{n(x)\cdot u>0} |\partial f^m(s,x,u)| \mu(u)^{1/4}\{n(x)\cdot u\}\mathrm{d}u\right]^p \mathrm{d}\gamma \mathrm{d}s \qquad (89)\\
&\quad + ||e^{\zeta|v|^2} f_0||_\infty^p \int_0^t \int_{\gamma_-} \frac{\mathbf{d}(s,x,v)^{\lambda p}}{|n(x)\cdot v|^p} e^{-\frac{\zeta p}{4}|v|^2} \mathrm{d}\gamma \mathrm{d}s.
\end{aligned}$$

Using $\mathbf{d}(s,x,v) \le e^{-\frac{l\langle v\rangle}{2}s} |\nabla_x \xi(x)\cdot v|$ for $x\in\partial\Omega$, the last term is bounded by

$$C_\Omega ||e^{\zeta|v|^2} f_0||_\infty^p \int_0^t \int_{\partial\Omega}\int_{\mathbb{R}^3} |n(x)\cdot v|^{\lambda p - p+1} e^{-\frac{\zeta p}{4}|v|^2} \mathrm{d}v \mathrm{d}S_x \mathrm{d}s \lesssim_{\Omega,p,\zeta} t||e^{\zeta|v|^2} f_0||_\infty^p,$$

for $\lambda > \frac{p-2}{p}$ so that $\lambda p - p + 1 > -1$.

For the first term in (89) we split as

$$\left[\int_{n(x)\cdot u>0} \cdots \mathrm{d}u\right]^p \lesssim_p \left[\int_{(x,u)\in\gamma_+^\varepsilon} \cdots \mathrm{d}u\right]^p + \left[\int_{(x,u)\in\gamma_+\setminus\gamma_+^\varepsilon} \cdots \mathrm{d}u\right]^p.$$

The γ_+^ε contribution (grazing part) of (89) is bounded by

$$\begin{aligned}
&C_p \int_0^t \int_{\gamma_-} \mathbf{d}^{\lambda p}\sqrt{\mu}^p \Big(|n\cdot v| + \frac{\langle v\rangle^p}{|n\cdot v|^{p-1}}\Big) \left|\int_{(x,u)\in\gamma_+^\varepsilon} \mathbf{d}^\lambda \partial f^m \{n\cdot u\}^{1/p} \frac{\{n\cdot u\}^{1/q}\mu^{1/4}}{\mathbf{d}(s,x,u)^\lambda}\mathrm{d}u\right|^p \mathrm{d}v\mathrm{d}S_x\mathrm{d}s \\
&\lesssim_{\Omega,p} \int_0^t \int_{\gamma_-} \mathbf{d}^{\lambda p}\Big(|n\cdot v| + \frac{\langle v\rangle^p}{|n\cdot v|^{p-1}}\Big)\sqrt{\mu}^p \\
&\qquad \times \left[\int_{(x,u)\in\gamma_+} \mathbf{d}^{\lambda p}|\partial f^m|^p\{n\cdot u\}\mathrm{d}u\right] \left[\int_{(x,u)\in\gamma_+^\varepsilon} \mathbf{d}(s,x,u)^{-\lambda q}\mu^{q/4}\{n\cdot u\}\mathrm{d}u\right]^{p/q} \mathrm{d}v\mathrm{d}S_x\mathrm{d}s, \\
&\lesssim_{\Omega,p,l,\lambda} \varepsilon^a e^{C_{l,\lambda,p}t^2} \int_0^t |\mathbf{d}^\lambda \partial f^m(s)|_{\gamma_+,p}^p \mathrm{d}s,
\end{aligned}$$

where we used $\mathbf{d}(s,x,v) \le e^{-\frac{l\langle v\rangle}{2}s}|\nabla\xi(x)\cdot v| \lesssim_\Omega |n(x)\cdot v|$ and, for $\lambda > \frac{p-2}{p}$,

$$\mathbf{d}^{\lambda p}\Big(|n\cdot v| + \frac{\langle v\rangle^p}{|n\cdot v|^{p-1}}\Big)\sqrt{\mu}^p \lesssim_\Omega \Big(|n(x)\cdot v|^{1+\lambda p} + \langle v\rangle^p |n(x)\cdot v|^{(\lambda-1)p+1}\Big)\sqrt{\mu(v)}^p \in L^1(\{v\in\mathbb{R}^3\}),$$

and $a>0$ is determined by (13) at the boundary with $\frac{p-1}{p} = \frac{1}{q}$,

$$\begin{aligned}
&\int_{\gamma_+^\varepsilon} \mathbf{d}(s,x,u)^{-\frac{\lambda p}{p-1}} \mu^{\frac{p}{4(p-1)}} \{n\cdot u\}\mathrm{d}u \lesssim_\Omega \int_{\gamma_+^\varepsilon} \left[e^{-\frac{l\langle u\rangle s}{2}}|u\cdot\nabla\xi(x)|\right]^{-\frac{\lambda p}{p-1}} e^{-\frac{p}{4(p-1)}|u|^2}|n\cdot u|\mathrm{d}u \\
&\lesssim_\Omega \int_{\gamma_+^\varepsilon} |u\cdot n|^{1-\frac{\lambda p}{p-1}} e^{\frac{l\lambda}{2(p-1)}\langle u\rangle s} e^{-\frac{p}{4(p-1)}|u|^2}\mathrm{d}u \lesssim_\Omega e^{C_{l,\lambda,p}s^2}\int_{\gamma_+^\varepsilon} |u\cdot n|^{1-\frac{\lambda p}{p-1}} e^{-\frac{p}{8(p-1)}|u|^2}\mathrm{d}u \\
&\lesssim_{\Omega,p} \varepsilon^a e^{C_{l,\lambda,p}t^2},
\end{aligned}$$

for some $a>0$ since $1-\frac{\lambda p}{p-1} > -1$.

On the other hand, for the non-grazing contribution $\gamma_+\backslash\gamma_+^\varepsilon$, we use a similar estimate to get

$$\begin{aligned}
&\int_0^t\int_{\gamma_-}\mathbf{d}^{\lambda p}\sqrt{\mu}^p\Big(1+\frac{\langle v\rangle}{|n(x)\cdot v|}\Big)^p\left[\int_{\gamma_+\backslash\gamma_+^\varepsilon}|\partial f^m(s,x,u)|\mu(u)^{1/4}\{n(x)\cdot u\}\mathrm{d}u\right]^p\mathrm{d}\gamma\mathrm{d}s\\
&\lesssim_\Omega\int_0^t\int_{\partial\Omega}\int_{\mathbb{R}^3}\mathbf{d}^{\lambda p}(s,x,v)\Big(|n\cdot v|+\frac{\langle v\rangle^p}{|n\cdot v|^{p-1}}\Big)\sqrt{\mu}^p\left[\int_{\gamma_+\backslash\gamma_+^\varepsilon}\mathbf{d}^\lambda|\partial f^m(s,x,u)|\{n\cdot u\}^{1/p}\frac{\{n\cdot u\}^{1/q}\mu(u)^{1/4}}{\mathbf{d}(s,x,u)^\lambda}\mathrm{d}u\right]^p\mathrm{d}v\mathrm{d}S_x\mathrm{d}s\\
&\lesssim_\Omega\int_0^t\int_{\gamma_-}\mathbf{d}^{\lambda p}\Big(|n\cdot v|+\frac{\langle v\rangle^p}{|n\cdot v|^{p-1}}\Big)\sqrt{\mu}^p\left[\int_{\gamma_+\backslash\gamma_+^\varepsilon}\mathbf{d}^{\lambda p}|\partial f^m|^p\{n\cdot u\}\mathrm{d}u\right]\left[\int_{\gamma_+}\mathbf{d}(s,x,u)^{-\lambda q}\mu^{q/4}\{n\cdot u\}\mathrm{d}u\right]^{p/q}\mathrm{d}v\mathrm{d}S_x\mathrm{d}s\\
&\lesssim_\Omega e^{C_{l,\lambda,p}t^2}\int_0^t\int_{\gamma_+\backslash\gamma_+^\varepsilon}\mathbf{d}^{\lambda p}|\partial f^m(s)|^p\mathrm{d}\gamma\mathrm{d}s,
\end{aligned}$$

where we used $\frac{p-2}{p}<\lambda<\frac{p-1}{p}$ and

$$\int_{\gamma_+}\mathbf{d}(s,x,u)^{-\lambda q}\mu(u)^{q/4}\{n(x)\cdot u\}\mathrm{d}u=\int_{\gamma_+}\mathbf{d}(s,x,u)^{-\frac{\lambda p}{p-1}}\mu(u)^{\frac{p}{4(p-1)}}\{n\cdot u\}\mathrm{d}u\lesssim_{\Omega,p}e^{C_{l,\lambda,p}t^2}.$$

