

Entropy, Duality and Cross Diffusion

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Abstract

This paper is devoted to the use of the entropy and duality methods for the existence theory of reaction-cross diffusion systems consisting of two equations, in any dimension of space. Those systems appear in population dynamics when the diffusion rates of individuals of two species depend on the concentration of individuals of the same species (self-diffusion), or of the other species (cross diffusion).

1 Introduction

We are interested in populations consisting of individuals belonging to two distinct species (or two classes in the same species) of (typically) animals, and interacting through diffusion and competition.

The unknowns are the concentration (number density) of individuals of the first species $u_1 := u_1(t, x) \geq 0$ and of the second species $u_2 := u_2(t, x) \geq 0$.

In absence of competition, the respective populations u_1 and u_2 would grow at a (strictly positive) rate r_1 and r_2 .

The competition is taken into account through logistic-type terms, in such a way that the growth rate becomes $r_1 - s_{11}(u_1) - s_{12}(u_2)$ for the first species, and $r_2 - s_{22}(u_2) - s_{21}(u_1)$ for the second one. The (nonnegative) terms $s_{11}(u_1)$ and $s_{22}(u_2)$ are called intraspecific competition, while the (nonnegative) terms $s_{12}(u_2)$ and $s_{21}(u_1)$ are by definition the interspecific competition. Since we are interested here in situations when there is no cooperation (or predator-prey type interaction), we shall assume in the sequel that all functions s_{ij} are nonnegative and increasing (see [22], p.94).

We also assume that the individuals of the two species diffuse with a rate that depends on the number density of both species: we denote $d_{11}(u_1) + a_{12}(u_2)$ the diffusion rate of the first species, and $d_{22}(u_2) + a_{21}(u_1)$ the diffusion rate of the second species. The diffusion terms related to $d_{11}(u_1)$ and $d_{22}(u_2)$ are called self diffusion, those related to $a_{12}(u_2)$ and $a_{21}(u_1)$ are called cross diffusion. We are interested in this paper in the case when all the functions d_{ii} , a_{ij} are nondecreasing (that is, all individuals try to leave the points where the competition is highest). This type of model were introduced in [24].

In the sequel, we systematically denote

$$a_{ii}(u_i) = u_i d_{ii}(u_i). \quad (1)$$

We are led to write down the following system (on $\mathbb{R}_+ \times \Omega$, where Ω is a smooth bounded open subset of \mathbb{R}^N with outward unit normal vector $n(x)$ at point $x \in \partial\Omega$):

$$\partial_t u_1 - \Delta[a_{11}(u_1) + u_1 a_{12}(u_2)] = u_1(r_1 - s_{11}(u_1) - s_{12}(u_2)) := R_{12}(u_1, u_2), \quad (2)$$

$$\partial_t u_2 - \Delta[a_{22}(u_2) + u_2 a_{21}(u_1)] = u_2(r_2 - s_{22}(u_2) - s_{21}(u_1)) := R_{21}(u_2, u_1), \quad (3)$$

with the homogeneous Neumann boundary conditions

$$\nabla u_1 \cdot n(x) = \nabla u_2 \cdot n(x) = 0 \quad \text{on} \quad \mathbb{R}_+ \times \partial\Omega, \quad (4)$$

and the initial condition

$$u_1(0, \cdot) = u_1^{in} \geq 0, \quad u_2(0, \cdot) = u_2^{in} \geq 0. \quad (5)$$

The case treated in this paper corresponds to a situation where the reaction terms are strictly subquadratic or dominated by the self diffusion, and where the cross diffusions are subquadratic. Extensions to other cases with faster growth of reaction terms or cross diffusion pressure will be studied in a future work.