By Lemma 4 and (80), the non-grazing part is further bounded by

$$\begin{aligned}
\int_0^t\int_{\gamma_+\backslash\gamma_+^\varepsilon}&\lesssim_\varepsilon\int_0^t||\mathbf{d}^\lambda\partial f_0||_p^p+\int_0^t||\mathbf{d}^\lambda\partial f^m||_p^p+\int_0^t\iint_{\Omega\times\mathbb{R}^3}|\{\partial_t+v\cdot\nabla_x+\nu_{l,\lambda}\}(\mathbf{d}^\lambda\partial f^m)^p|\\
&\lesssim\int_0^t||\mathbf{d}^\lambda\partial f_0||_p^p+\int_0^t||\mathbf{d}^\lambda\partial f^m||_p^p+\int_0^t\iint\mathbf{d}^{\lambda p}|K\partial f^{m-1}||\partial f^m|^{p-1}\\
&\quad+\int_0^t\iint\mathbf{d}^{\lambda p}\{|\Gamma(f^{m-1},\partial f^{m-1})|+|\Gamma(\partial f^{m-1},f^{m-1})|\}|\partial f^m|^{p-1}\\
&\quad+t\sup_{0\le s\le t}||\mathbf{d}^\lambda\partial f^m(s)||_p^p+(1+t)||e^{\zeta|v|^2}f_0||_\infty^p(1+||e^{\zeta|v|^2}f_0||_\infty^p).
\end{aligned}$$

In summary, for small $T_*\ll1$, the boundary contribution of (84) is controlled by, for all $0\le t\le T_*$,

$$\begin{aligned}
&\int_0^t|\mathbf{d}^\lambda\partial f^m(s)|_{\gamma_-,p}^p\mathrm{d}s\\
&\lesssim\int_0^{T_*}||\mathbf{d}^\lambda\partial f_0||_p^p+\varepsilon^a\int_0^{T_*}|\mathbf{d}^\lambda\partial f^m|_{\gamma_+,p}^p+T_*\max_{i=m-1,m}\sup_{0\le t\le T_*}||\mathbf{d}^\lambda\partial f^i(t)||_p^p+||e^{\zeta|v|^2}f_0||_\infty^p(1+||e^{\zeta|v|^2}f_0||_\infty^p)\\
&\quad+C_{\Omega,T_*}\Big\{\int_0^{T_*}||\mathbf{d}^\lambda\partial f^m||_p^p+\int_0^{T_*}\iint\mathbf{d}^{\lambda p}|K\partial f^{m-1}||\partial f^m|^{p-1}\\
&\qquad+\int_0^{T_*}\iint\mathbf{d}^{\lambda p}\{|\Gamma(f^{m-1},\partial f^{m-1})|+|\Gamma(\partial f^{m-1},f^{m-1})|\}|\partial f^m|^{p-1}\Big\}.
\end{aligned}$$

Applying (87) and (88) to the boundary estimates for $m-1$, then putting together the estimates (90), (87) and (88) we deduce from (84)

$$\begin{aligned}
&\sup_{0\le t\le T_*}||\mathbf{d}^\lambda\partial f^{m+1}(t)||_p^p+\int_0^{T_*}|\mathbf{d}^\lambda\partial f^{m+1}|_{\gamma_+,p}^p+\int_0^{T_*}||\nu_{l,\lambda}^{1/p}\mathbf{d}^\lambda\partial f^{m+1}||_p^p\\
\le\;&C_{T_*,\Omega}\{||\mathbf{d}^\lambda\partial f_0||_p^p+||e^{\zeta|v|^2}f_0||_\infty^p\}+\{\varepsilon+\delta+||e^{\zeta|v|^2}f_0||_\infty^p+T_*e^{C_{l,\lambda,p}(T^*)^2}\}\\
&\times\max_{i=m,m-1}\Big\{\sup_{0\le t\le T_*}||\mathbf{d}^\lambda\partial f^i(t)||_p^p+\int_0^{T_*}|\mathbf{d}^\lambda\partial f^i|_{\gamma_+,p}^p+\int_0^{T_*}||\nu_{l,\lambda}^{1/p}\mathbf{d}^\lambda\partial f^i||_p^p\Big\}.
\end{aligned}$$

Recall $C_{l,\lambda,p}$ from (85). Choose $T_*\ll1,||e^{\zeta|v|^2}f_0||_\infty\ll1$, and $\varepsilon\ll1,\ \delta\ll1$ and hence

$$\begin{aligned}
&\sup_{0\le t\le T_*}||\mathbf{d}^\lambda\partial f^{m+1}(t)||_p^p+\int_0^{T_*}|\mathbf{d}^\lambda\partial f^{m+1}|_{\gamma_+,p}^p+\int_0^{T_*}||\nu_{l,\lambda}^{1/p}\mathbf{d}^\lambda\partial f^{m+1}||_p^p\\
\le\;&C_{T_*,\Omega}\{||\mathbf{d}^\lambda\partial f_0||_p^p+||e^{\zeta|v|^2}f_0||_\infty^p\}+\frac{1}{8}\max_{i=m,m-1}\Big\{\sup_{0\le t\le T_*}||\mathbf{d}^\lambda\partial f^i(t)||_p^p+\int_0^{T_*}|\mathbf{d}^\lambda\partial f^i|_{\gamma_+,p}^p+\int_0^{T_*}||\nu_{l,\lambda}^{1/p}\mathbf{d}^\lambda\partial f^i||_p^p\Big\}.
\end{aligned}$$

Set $k = 2$ and

$$
\begin{aligned}
a_i &= \sup_{0\le t\le T_*} ||\mathbf{d}^\lambda \partial f^{m+1}(t)||_p^p + \int_0^{T_*} |\mathbf{d}^\lambda \partial f^{m+1}|_{\gamma_+,p}^p + \int_0^{T_*} ||\nu_{l,\lambda}^{1/p} \mathbf{d}^\lambda \partial f^{m+1}||_p^p, \\
D &= C_{T_*,\Omega}\{||\mathbf{d}^\lambda \partial f_0||_p^p + ||e^{\zeta|v|^2} f_0||_\infty^p\}.
\end{aligned}
$$

Apply (76) to complete the proof. □

6 Weighted $\mathbf{C}^1$ Estimate

We start with the same iterative sequences (80) with $\lambda = 1$. For $(x, v) \in \gamma$, note that $\mathbf{d}(t, x, v) \backsim e^{-\frac{l\langle v\rangle}{2}t}|n(x)\cdot v|$. From (82) with $\lambda = 1$, we have, for $(x, v) \in \gamma_-$,

$$
\begin{aligned}
\mathbf{d}|\partial f^{m+1}(t,x,v)| \lesssim\ & \langle v\rangle c_\mu \sqrt{\mu(v)} \int_{n(x)\cdot u>0} \mathbf{d}|\partial f^m(t,x,u)| e^{\frac{l\langle u\rangle}{2}t} \langle u\rangle \sqrt{\mu(u)} \mathrm{d}u \\
&+ e^{-\frac{\zeta}{4}|v|^2} ||e^{\zeta|v|^2} f_0||_\infty (1 + ||e^{\zeta|v|^2} f_0||_\infty).
\end{aligned} \tag{90}
$$

We use the stochastic cycles in [7, 2] : For (t, x, v) with $(x, v) \notin \gamma_0$ and let $(t_0, x_0, v_0) = (t, x, v)$. For $v_k \cdot n(x_{k+1}) > 0$ we define the $(k + 1)-$component of the back-time cycle as

$$
(t_{k+1}, x_{k+1}, v_{k+1}) = (t_k - t_{\mathbf{b}}(x_k, v_k), x_{\mathbf{b}}(x_k, v_k), v_{k+1}). \tag{91}
$$

Lemma 10. *If $t_1 < 0$ then*

$$
|\mathbf{d}\partial f^{m+1}(t,x,v)| \lesssim\ ||\mathbf{d}\partial f_0||_\infty + \int_0^t |\mathcal{N}^m(s, x-(t-s)v, v)| ds. \tag{92}
$$

If $t_1 > 0$ then

$$
\begin{aligned}
&|\mathbf{d}\partial f^{m+1}(t,x,v)| \\
\lesssim\ & \int_{t_1}^t |\mathcal{N}^m(s, x-(t-s)v, v)| \mathrm{d}s + e^{-\frac{\zeta}{4}|v|^2} ||e^{\zeta|v|^2} f_0||_\infty (1 + ||e^{\zeta|v|^2} f_0||_\infty) \\
&+ \frac{1}{w(v)} \int_{\prod_{j=1}^{k-1}\mathcal{V}_j} \sum_{i=1}^{k-1} 1_{\{t_{i+1}<0<t_i\}}\ |\mathbf{d}\partial f^{m+1-i}(0, x_i - t_i v_i, v_i)|\ \mathrm{d}\Sigma_i^{k-1} \\
&+ \frac{1}{w(v)} \int_{\prod_{j=1}^{k-1}\mathcal{V}_j} \sum_{i=1}^{k-1} 1_{\{t_{i+1}<0<t_i\}} \int_0^{t_i} |\mathcal{N}^{m-i}(s, x_i - (t_i - s)v_i, v_i)|\ \mathrm{d}s\ \mathrm{d}\Sigma_i^{k-1} \\
&+ \frac{1}{w(v)} \int_{\prod_{j=1}^{k-1}\mathcal{V}_j} \sum_{i=1}^{k-1} 1_{\{t_{i+1}<0\}} \int_{t_{i+1}}^{t_i} |\mathcal{N}^{m-i}(s, x_i - (t_i - s)v_i, v_i)|\ \mathrm{d}s\ \mathrm{d}\Sigma_i^{k-1} \\
&+ \frac{1}{w(v)} \int_{\prod_{j=1}^{k-1}\mathcal{V}_j} \sum_{i=2}^{k-1} 1_{\{t_{i-1}<0\}} e^{-\frac{\zeta}{4}|v_{i-1}|^2} ||e^{\zeta|v|^2} f_0||_\infty (1 + ||e^{\zeta|v|^2} f_0||_\infty)\ \mathrm{d}\Sigma_{i-1}^{k-1} \\
&+ \frac{1}{w(v)} \int_{\prod_{j=1}^{k-1}\mathcal{V}_j} \mathbf{1}_{\{t_k>0\}} |\mathbf{d}\partial f^{m+1-k}(t_k, x_k, v_{k-1})|\ \mathrm{d}\Sigma_{k-1}^{k-1},
\end{aligned} \tag{93}
$$

where $\mathcal{V}_j = \{v_j \in \mathbb{R}^3 : n(x_j)\cdot v_j > 0\}$ and

$$
w(v) = \frac{c_\mu}{\langle v\rangle \sqrt{\mu(v)}}, \tag{94}
$$

and

$$
\mathrm{d}\Sigma_i^{k-1} = \Big\{\Pi_{j=i+1}^{k-1} \mu(v_j) c_\mu |n(x_j)\cdot v_j| \mathrm{d}v_j\Big\}\Big\{w(v_i) e^{\frac{l\langle v_i\rangle}{2}t_i} \langle v_i\rangle^2 c_\mu \mu(v_i) \mathrm{d}v_i\Big\}\Big\{\Pi_{j=1}^{i-1} e^{\frac{l\langle v_j\rangle}{2}t_j} \langle v_j\rangle^2 c_\mu \mu(v_j) \mathrm{d}v_j\Big\}.
$$

Remark that $\mathrm{d}\Sigma_i^{k-1}$ is not a probability measure!

Proof. For $t_1 < 0$ we use (80) with $\lambda = 1$ to obtain

$$\mathbf{d}\partial f^{m+1}(t,x,v) = e^{-\nu_{l,1}(v)t}\mathbf{d}\partial f_0(x-tv,v) + \int_0^t e^{-\nu_{l,1}(v)(t-s)}\mathcal{N}^m(s,x-(t-s)v,v)\mathrm{d}s.$$

Consider the case of $t_1 > 0$. We prove by the induction on k, the number of iterations. First for $k=1$, along the characteristics, for $t_1 > 0$, we have

$$\mathbf{d}\partial f^{m+1}(t,x,v) = e^{-\nu_{l,1}(t-t_1)}\mathbf{d}\partial f^{m+1}(t_1,x_1,v) + \int_{t_1}^t e^{-\nu_{l,1}(t-s)}\mathcal{N}^m(s,x-(t-s)v,v)\mathrm{d}s.$$

Now we apply (90) to the first term above to further estimate

$$\begin{aligned}
\mathbf{d}|\partial f^{m+1}(t,x,v)| \lesssim\ & e^{-\nu_{l,1}(v)(t-t_1)}e^{-\frac{\zeta}{4}|v|^2}||e^{\zeta|v|^2}f_0||_\infty(1+||e^{\zeta|v|^2}f_0||_\infty) \\
& + e^{-\nu_{l,1}(v)(t-t_1)}\langle v\rangle c_\mu\sqrt{\mu(v)}\int_{n\cdot v_1>0}\mathbf{d}|\partial f^m(t_1,x_1,v_1)|e^{\frac{l\langle v_1\rangle}{2}t_1}\langle v_1\rangle\sqrt{\mu(v_1)}\mathrm{d}v_1 \\
& + \int_{t_1}^t e^{-\nu_{l,1}(v)(t-s)}|\mathcal{N}^m(s,x-(t-s)v,v)|\mathrm{d}s \\
& \lesssim e^{-\frac{\zeta}{4}|v|^2}||e^{\zeta|v|^2}f_0||_\infty(1+||e^{\zeta|v|^2}f_0||_\infty) \\
& + \frac{c_\mu}{w(v)}\int_{\mathcal{V}_1}\mathbf{d}|\partial f^m(t_1,x_1,v_1)|e^{\frac{l\langle v_1\rangle}{2}t_1}w(v_1)\langle v_1\rangle^2\mu(v_1)\mathrm{d}v_1 + \int_{t_1}^t|\mathcal{N}^m(s,x-(t-s)v,v)|\mathrm{d}s,
\end{aligned} \tag{95}$$

where $w(v)$ is in (94). Now we continue to express $\partial f^m(t_1,x_1,v_1)$ via backward trajectory to get

$$\begin{aligned}
\mathbf{d}|\partial f^m(t_1,x_1,v_1)| \le\ & \mathbf{1}_{\{t_2<0<t_1\}}\Big\{\mathbf{d}|\partial f^m(0,x_1-t_1v_1,v_1)| + \int_0^{t_1}|\mathcal{N}^{m-1}(s,x_1-(t_1-s)v_1,v_1)|\mathrm{d}s\Big\} \\
& + \mathbf{1}_{\{t_2>0\}}\Big\{\mathbf{d}|\partial f^m(t_2,x_2,v_1)| + \int_{t_2}^{t_1}|\mathcal{N}^{m-1}(s,x_1-(t_1-s)v_1,v_1)|\mathrm{d}s\Big\}.
\end{aligned}$$

Therefore we conclude from (95) that

$$\begin{aligned}
& \mathbf{d}|\partial f^{m+1}(t,x,v)| \\
& \lesssim \int_{t_1}^t|\mathcal{N}^m(s,x-(t-s)v,v)|\mathrm{d}s + e^{-\frac{\zeta}{4}|v|^2}||e^{\zeta|v|^2}f^m||_\infty(1+||e^{\zeta|v|^2}f^m||_\infty) \\
& + \frac{1}{w(v)}\int_{\mathcal{V}_1}\mathbf{1}_{\{t_2<0<t_1\}}\mathbf{d}|\partial f(0,x_1-t_1v_1,v_1)|e^{\frac{l\langle v_1\rangle}{2}t_1}w(v_1)\langle v_1\rangle^2c_\mu\mu(v_1)\mathrm{d}v_1 \\
& + \frac{1}{w(v)}\int_{\mathcal{V}_1}\mathbf{1}_{\{t_2<0<t_1\}}\int_0^{t_1}|\mathcal{N}^{m-1}(s,x_1-(t_1-s)v_1,v_1)|\mathrm{d}se^{\frac{l\langle v_1\rangle}{2}t_1}w(v_1)\langle v_1\rangle^2c_\mu\mu(v_1)\mathrm{d}v_1 \\
& + \frac{1}{w(v)}\int_{\mathcal{V}_1}\mathbf{1}_{\{t_2>0\}}\int_{t_2}^{t_1}|\mathcal{N}^{m-1}(s,x_1-(t_1-s)v_1,v_1)|\mathrm{d}se^{\frac{l\langle v_1\rangle}{2}t_1}w(v_1)\langle v_1\rangle^2c_\mu\mu(v_1)\mathrm{d}v_1 \\
& + \frac{1}{w(v)}\int_{\mathcal{V}_1}\mathbf{1}_{\{t_2>0\}}\mathbf{d}|\partial f^m(t_2,x_2,v_1)|e^{\frac{l\langle v_1\rangle}{2}t_1}w(v_1)\langle v_1\rangle^2c_\mu\mu(v_1)\mathrm{d}v_1,
\end{aligned}$$

and it equals (93) for $k=2$.