We detail below the set of mathematical assumptions that will be imposed on the coefficients a_{ij} , s_{ij} , and which correspond to the case described above:

H1 For $i, j \in \{1, 2\}$, $r_i \in \mathbb{R}_+$ and s_{ij} is a nonnegative continuous function on \mathbb{R}_+ which is either strictly sublinear: $\lim_{x \rightarrow +\infty} \frac{s_{ij}(x)}{x} = 0$; or dominated by the self diffusion in the following sense: $\lim_{x \rightarrow +\infty} \frac{s_{ii}(x)}{x d_{ii}(x)} = 0$ and (for $i \neq j$) $\lim_{x \rightarrow +\infty} \frac{s_{ij}(x)}{x \sqrt{d_{jj}(x) + a_{ij}(x)}} = 0$;

H2 For $i \neq j \in \{1, 2\}$, a_{ij} is continuous on \mathbb{R}_+ and belongs to $\mathcal{C}^2([0, +\infty[)$. It is also nonnegative, nondecreasing, concave and vanishes at point 0. Furthermore, there exists $\alpha \in]0, 1[$, $C > 0$, such that

$$\forall x \in]0, +\infty[, \quad x^\alpha a'_{ij}(x) \geq C.$$

H3 The self diffusion rate d_{ii} (recall notation (1)) is continuous on \mathbb{R}_+ and belongs to $\mathcal{C}^1([0, +\infty[)$. It is also nonnegative, nondecreasing and such that $d_{ii}(0) > 0$ (note in particular that a'_{ii} is bounded below by a strictly positive constant). In the proof of our main Theorem, we use $d_{ii} \geq 1$ for the sake of readability.

The union of the assumptions **H1**, **H2** and **H3** on the parameters [for $i, j \in \{1, 2\}$] r_i , s_{ij} , a_{ij} (and thus d_{ii}) will be called in the sequel the **H** assumptions.

Since we consider the homogeneous Neumann boundary condition, it is useful to introduce the following notation :

Definition 1.1. for any space of functions defined on Ω whose gradient has a well-defined trace on the boundary $\partial\Omega$ (such as $\mathcal{C}^\infty(\bar{\Omega})$ or $H^2(\Omega)$ for instance), we add the subscript ν (the former spaces hence become $\mathcal{C}_\nu^\infty(\bar{\Omega})$ or $H_\nu^2(\Omega)$) to describe the subspace of functions satisfying the homogeneous Neumann boundary condition.

Our main Theorem reads:

Theorem 1.2. Let Ω be a smooth (\mathcal{C}^2) bounded open subset of \mathbb{R}^N ($N \geq 1$). Let $\mathbf{u}^0 := (u_1^{in}, u_2^{in}) \in L^2(\Omega)^2$ be a couple of nonnegative functions and assume that assumptions **H** are satisfied on the coefficients of the system.

Then, for any $T > 0$, there exists a couple $\mathbf{u} := (u_1, u_2)$ of nonnegative weak solutions to (2)–(3)–(4) on $[0, T]$ in the following sense: for $i = 1, 2$ ($j \neq i$),

$$\int_0^T \int_\Omega \left[d_{ii}(u_i(t, x)) + a_{ij}(u_j(t, x)) \right] |u_i(t, x)|^2 dx dt < +\infty, \quad (6)$$

and for any $\theta \in \mathcal{D}([0, T]; \mathcal{C}_\nu^\infty(\bar{\Omega}))$, we have the weak formulation

$$\begin{aligned} & - \int_\Omega u_1^{in}(x) \theta(0, x) dx - \int_0^\infty \int_\Omega u_1(t, x) \partial_t \theta(t, x) dx dt \\ & - \int_0^\infty \int_\Omega \Delta \theta(t, x) \left[a_{11}(u_1(t, x)) + u_1(t, x) a_{12}(u_2(t, x)) \right] dx dt \\ & = \int_0^\infty \int_\Omega R_{12}(u_1(t, x), u_2(t, x)) dx dt, \end{aligned}$$

and

$$\begin{aligned} & - \int_\Omega u_2^{in}(x) \theta(0, x) dx - \int_0^\infty \int_\Omega u_2(t, x) \partial_t \theta(t, x) dx dt \\ & - \int_0^\infty \int_\Omega \Delta \theta(t, x) \left[a_{22}(u_2(t, x)) + u_2(t, x) a_{21}(u_1(t, x)) \right] dx dt \\ & = \int_0^\infty \int_\Omega R_{21}(u_1(t, x), u_2(t, x)) dx dt. \end{aligned}$$

Moreover, denoting $Q_T := [0, T] \times \Omega$, the following bounds hold:

$$\begin{aligned} & \left\| \frac{1}{u_1} a'_{21}(u_1) \nabla u_1 \right\|_{L^2(Q_T)} + \left\| \frac{1}{u_2} a'_{12}(u_2) \nabla u_2 \right\|_{L^2(Q_T)} \leq K_T (1 + \|\mathbf{u}^0\|_{L^2(\Omega)}), \\ & \left\| \nabla \sqrt{a_{21}(u_1)} \right\|_{L^2(Q_T)} + \left\| \nabla \sqrt{a_{12}(u_2)} \right\|_{L^2(Q_T)} + \left\| \nabla \sqrt{a_{21}(u_1) a_{12}(u_2)} \right\|_{L^2(Q_T)} \leq K_T (1 + \|\mathbf{u}^0\|_{L^2(\Omega)}), \end{aligned}$$

for some positive constant K_T depending only on T and the data of the equations (a_{ij} , r_i , s_{ij}).

Note that all the terms in the weak formulation above are well-defined thanks to assumptions **H**. Indeed, remembering that d_{ii} is bounded below, estimate (6) implies $u_i \in L^2([0, T] \times \Omega)$ so that the reaction terms are integrable (thanks to **H1**). Using the identity

$$s_{12}(u_2) u_1 = \left(\frac{s_{12}(u_2)}{u_2 \sqrt{d_{22}(u_1) + a_{12}(u_2)}} \right) \left(\frac{u_2 \sqrt{d_{22}(u_2) + a_{12}(u_2)}}{\sqrt{a_{12}(u_2)}} \right) u_1 \sqrt{a_{12}(u_2)},$$

the terms coming out of cross diffusion are also well defined due to concavity assumptions **H2**. Those coming out of self diffusion are integrable thanks to (6).

We now comment the assumptions on the initial data and coefficients of system (2)–(3). The requirement that $\mathbf{u}^0 \in (L^2(\Omega))^2$ can certainly be relaxed to $\mathbf{u}^0 \in H^{-1}(\Omega)^2$ and $\psi_i(u_i^{i_n}) \in L^1(\Omega)$, with ψ_i defined later in Definition 2.1 (this should be enough to keep estimate (6)). It is most likely possible to relax assumption **H1**, provided that **H2** and **H3** are reinforced. Finally, there is some hope of treating the special case when reaction terms are exactly quadratic, thanks to recent improvements in the theory of duality Lemmas [5].

We restricted in this paper our study to the case when both a_{ij} , $i \neq j$ are concave, whereas our methods should adapt in the case when one of them is concave and the other one is convex. The corresponding theory is then quite different and will be left to a future work. Note that when both a_{ij} , $i \neq j$ are convex, our feeling is that existence of weak global solutions does not hold in general.

Our opinion is that for systems involving cross diffusion and consisting of more than two equations, the type of Lyapounov functional that we build in the sequel can exist only for a very small class of cross diffusion terms (that is, strong algebraic constraints have to be assumed on the cross diffusion coefficients).

Finally, we do not treat the case when one cross diffusion term is missing (sometimes called the “triangular case”), since the Lyapounov functional that we introduce in the sequel degenerates in this case.