Assume (93) is valid for $k\in\mathbb{N}$. We use (90) and express the last term of (93) as

$$\begin{aligned}
& \mathbf{1}_{\{t_k>0\}}\mathbf{d}|\partial f^{m+1-k}(t_k,x_k,v_{k-1})| \\
& \lesssim \langle v_{k-1}\rangle c_\mu\sqrt{\mu(v_{k-1})}\int_{\mathcal{V}_k}\mathbf{1}_{\{t_k>0\}}\mathbf{d}|\partial f^{m+1-(k+1)}(t_k,x_k,v_k)|e^{\frac{l\langle v_k\rangle}{2}t_k}\langle v_k\rangle\sqrt{\mu(v_k)}\mathrm{d}v_k \\
& + e^{-\frac{\zeta}{4}|v_{k-1}|^2}||e^{\zeta|v|^2}f_0||_\infty(1+||e^{\zeta|v|^2}f_0||_\infty).
\end{aligned} \tag{96}$$

Then we decompose $\mathbf{1}_{\{t_k>0\}}\mathbf{d}|\partial f^{m+1-(k+1)}(t_k,x_k,v_k)| = \mathbf{1}_{\{t_{k+1}<0<t_k\}} + \mathbf{1}_{\{t_{k+1}>0\}}$, where the first part hits the initial plane as

$$\mathbf{1}_{\{t_{k+1}<0<t_k\}}\mathbf{d}|\partial f^{m+1-(k+1)}(t_k,x_k,v_k)| \lesssim \mathbf{d}|\partial f_0(x_k-t_kv_k,v_k)|+\int_0^{t_k}|\mathcal{N}^{m+1-(k+2)}(s,x_k-(t_k-s)v_k,v_k)|\mathrm{d}s, \tag{97}$$

and the second part hits at the boundary as

$$\mathbf{1}_{\{t_{k+1}>0\}}\mathbf{d}|\partial f^{m+1-(k+1)}(t_k,x_k,v_k)| \lesssim \mathbf{d}|\partial f^{m+1-(k+1)}(t_{k+1},x_{k+1},v_k)|+\int_{t_{k+1}}^{t_k}|\mathcal{N}^{m+1-(k+2)}(s,x_k-(t_k-s)v_k,v_k)|\mathrm{d}s. \tag{98}$$

To summarize, from (96) upon integrating over $\prod_{j=1}^{k-1}\mathcal{V}_j$, we obtain a bound for the last term of (93) as

$$\begin{aligned}
&\frac{1}{w(v)}\int_{\prod_{j=1}^{k-1}\mathcal{V}_j}\mathbf{1}_{\{t_k>0\}}|\mathbf{d}\partial f^{m+1-k}(t_k,x_k,v_{k-1})|\mathrm{d}\Sigma_{k-1}^{k-1}\\
&\lesssim \frac{1}{w(v)}\int_{\prod_{j=1}^{k-1}\mathcal{V}_j}\mathbf{1}_{\{t_k>0\}}e^{-\frac{\zeta}{4}|v_{k-1}|^2}||e^{\zeta|v|^2}f_0||_\infty(1+||e^{\zeta|v|^2}f_0||_\infty)\mathrm{d}\Sigma_{k-1}^{k-1}\\
&\quad+\frac{1}{w(v)}\int_{\prod_{j=1}^{k}\mathcal{V}_j}\mathbf{1}_{\{t_k>0\}}\mathbf{d}|\partial f^{m+1-(k+1)}(t_k,x_k,v_k)|\mathrm{d}\Sigma_k^k,
\end{aligned}$$

where by (97) and (98), the last term is bounded by

$$\begin{aligned}
&\frac{1}{w(v)}\int_{\prod_{j=1}^{k}\mathcal{V}_j}\langle v_{k-1}\rangle c_\mu\sqrt{\mu(v_{k-1})}\sqrt{\mu(v_k)}\langle v_k\rangle e^{\frac{l\langle v_k\rangle}{2}t_k}\mathrm{d}v_k\\
&\times\prod_{j=1}^{k-2}\Big\{e^{\frac{l\langle v_j\rangle}{2}t_j}\langle v_j\rangle^2c_\mu\mu(v_j)\mathrm{d}v_j\Big\}\Big\{w(v_{k-1})e^{\frac{l\langle v_{k-1}\rangle}{2}t_{k-1}}\langle v_{k-1}\rangle^2\mu(v_{k-1})\mathrm{d}v_{k-1}\Big\}\\
&\times\Big\{\mathbf{1}_{\{t_{k+1}<0<t_k\}}\Big[\mathbf{d}|\partial f(0,x_k-t_kv_k,v_k)|+\int_0^{t_k}|\mathcal{N}^{m-k-2}(s,x_k-(t_k-s)v_k,v_k)|\mathrm{d}s\Big]\\
&\quad+\mathbf{1}_{\{t_{k+1}>0\}}\Big[\mathbf{d}|\partial f^{m-k-1}(t_{k+1},x_{k+1},v_k)|+\int_{t_{k+1}}^{t_k}|\mathcal{N}^{m-k-2}(s,x_k-(t_k-s)v_k,v_k)|\mathrm{d}s\Big]\Big\}.
\end{aligned}$$

Now we use (94) to conclude Lemma 10. ☐

Lemma 11. *There exists $k_0(\varepsilon)>0$ such that for $k\geq k_0$ and for all $(t,x,v)\in[0,1]\times\overline{\Omega}\times\mathbb{R}^3$, we have*

$$\int_{\prod_{j=1}^{k-1}\mathcal{V}_j}\mathbf{1}_{\{t_k(t,x,v,v_1,\cdots,v_{k-1})>0\}}\mathrm{d}\Sigma_{k-1}^{k-1}\quad\lesssim_\Omega\quad\left(\frac{1}{2}\right)^{-k/5}.$$

Proof. The proof is based on [7]. We note that, for some fixed constant $C_0>0$,

$$\begin{aligned}
d\Sigma_{k-1}^{k-1} &\leq w(v_{k-1})e^{\frac{l\langle v_{k-1}\rangle}{2}t_{k-1}}\langle v_{k-1}\rangle^2c_\mu\mu(v_{k-1})\Pi_{j=1}^{k-2}e^{\frac{l\langle v_j\rangle}{2}t_j}\langle v_j\rangle^2c_\mu\mu(v_j)\mathrm{d}v_1\ldots\mathrm{d}v_{k-1}\\
&\leq \Pi_{j=1}^{k-1}\{C'e^{C't^2}\mu(v_j)^{\frac14}\}\ \mathrm{d}v_1\ldots\mathrm{d}v_{k-1}\leq\{C_0\}^k\Pi_{j=1}^{k-1}\mu(v_j)^{\frac14}\mathrm{d}v_j.
\end{aligned}$$

Choose $\delta=\delta(C_0)>0$ small and define

$$\mathcal{V}_j^\delta\equiv\{v_j\in\mathcal{V}_j: v_j\cdot n(x_j)\geq\delta,\ |v_j|\leq\delta^{-1}\},$$

where we have $\int_{\mathcal{V}_j\backslash\mathcal{V}_j^\delta}C_0\mu(v_j)^{\frac14}\lesssim\delta$ for some $C_0>0$. Choose sufficiently small $\delta>0$.

On the other hand if $v_j\in\mathcal{V}_j^\delta$ then by Lemma 6 of [7], $(t_j-t_{j+1})\ \geq\ \delta^3/C_\Omega$. Therefore if $t_k\geq0$ then there can be at most $\{[\frac{C_\Omega}{\delta^3}]+1\}$ numbers of $v_m\in\mathcal{V}_m^\delta$ for $1\leq m\leq k-1$. Equivalently there are at least $k-2-[\frac{C_\Omega}{\delta^3}]$ numbers of $v_{m_i}\in\mathcal{V}_{m_i}\backslash\mathcal{V}_{m_i}^\delta$. Hence from $\{C_0\}^k=\{C_0\}^m\times\{C_0\}^{k-1-m}$, we have

$$
\begin{aligned}
&\int_{\prod_{j=1}^{k-1}\mathcal{V}_j} 1_{\{t_k(t,x,v,v_1,\cdots,v_{k-1})>0\}} d\Sigma_{k-1}^{k-1} \\
\le\ & \sum_{m=1}^{\left[\frac{C_\Omega}{\delta^3}\right]+1} \int_{\left\{\substack{\text{there are exactly } m \text{ of } v_{m_i}\in\mathcal{V}_{m_i}^\delta \\ \text{and } k-1-m \text{ of } v_{m_i}\in\mathcal{V}_{m_i}\backslash\mathcal{V}_{m_i}^\delta}\right\}} \prod_{j=1}^{k-1} C_0\mu(v_j)^{1/4}\mathrm{d}v_j \\
\le\ & \sum_{m=1}^{\left[\frac{C_\Omega}{\delta^3}\right]+1} \begin{pmatrix} k-1 \\ m \end{pmatrix} \left\{\int_{\mathcal{V}} C_0\mu(v)^{1/4}\mathrm{d}v\right\}^m \left\{\int_{\mathcal{V}\backslash\mathcal{V}^\delta} C_0\mu(v)^{1/4}\mathrm{d}v\right\}^{k-1-m} \\
\le\ & \left(\left[\frac{C_\Omega}{\delta^3}\right]+1\right)\{k-1\}^{\left[\frac{C_\Omega}{\delta^3}\right]+1}\{\delta\}^{k-2-\left[\frac{C_\Omega}{\delta^3}\right]}\left\{\int_{\mathcal{V}} C_0\mu(v)^{1/4}dv\right\}^{\left[\frac{C_\Omega}{\delta^3}\right]+1} \lesssim \frac{k}{N}\{Ck\}^{\frac{k}{N}}\left(\frac{k}{N}\right)^{-\frac{Nk}{10}} \\
\le\ & \{CN\}^{\frac{k}{N}}\left(\frac{k}{N}\right)^{\frac{k}{N}}\left(\frac{k}{N}\right)^{-\frac{k}{N}\frac{N^2}{20}} \le \left(\frac{k}{N}\right)^{\frac{k}{N}\left(-\frac{N^2}{20}+3\right)} \le \left(\frac{1}{k/N}\right)^{\frac{-\frac{N^2}{20}+3}{N}k} \le \left(\frac{1}{2}\right)^{-k},
\end{aligned}
$$

where we have chosen $k = N\times\left(\left[\frac{C_\Omega}{\delta^3}\right]+1\right)$ and $N = \left(\left[\frac{C_\Omega}{\delta^3}\right]+1\right) \gg C > 1$. □

We now need a key lemma to overcome the singularity of $\frac{1}{\mathbf{d}}$ along the integration over a characteristic line.

Lemma 12. *Assume Ω is strictly convex (2). Recall $\mathbf{k}_\zeta$ in (26) and $\mathbf{k} = \mathbf{k}_\zeta$, if $\zeta = \frac{1}{4}$. For $\zeta \in (0, \frac{1}{4}]$ there exists small $\delta_1 > 0$ such that*

$$
\begin{aligned}
&\mathbf{1}_{\{t_1(t,x,v)\ge 0\}}\int_{t_1}^t\int_{\mathbb{R}^3}\frac{\mathbf{d}(s,x-(t-s)v,v)}{\mathbf{d}(s,x-(t-s)v,u)}|\mathbf{k}_\zeta(v,u)|\ \mathrm{d}u\mathrm{d}s, \\
&\mathbf{1}_{\{t_1(t,x,v)< 0\}}\int_{0}^t\int_{\mathbb{R}^3}\frac{\mathbf{d}(s,x-(t-s)v,v)}{\mathbf{d}(s,x-(t-s)v,u)}|\mathbf{k}_\zeta(v,u)|\ \mathrm{d}u\mathrm{d}s, \\
&\mathbf{1}_{\{t_m\ge 0, t_{m+1}\ge 0\}}\int_{t_{m+1}}^{t_m}\int_{\mathbb{R}^3}\frac{\mathbf{d}(s,x_m-(t_m-s)v_m,v_m)}{\mathbf{d}(s,x_m-(t_m-s)v_m,u)}|\mathbf{k}_\zeta(v_m,u)|\ \mathrm{d}u\mathrm{d}s, \\
&\mathbf{1}_{\{t_m\ge 0, t_{m+1}< 0\}}\int_{0}^{t_m}\int_{\mathbb{R}^3}\frac{\mathbf{d}(s,x_m-(t_m-s)v_m,v_m)}{\mathbf{d}(s,x_m-(t_m-s)v_m,u)}|\mathbf{k}_\zeta(v_m,u)|\ \mathrm{d}u\mathrm{d}s,
\end{aligned}
$$

are bounded by

$$
C_\Omega\{te^{Ct^2}|\delta_1|^{-\frac{1}{2}} + e^{C|\delta_1|^2}|\delta_1|^{\frac{1}{2}}\}.
$$

Proof. Notice that we use the estimate (26). Without loss of generality, we only consider the third estimate, by the definition of $\mathbf{d}$,

$$
\begin{aligned}
\mathbf{1}_{\{t_m\ge 0, t_{m+1}\ge 0\}}\int_{t_{m+1}}^{t_m}\cdots \le \int_{t_{m+1}}^{t_m}\mathrm{d}s\int_{\mathbb{R}^3}\mathrm{d}u\ &\left\{|v_m-u|^\gamma + \frac{1}{|v_m-u|^{2-\gamma}}\right\}e^{-C_\zeta|v_m-u|^2} \\
&\times e^{\frac{l|v_m-u|}{2}s}\frac{\left\{|\nabla\xi\cdot v_m|^2 - 2\big(v_m\cdot\nabla^2\xi\cdot v_m\big)\ \xi\right\}^{1/2}}{\left\{|\nabla\xi\cdot u|^2 - 2\big(u\cdot\nabla^2\xi\cdot u\big)\ \xi\right\}^{1/2}},
\end{aligned}
\tag{99}
$$

where $\xi = \xi(x_m - (t_m-s)v_m)$ and we have used $-\langle v_m\rangle s + \langle u\rangle s \le |v_m-u|s$ in the exponent of $e^{-\frac{l\langle v_m\rangle}{2}s}e^{\frac{l\langle u\rangle}{2}s} = e^{-\frac{l}{2}(\langle v_m\rangle - \langle u\rangle)s} \le e^{\frac{l}{2}|v_m-u|s}$. In order to bound $e^{\frac{l|v_m-u|}{2}s}$ we split the exponent

$$
\begin{aligned}
\frac{l|v_m-u|}{2}s &= \frac{\sigma l^2}{4}|v_m-u|^2 + \frac{s^2}{4\sigma} - \frac{1}{4\sigma}\{s^2 - 2\sigma l|v_m-u|s + \sigma^2l^2|v_m-u|^2\} \\
&= \frac{\sigma l^2}{4}|v_m-u|^2 + \frac{s^2}{4\sigma} - \frac{1}{4\sigma}[s-\sigma l|v_m-u|]^2,
\end{aligned}
$$

to have, for sufficiently small $\sigma > 0$,

$$
e^{\frac{l|v_m-u|}{2}s}e^{-C_{\zeta,l}|v_m-u|^2} \lesssim e^{Cs^2}e^{-C_{\zeta,l}|v_m-u|^2}.
$$

Therefore

$$(99) \quad \lesssim \quad \int_{t_{m+1}}^{t_m} \int_{\mathbb{R}^3} e^{Cs^2} \big\{ 1 + |v_m - u|^{\gamma-2} \big\} e^{-C_{\zeta,l}|v_m-u|^2} \frac{\big\{ |\nabla\xi \cdot v_m|^2 - 2\big(v_m \cdot \nabla^2\xi \cdot v_m\big)\ \xi \big\}^{1/2}}{\big\{ |\nabla\xi \cdot u|^2 - 2\big(u \cdot \nabla^2\xi \cdot u\big)\ \xi \big\}^{1/2}} \mathrm{d}u\mathrm{d}s.$$

We now separate three different cases.