Let us also explain the meaning of assumptions **H** when all the functions appearing in (2)–(3) are of the form $x \mapsto x^q$ (or $x \mapsto x + x^q$ for the self diffusion), with $q \in \mathbb{R}$. In this case, if $s_{ij}(x) = S_{ij} x^{\sigma_{ij}}$ and $a_{ij}(x) = D_i x \delta_{ij} + A_{ij} x^{\alpha_{ij}}$, with $S_{ij}, A_{ij}, D_i > 0$ and $\sigma_{ij}, \alpha_{ij} \in \mathbb{R}$, those assumptions become:

$$\begin{aligned} \forall i = 1, 2, \quad 0 \leq \sigma_{ii} < \sup(1, \alpha_{ii}); \\ \forall i \neq j, \quad 0 \leq \sigma_{ij} < \sup\left(\frac{\alpha_{jj} + 1}{2}, 1 + \frac{\alpha_{ij}}{2}\right), \\ 0 < \alpha_{ij} < 1. \end{aligned}$$

As a consequence, our Theorem provides existence of global weak solutions for systems like

$$\partial_t u_1 - \Delta[(D_1 + A_{11} u_1^\alpha + A_{12} u_2^\beta) u_1] = u_1(r_1 - S_{11} u_1 - S_{12} u_2), \quad (7)$$

$$\partial_t u_2 - \Delta[(D_2 + A_{21} u_1^\gamma + A_{22} u_2^\delta) u_2] = u_2(r_2 - S_{21} u_1 - S_{22} u_2), \quad (8)$$

with $0 \leq \alpha, \delta < 1$; $\beta, \gamma > 0$, and $r_i, D_i, A_{ij}, S_{ij} > 0$.

Let us now describe how our work fits in the existing literature.

The question of local and global existence of classical solutions of systems like (2)–(3) was treated in many particular cases. Most of them deal either with the case when the system is in fact parabolic (that is, when the diffusion matrix is elliptic), which amounts to assume that self diffusion is dominant w.r.t. cross diffusion (for instance when $8A_{11} \geq A_{12}, 8A_{22} \geq A_{21}$ in eq. (7), (8) with $\alpha = \beta = \gamma = \delta = 1$, cf. [26]), or in the triangular case (when $a_{21}(u_1) = 0$, cf. [20]). The question of local existence is usually treated using Amann’s theorem [1], and extension to global existence requires additional structure to preserve boundedness of solutions see [8, 12] for instance.

Our work is concerned with situations that are neither parabolic, nor triangular. It extends to a very large class of problems the result of [7], dedicated to the cross diffusion model for population dynamics introduced by Shigesada, Kawasaki and Teramoto in [24], where the cross diffusions have a linear form, that is $\beta = \gamma = 1$ in eq. (7), (8). Note that this system can be seen as a limiting case of the equations that we treat.

Our results rely on two main ingredients: entropy structure and duality Lemmas. We show that our systems possess a hidden entropy-like structure, strongly reminiscent of the entropy structure exhibited in [7]. In general, this structure however gives less estimates than in [7]. We therefore need another ingredient in order to recover existence of weak global solutions, namely duality Lemmas: we recall that duality Lemmas enable to recover L^2 type estimate for solutions to linear singular parabolic equations (with variable coefficients) when the diffusion rate is inside the Laplacian. This is how estimate (6) is derived.

For the use of an underlying entropy structure, its link with symmetrization, and its applications to existence of weak solutions, we refer to [18, 9]. The possibility to use such a structure in the case of cross diffusion was first noticed in [13], and exploited in [7, 6, 17, 15]. The duality estimate that we use comes from [23], and was applied together with entropy methods in the framework of reaction-diffusion systems in [11]. It was also applied in the framework of cross diffusion or similar models in [2, 3, 4].

We finally quote some works dealing with other aspects of cross diffusion models. For modeling issues, we refer to [16, 10, 3, 2]. For the analysis of equilibria, we cite [16, 19] for instance.

Unfortunately, as often in papers dealing with cross diffusion, the process of approximation enabling to make the estimates and structures rigorous is quite involved, and gives rise to various difficulties which explain the length and technicality of the proofs.