CASE 1 : $\xi(x_m - (t_m - s)v_m) < -\delta_1$ (interior). By the convexity (2), $u \cdot \nabla^2\xi \cdot u \gtrsim |u|^2$,

$$|\nabla\xi \cdot u|^2 - 2(u \cdot \nabla^2\xi \cdot u)\xi \geq -2(u \cdot \nabla^2\xi(x_m - (t_m - s)v_m) \cdot u)\xi(x_m - (t_m - s)v_m) \geq C_\Omega\ \delta_1\ |u|^2.$$

Therefore, since $\big\{ |\nabla\xi \cdot v_m|^2 - 2\big(v_m \cdot \nabla^2\xi \cdot v_m\big)\ \xi \big\}^{1/2} \lesssim |v_m|$, (99) is bounded in this case by

$$\begin{aligned} &C_\Omega \int_0^t \int_{\mathbb{R}^3} e^{Cs^2} \big\{ 1 + |v_m - u|^{\gamma-2} \big\} e^{-C_{\zeta,l}|v_m-u|^2} \frac{|v_m|}{|u|\delta_1^{1/2}} \mathbf{1}_{\{\xi < -\delta_1\}} \mathrm{d}u\mathrm{d}s \\ &\lesssim_\Omega \delta_1^{-1/2} t e^{Ct^2} |v_m| \int_{\mathbb{R}^3} \frac{1}{|\eta + v_m||\eta|^{2-\gamma}} e^{-C_{\zeta,l}|\eta|^2} \mathrm{d}\eta, \end{aligned}$$

where we have used a change of variable $\eta = u - v_m$.

Now we claim that, for $0 \leq \gamma \leq 1$

$$\int_{\mathbb{R}^3} \frac{|v_m|}{|\eta + v_m||\eta|^{2-\gamma}} e^{-C_{\zeta,l}|\eta|^2} \mathrm{d}\eta\ \leq C_{\zeta,l,\gamma}. \tag{100}$$

First consider the case of $|v_m| \leq 1$. We define $\eta_\parallel = \eta \cdot \frac{v_m}{|v_m|}$ and $\eta_\perp = \eta - \eta_\parallel \frac{v_m}{|v_m|}$. Then (100) is bounded by

$$\int_{\mathbb{R}} \mathrm{d}\eta_\parallel \int_{\mathbb{R}^2} \mathrm{d}\eta_\perp\ \frac{|v_m| e^{-C_{\zeta,l}|\eta_\parallel|^2} e^{-C_{\zeta,l}|\eta_\perp|^2}}{[(\eta_\parallel + |v_m|)^2 + |\eta_\perp|^2]^{1/2}\ [\eta_\parallel^2 + |\eta_\perp|^2]^{1-\gamma/2}}.$$

If $\big|\eta_\parallel + |v_m|\big| \geq \frac{|v_m|}{2}$, then (100) is bounded by

$$4 \int_{\mathbb{R}} \mathrm{d}\eta_\parallel \int_{\mathbb{R}^2} \mathrm{d}\eta_\perp \frac{e^{-C_{\zeta,l}|\eta_\parallel|^2} e^{-C_{\zeta,l}|\eta_\perp|^2}}{[\eta_\parallel^2 + |\eta_\perp|^2]^{1-\gamma/2}} \lesssim_{\zeta,l,\gamma} 1.$$

If $\big|\eta_\parallel + |v_m|\big| \leq \frac{|v_m|}{2}$, then $|\eta_\parallel| \geq \frac{|v_m|}{2}$ so that

$$[(\eta_\parallel)^2 + |\eta_\perp|^2]^{1-\gamma/2} \geq |\eta_\parallel|^{2-\gamma} \geq \frac{|v_m|^{2-\gamma}}{4}.$$

We define $\tilde{\eta}_\parallel = |v_m|^{-1}\eta_\parallel \in \mathbb{R}$ and $\mathrm{d}\eta_\parallel = |v_m|\mathrm{d}\tilde{\eta}_\parallel$. Remark that $|\tilde{\eta}_\parallel| \leq \frac{3}{2}$ since $|\eta_\parallel| \leq \frac{3}{2}|v_m|$. Using $[(\eta_\parallel + |v_m|)^2 + |\eta_\perp|^2]^{1/2} \geq |\eta_\perp|$, we bound (100) by

$$4 \int_{|\tilde{\eta}_\parallel| \leq \frac{3}{2}} \mathrm{d}\tilde{\eta}_\parallel |v_m| \int_{\mathbb{R}^2} \mathrm{d}\eta_\perp \frac{|v_m| e^{-C_{\zeta,l}|\eta_\perp|^2}}{|\eta_\perp||v_m|^{2-\gamma}} \lesssim_{\zeta,l,\gamma} |v_m|^\gamma \lesssim_{\zeta,l,\gamma} 1.$$

On the other hand, if $|v_m| \geq 1$, we divide the integration as $\int_{|\eta+v_m| \geq \frac{|v_m|}{2}} \mathrm{d}\eta + \int_{|\eta+v_m| \leq \frac{|v_m|}{2}} \mathrm{d}\eta$. The first term is easily bounded by

$$\int_{|\eta+v_m| \geq \frac{|v_m|}{2}} \mathrm{d}\eta\ \lesssim\ \int_{\mathbb{R}^3} \frac{1}{|\eta|^{2-\gamma}} e^{-C_{\zeta,l}|\eta|^2} \mathrm{d}\eta\ \lesssim_{\zeta,l,\gamma} 1.$$

In the case of $|\eta + v_m| \leq \frac{|v_m|}{2}$ then $|\eta| \geq \frac{|v_m|}{2}$ and then

$$\begin{aligned} \int_{|\eta+v_m| \leq \frac{|v_m|}{2}} \mathrm{d}\eta &\lesssim \int_{|\eta+v_m| \leq \frac{|v_m|}{2}} \frac{|v_m|}{|\eta + v_m||\eta|^{2-\gamma}} e^{-C_{\zeta,l}|\eta|^2} d\eta \lesssim \int_{|\eta+v_m| \leq \frac{|v_m|}{2}} \frac{e^{-C_{\zeta,l}|\eta|^2}}{|v_m + \eta||v_m|^{1-\gamma}} \mathrm{d}\eta \\ &\lesssim \frac{1}{|v_m|^{1-\gamma}} e^{-\frac{C_{\zeta,l}}{4}|v_m|^2} \int_{|\tilde{\eta}| \leq 2|v_m|} \frac{1}{|\tilde{\eta}|} \mathrm{d}\tilde{\eta} \lesssim |v_m|^3 e^{-\frac{C_{\zeta,l}}{4}|v_m|^2} \lesssim_{\zeta,l,\gamma} 1. \end{aligned}$$

This proves the claim (100). We conclude the first case such that (99) is

$$C_\Omega \delta_1^{-1/2} t e^{Ct^2}.$$

Now we further separate the situation near the boundary region of $-\delta_1 \leq \xi \leq 0$ into two cases.

CASE 2 : $-\delta_1 \leq \xi(x_m - (t_m - s)v_m) \leq 0$ and $|v_m \cdot \nabla \xi(x_m - (t_m - s)v_m)| \leq |v_m| \sqrt{-\xi(x_m - (t_m - s)v_m)}$.
In this case, for $\xi = \xi(x_m - (t_m - s)v_m)$, from our assumption,

$$|\nabla \xi \cdot v_m|^2 - 2(v_m \cdot \nabla^2 \xi \cdot v_m)\xi \ \lesssim_\Omega \ |\nabla \xi \cdot v_m|^2 \ \lesssim \ -|v_m|^2 \xi(x_m - (t_m - s)v_m).$$

By the convexity of ξ we also have

$$|\nabla \xi \cdot u|^2 - 2(u \cdot \nabla^2 \xi \cdot u)\xi \ \gtrsim_\Omega \ -|u|^2 \xi(x_m - (t_m - s)v_m),$$

so that (99) is bounded by

$$\begin{aligned}
&\int_{t_{m+1}}^{t_m} \mathrm{d}s \int_{\mathbb{R}^3} \mathrm{d}u \mathbf{1}_{\{-\delta_1 \leq \xi \leq 0, |v_m \cdot \nabla \xi| \leq |v_m \sqrt{\xi}|\}} \ e^{Cs^2} \big\{1 + |v_m - u|^{\gamma - 2}\big\} e^{-C_{\varsigma,l}|v_m - u|^2} \frac{|v_m| \sqrt{-\xi(x_m - (t_m - s)v_m)}}{|u| \sqrt{-\xi(x_m - (t_m - s)v_m)}} \\
\lesssim \ & te^{Ct^2} \langle v_m \rangle \int_{\mathbb{R}^3} \frac{1}{|u|} \big\{1 + |v_m - u|^{\gamma-2}\big\} e^{-C_{\varsigma,l}|v_m - u|^2} \mathrm{d}u \lesssim \ t \ e^{Ct^2},
\end{aligned}$$

where we used (100).

CASE 3 : $-\delta_1 \leq \xi(x_m - (t_m - s)v_m) \leq 0$ and $|v_m \cdot \nabla \xi(x_m - (t_m - s)v_m)| > |v_m| \sqrt{-\xi(x_m - (t_m - s)v_m)}$. For this case we shall apply a change of variables, at most twice, near the boundary. We need a geometric fact that a straight line intersects a convex domain at most two points. Precisely we claim that *for a strictly convex domain Ω there exists $\delta_0 > 0$ such that for all $0 < \delta_1 \leq \delta_0$*

$$\big\{s \in [t_{m+1}, t_m] \ : \ \xi(x_m - (t_m - s)v_m) \in [-\delta_1, \ 0\ ]\big\} \ \subset \ [t_{m+1}, t_{m+1} + \sigma_1] \ \cup \ [t_m - \sigma_2, t_m],$$

where $\sigma_1 = \sigma_1(t_m, x_m, v_m, \delta_0), \sigma_2 = \sigma_2(t_m, x_m, v_m, \delta_0) > 0$ *and*

$$\begin{aligned}
v_m \cdot \nabla \xi(x_m - (t_m - s)v_m) \quad &\geq \quad 0 \quad \mathit{for} \quad s \in [t_m - \sigma_2, t_m], \\
v_m \cdot \nabla \xi(x_m - (t_m - s)v_m) \quad &\leq \quad 0 \quad \mathit{for} \quad s \in [t_{m+1}, t_{m+1} + \sigma_1].
\end{aligned}$$

In fact, we define $Z(s) \equiv \xi(x_m - (t_m - s)v_m)$ for $t_{m+1} \leq s \leq t_m$. Note that $Z'(s) = v_m \cdot \nabla \xi$ and $Z''(s) = v_m \cdot \nabla^2 \xi \cdot v_m > 0$ from convexity (2) for $v_m \neq 0$. Hence $Z_m(s)$ is a strictly convex function and $Z(t_{m+1}) = Z(t_m) = 0$. Denote the unique minimizer t_* such that

$$Z(t_*) = \min_{t_{m+1} \leq s \leq t_m} Z(s) < 0, \tag{101}$$

such that $Z' < 0$ for $s < t_*$ and $Z' > 0$ for $s > t_*$. If $-\delta_1 \leq Z(t_*)$, then we can choose $\sigma_1 = t_* - t_{m+1}$ and $\sigma_2 = t_m - t_*$. On the other hand, if $-\delta_1 > Z(t_*)$, we simply choose σ_1 and σ as two unique numbers such that

$$Z(t_{m+1} + \sigma_1) = Z(t_m - \sigma_2) = -\delta_1,$$

where $t_{m+1} + \sigma_1 < t_* < t_m - \sigma_2$. This proves the claim.
In this case (99) is bounded by, from our assumption,

$$\int_{t_{m+1}}^{t_m} \int_{\mathbb{R}^3} e^{Cs^2} \big\{1 + |v_m - u|^{\gamma-2}\big\} e^{-C_{\varsigma,l}|v_m - v'|^2} \frac{|\nabla \xi(x_m - (t_m - s)v_m) \cdot v_m|}{|u| \sqrt{-\xi(x_m - (t_m - s)v_m)}} \mathrm{d}u \ \mathrm{d}s = \int_{t_m - \sigma_2}^{t_m} + \int_{t_{m+1}}^{t_{m+1} + \sigma_1}.$$

Apply the change of variables for $s \in [t_{m+1}, t_{m+1} + \sigma_1]$ and for $s \in [t_m - \sigma_2, t_m]$ as

$$s \ \to \ \xi \ = \ \xi(x_m - (t_m - s)v_m), \tag{102}$$

where the Jacobian is $\left|\frac{d\xi}{ds}\right| = |v_m \cdot \nabla\xi(x_m - (t_m - s)v_m)|$. Therefore

$$\int_{t_m-\sigma_2}^{t_m} + \int_{t_{m+1}}^{t_{m+1}+\sigma_1} \le 2\int_{-\delta_1}^{0} \mathrm{d}\xi \int_{\mathbb{R}^3} \mathrm{d}u \; e^{C|\delta_1|^2}\big\{1 + |v_m - u|^{\gamma-2}\big\} e^{-C_{\zeta,l}|v_m-u|^2} \frac{1}{|u||\xi|^{1/2}}$$

$$\lesssim e^{C|\delta_1|^2} \int_0^{\delta_1} |\xi|^{-\frac{1}{2}} \mathrm{d}|\xi| \int_{\mathbb{R}^3} |u|^{-1}\big\{1 + |v_m - u|^{\gamma-2}\big\} e^{-C_{\zeta,l}|v_m-u|^2} \mathrm{d}u \lesssim_{\Omega,\zeta,l,\gamma} |\delta_1|^{\frac{1}{2}} e^{C|\delta_1|^2},$$

where we used (100). □

Now we are ready to prove the weighted C^1 part of the main theorem :

Proof of weighted C^1 part in Theorem 2. First we show $W^{1,\infty}$ estimate. Recall that we use the same sequences (80) with $\lambda = 1$ used for the weighted $W^{1,p}$ estimate ($2 \le p < \infty$). We estimate along the stochastic cycles with (92) and (93). For $t_1 < 0$, the backward trajectory first hits $t = 0$. From Lemma 10 and Lemma 12, and Lemma 2 for (81), we deduce

$$\begin{aligned} &\sup_{0\le t\le T_*} ||\mathbf{1}_{\{t_1<0\}}\mathbf{d}\partial f^{m+1}(t)||_\infty \le ||\mathbf{d}\partial f_0||_\infty + ||e^{\zeta|v|^2} f_0||_\infty (1 + ||e^{\zeta|v|^2} f_0||_\infty) \\ &+ T_* \sup_{0\le t\le T_*} ||\mathbf{d}\partial f^{m+1}(t)||_\infty + \big\{T_* e^{CT_*^2}|\delta_1|^{-\frac{1}{2}} + e^{C|\delta_1|^2}|\delta_1|^{\frac{1}{2}}\big\}(1 + ||e^{\zeta|v|^2} f_0||_\infty) \sup_{0\le t\le T_*} ||\mathbf{d}\partial f^m(t)||_\infty. \end{aligned} \tag{103}$$

If $t_1(t,x,v) \ge 0$, the backward trajectory first hits the boundary, then from (93) we have the following line-by-line estimate

$$\begin{aligned} &|\mathbf{d}\partial f^{m+1}(t,x,v)| \\ &\le t \sup_{0\le s\le t} ||\mathbf{d}\partial f^{m+1}(s)||_\infty + tk \max_{1\le i\le k-1} \sup_{0\le s\le t} ||\mathbf{d}\partial f^{m+1-i}(t)||_\infty \\ &\quad + (1 + ||e^{\zeta|v|^2} f_0||_\infty)\{te^{Ct^2}|\delta_1|^{-\frac{1}{2}} + e^{C|\delta_1|^2}|\delta_1|^{\frac{1}{2}}\} \sup_{0\le s\le t} ||\mathbf{d}\partial f^m(s)||_\infty + e^{-\frac{\zeta}{4}|v|^2}||e^{\zeta|v|^2} f_0||_\infty (1 + ||e^{\zeta|v|^2} f_0||_\infty) \\ &\quad + k \max_{1\le i\le k-1} ||\mathbf{d}\partial f^{m+1-i}(0)||_\infty \sup_i \Big(\int_{\prod_{j=1}^{k-1}\mathcal{V}_j} \mathrm{d}\Sigma_i^{k-1}\Big) \\ &\quad + k(1 \max_j ||e^{\zeta|v|^2} f^j||_\infty)\{te^{Ct^2}|\delta_1|^{-\frac{1}{2}} + e^{C|\delta_1|^2}|\delta_1|^{\frac{1}{2}}\} \times \max_{1\le i\le k-1} \sup_{0\le s\le t} ||\mathbf{d}\partial f^{m-i}(s)||_\infty \sup_i \Big(\int_{\prod_{j=1}^{k-1}\mathcal{V}_j} \mathrm{d}\Sigma_i^{k-1}\Big) \\ &\quad + Ct \max_j ||e^{\zeta|v|^2} f_0||_\infty (1 + \max_j ||e^{\zeta|v|^2} f_0||_\infty)\Big(\int_{\prod_{j=1}^{k-1}\mathcal{V}_j} \mathrm{d}\Sigma_i^{k-1}\Big) + \Big(\frac{1}{2}\Big)^{-\frac{k}{5}} \sup_{0\le s\le t} ||\mathbf{d}\partial f^{m+1-k}(s)||_\infty, \end{aligned}$$

where we have used (80), Lemma 12, Lemma 11, and Lemma 2 for (81). Further we use the proof of Lemma 11 to show that

$$\sup_i \int_{\prod_{j=1}^{k-1}\mathcal{V}_j} \mathrm{d}\Sigma_i^{k-1} \le \Big(C\int_{\mathbb{R}^3} \mu(v)^{\frac{1}{4}} \mathrm{d}v\Big)^k \le C_k.$$