Our paper is structured as follows: in Section 2, we introduce notations which are used in the proof of our main Theorem, especially those related to the Lyapounov functional that we systematically use. We also present some classical lemmas used in the sequel. Then, Section 3 is devoted to the proof of existence of a solution to a finite-dimensional (Galerkin) approximation of a discrete time version of our (smoothed) system. *A priori* estimates (and their dependence with respect to the various approximations) are provided for such solutions. We let the dimension of the Galerkin approximation go to infinity in Section 4. The duality estimate is presented and proven in Section 5. The last section (that is, Section 6) is devoted to the relaxation of all remaining approximations, which leads to the proof of our main Theorem.

2 Preliminaries

2.1 Entropy structure

We first introduce some notations which enable to rewrite our system under a form in which a Lyapounov functional naturally appears.

Definition 2.1. For given cross diffusion parameters a_{12} and a_{21} (satisfying assumption **H2**), we introduce φ_i, ψ_i (for $i = 1, 2$), as:

$$\begin{aligned}\varphi_1(x) &:= \int_1^x \frac{a'_{21}(t)}{t} dt, \quad \varphi_2(x) := \int_1^x \frac{a'_{12}(t)}{t} dt, \\ \psi_i(x) &:= \int_1^x \varphi_i(t) dt.\end{aligned}$$

We also define

$$w_i(t, x) := \varphi_i(u_i(t, x)).$$

One can then rewrite the system in a symmetric form:

$$\partial_t \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} - \operatorname{div} \left(\overbrace{\begin{pmatrix} \frac{a'_{11}(u_1)+a_{12}(u_2)}{a'_{21}(u_1)} u_1 & u_1 u_2 \\ u_1 u_2 & \frac{a'_{22}(u_2)+a_{21}(u_1)}{a'_{12}(u_2)} u_2 \end{pmatrix}}^{:=A(u_1, u_2)} \right) \begin{pmatrix} \nabla w_1 \\ \nabla w_2 \end{pmatrix} = \begin{pmatrix} R_{12}(u_1, u_2) \\ R_{21}(u_2, u_1) \end{pmatrix}. \quad (9)$$

The terms ∇w_i have to be considered as scalars for the matrix product, the divergence being understood line by line (after the matrix product).

Multiplying the equations of the system by w_1, w_2 and integrating in space we obtain formally the entropy identity

$$\frac{d}{dt} \int_{\Omega} (\psi_1(u_1) + \psi_2(u_2)) + \int_{\Omega} (\nabla w_1, \nabla w_2) A(u_1, u_2) \begin{pmatrix} \nabla w_1 \\ \nabla w_2 \end{pmatrix} = (w_1, w_2) \begin{pmatrix} R_{12} \\ R_{21} \end{pmatrix}. \quad (10)$$

2.2 Properties of the symmetric matrix

For $u_1, u_2 > 0$, $\mathbf{u} := (u_1, u_2)$, the matrix $A(\mathbf{u})$ is defined by

$$A(u_1, u_2) := \begin{pmatrix} \frac{a'_{11}(u_1)+a_{12}(u_2)}{a'_{21}(u_1)} u_1 & u_1 u_2 \\ u_1 u_2 & \frac{a'_{22}(u_2)+a_{21}(u_1)}{a'_{12}(u_2)} u_2 \end{pmatrix}, \quad (11)$$

and we denote the associate quadratic form $\mathbf{Q}(\mathbf{u})$. For a function $\mathbf{u} = (u_1, u_2) : \Omega \rightarrow]0, +\infty[^2$, we defined in the previous subsection $\mathbf{w} = (w_1, w_2) = (\varphi_1(u_1), \varphi_2(u_2))$. We will have to deal with the expression

$$\mathbf{Q}(\mathbf{u})(\nabla \mathbf{w}),$$

since this term naturally appears in the entropy estimate. We establish the following proposition, that will be useful later.