Denote $T_* \equiv t$ and estimate

$$\begin{aligned} &\sup_{0\le t\le T_*} ||\mathbf{1}_{\{t_1\ge 0\}}\mathbf{d}\partial f^{m+1}(t)||_\infty \le O(T_*) \max_{0\le i\le k-1} \sup_{0\le t\le T_*} ||\mathbf{d}\partial f^{m+1-i}(t)||_\infty \\ &+ C_{k,T_*}||e^{\zeta|v|^2} f_0||_\infty (1 + ||e^{\zeta|v|^2} f_0||_\infty) \\ &+ C_k(1 + ||e^{\zeta|v|^2} f_0||_\infty)\Big\{T_* e^{CT_*}|\delta_1|^{-\frac{1}{2}} + e^{C|\delta_1|^2}|\delta_1|^{\frac{1}{2}}\Big\} \max_{1\le i\le k-1} \sup_{0\le t\le T_*} ||\mathbf{d}\partial f^{m+1-i}(t)||_\infty \\ &+ \Big(\frac{1}{2}\Big)^{-\frac{k}{5}} \sup_{0\le t\le T_*} ||\mathbf{d}\partial f^{m+1-k}(t)||_\infty. \end{aligned} \tag{104}$$

We put together (103) and (104) and choose $k \gg 1$ so that $\left(\frac{1}{2}\right)^{-k/5} \le \frac{1}{100}$. Then we further choose $T_* = \delta_1 \ll_k 1$ so that

$$C_k(1 + ||e^{\zeta|v|^2} f_0||_\infty)\Big\{T_* e^{CT_*}|\delta_1|^{-\frac{1}{2}} + e^{C|\delta_1|^2}|\delta_1|^{\frac{1}{2}}\Big\} \le \frac{1}{100}.$$

We conclude

$$\sup_{0\le t\le T_*} ||\mathbf{d}\partial f^{m+1}(t)||_\infty \quad \le \quad \frac{1}{8}\max_{1\le i\le k}\sup_{0\le t\le T_*}||\mathbf{d}\partial f^{m+1-i}(t)||_\infty + ||\mathbf{d}\partial f_0||_\infty + C||e^{\zeta|v|^2}f_0||_\infty.$$

Set $D = ||\mathbf{d}\partial f_0||_\infty + C||e^{\zeta|v|^2}f_0||_\infty$,

$$a_i = \sup_{0\le t\le T_*}||\mathbf{d}\partial f^i(t)||_\infty, \quad A_i = \max\{a_i, a_{i-1}, \cdots, a_{i-(k-1)}\},$$

then we have $a_{m+1} \le \frac{1}{8}A_m + D$. Use (76) to conclude

$$\sup_{0\le t\le T_*}||\mathbf{d}\partial f^{m+1}(t)||_\infty \lesssim ||\mathbf{d}\partial f_0||_\infty + ||e^{\zeta|v|^2}f_0||_\infty.$$

The existence and uniqueness and the estimate in Theorem 2 are clear for short time $T_* > 0$. We follow the same procedure for $t \in [T_*, 2T_*]$ to conclude

$$\sup_{T_*\le t\le 2T_*}||\mathbf{d}\partial f(t)||_\infty \lesssim_{\Omega,T_*} ||\mathbf{d}\partial f(T_*)||_\infty + ||e^{\zeta|v|^2}f_0||_\infty.$$

Then we conclude the weighted $W^{1,\infty}$ part of Theorem 2 following the same procedure for $[T_*, 2T_*], [2T_*, 3T_*], \cdots$.

Now we consider the continuity of $\mathbf{d}\partial f$. Remark that for each step $\mathbf{d}\partial f^m$ satisfies the condition of Proposition 3. Therefore we conclude $\mathbf{d}\partial f^m \in C^1([0,T_*]\times\bar{\Omega}\times\mathbb{R}^3)$. Now we follow $W^{1,\infty}$ estimate part for $\mathbf{d}[\partial f^{m+1} - \partial f^m]$ to show that $\mathbf{d}\partial f^m$ is Cauchy in L^∞. Then $\mathbf{d}\partial f^m \to \mathbf{d}\partial f$ strongly in L^∞ so that $\mathbf{d}\partial f \in C^0([0,T_*]\times\bar{\Omega}\times\mathbb{R}^3)$. □

Acknowledgements: This work was initiated during the Kinetic Program at ICERM, 2011. Y.Guo's research is supported in part by a NSF grant, FRG grant, a Chinese NSF grant as well as Beijing International Center for Mathematical Research. He thanks Nader Masmoudi for earlier discussions on the same subject. C. Kim's research is supported in part by the Herchel Smith fund. He thanks Division of Applied Mathematics, Brown University for the kind hospitality and support and also he thanks Clément Mouhot for helpful discussions. A. Trescases thanks the Division of Applied Mathematics, Brown University for its hospitality during her visit. The authors dedicate this work to the memory of Seiji Ukai.

References

[1] Cercignani, C.; Illner, R.; Pulvirenti, M.: *The mathematical theory of dilute gases.* Applied Mathematical Sciences, 106. Springer-Verlag, New York, (1994)

[2] Esposito, R.; Guo, Y.; Kim, C. ; Marra, R: Non-Isothermal Boundary in the Boltzmann Theory and Fourier Law, *preprint*

[3] Desvillettes, L.; Villani, C.: On the trend to global equilibrium for spatial inhomogeneous kinetic systems: the Boltzmann equation. *Invent. Math.* 159 (2005), no.2, 245–316.

[4] Guiraud, J. P.: An H-theorem for a gas of rigid spheres in a bounded domain.In: Pichon, G. (ed.) Theories cinetique classique et relativistes, CNRS, Paris, (1975) 29–58,

[5] Guo, Y.: Regularity of the Vlasov equations in a half space. *Indiana. Math. J.* 43 (1994) 255–320.

[6] Guo, Y.: Singular Solutions of the Vlasov-Maxwell System on a Half Line. *Arch. Rational Mech. Anal.* 131 (1995) 241–304.

[7] Guo, Y.: Decay and Continuity of Boltzmann Equation in Bounded Domains. *Arch. Rational Mech. Anal.* 197 (2010) 713–809.

[8] Glassey, R. T.: *The Cauchy problem in kinetic theory.* Society for Industrial and Applied Mathematics (SIAM), Philadelphia, 1996

[9] Grad, H.: *Asymptotic theory of the Boltzmann equation, II.* Rarefied gas dynamics. In: Proceedings of the 3rd international Symposium, pp. 26–59, Paris, 1962.

[10] Hwang, H-J, Velazquez, J.: Global Existence for the Vlasov-Poisson System in Bounded Domains. *Arch. Rational Mech. Anal.* 195 (2010) 763–796.

[11] Kim, C.: Formation and propagation of discontinuity for Boltzmann equation in non-convex domains. *Comm. Math. Phys.* 308 (2011) 641–701.

Division of Applied Mathematics, Brown University, Providence, RI 02812, U.S.A., Yan_Guo@brown.edu

Department of Pure Mathematics and Mathematical Statistics , University of Cambridge, Cambridge, CB3 0WB,UK, C.W.Kim@dpmms.cam.ac.uk

Institut de mathématiques de Jussieu, case 472, 4 Place de Jussieu, 75252, Paris, France, tonond@math.jussieu.fr

CMLA, ENS Cachan, 61, avenue du Président Wilson, 94235 Cachan, France, ariane.trescases@ens-cachan.fr