Proposition 2.2. Under the **H** assumptions, the application $A :]0, +\infty[^2 \rightarrow \mathcal{M}_2(\mathbb{R})$ (defined by (11)) belongs to $\mathcal{C}^0(]0, +\infty[^2)$ and takes its values in $S_2^{++}(\mathbb{R})$ (space of strictly positive symmetric matrices). We have furthermore the two following estimates on the quadratic form \mathbf{Q}

$$\mathbf{Q}(\mathbf{u})(\nabla \mathbf{w}) \geq \frac{1}{u_1} a'_{21}(u_1) |\nabla u_1|^2 + \frac{1}{u_2} a'_{12}(u_2) |\nabla u_2|^2, \quad (12)$$

$$\mathbf{Q}(\mathbf{u})(\nabla \mathbf{w}) \geq 4 \left\{ \left| \nabla \sqrt{a_{21}(u_1)} \right|^2 + \left| \nabla \sqrt{a_{12}(u_2)} \right|^2 + \left| \nabla \sqrt{a_{21}(u_1) a_{12}(u_2)} \right|^2 \right\}. \quad (13)$$

Proof. The fact that $A \in \mathcal{C}^0(]0, +\infty[^2)$ is an easy consequence of the regularity of the functions a_{ij} . As for the strict positiveness of the matrix, we just decompose A between self (clearly strictly positive) and cross diffusion:

$$A(u_1, u_2) = \overbrace{\begin{pmatrix} \frac{a'_{11}(u_1)}{a'_{21}(u_1)} u_1 & 0 \\ 0 & \frac{a'_{22}(u_2)}{a'_{12}(u_2)} u_2 \end{pmatrix}}^{B(u_1, u_2)} + \overbrace{\begin{pmatrix} \frac{a_{12}(u_2)}{a'_{21}(u_1)} u_1 & u_1 u_2 \\ u_1 u_2 & \frac{a_{21}(u_1)}{a'_{12}(u_2)} u_2 \end{pmatrix}}^{C(u_1, u_2)}.$$

We see that due to the assumptions on a_{12} and a_{21} , the cross diffusion matrix $C(u_1, u_2)$ is nonnegative. Indeed, for $u_1, u_2 > 0$

$$\begin{aligned}\frac{a_{12}(u_2)}{a'_{21}(u_1)} u_1 \times \frac{a_{21}(u_1)}{a'_{12}(u_2)} u_2 &\geq u_1 u_2 \times u_1 u_2 \\ &\Updownarrow \\ a_{12}(u_2) a_{21}(u_1) &\geq u_1 u_2 a'_{12}(u_2) a'_{21}(u_1),\end{aligned}$$

which is true by concavity. We get

$$\mathbf{Q}(\mathbf{u})(\nabla \mathbf{w}) = [{}^t \nabla \mathbf{w}] [A(\mathbf{u}) \nabla \mathbf{w}] \geq [{}^t \nabla \mathbf{w}] [B(\mathbf{u}) \nabla \mathbf{w}] = \frac{a'_{11}(u_1)}{a_{21}(u_1)} u_1 |\nabla w_1|^2 + \frac{a'_{22}(u_2)}{a'_{12}(u_2)} u_2 |\nabla w_2|^2.$$

Since $w_i = \varphi_i(u_i)$, with $\varphi'_i(x) = \frac{a'_{ji}(x)}{x}$, we end up with

$$\mathbf{Q}(\mathbf{u})(\nabla \mathbf{w}) \geq \frac{a'_{11}(u_1)}{u_1} a'_{21}(u_1) |\nabla u_1|^2 + \frac{a'_{22}(u_2)}{u_2} a'_{12}(u_2) |\nabla u_2|^2,$$

which, using assumption **H3**, leads to the first lower bound (12). On the other hand, expanding directly $\mathbf{Q}(\mathbf{u})(\nabla \mathbf{w})$ with the definitions of $A(\mathbf{u})$ and \mathbf{w} , we get

$$\begin{aligned} \mathbf{Q}(\mathbf{u})(\nabla \mathbf{w}) &= [a'_{11}(u_1) + a_{12}(u_2)] \frac{a'_{21}(u_1)}{u_1} |\nabla u_1|^2 + [a'_{22}(u_2) + a_{21}(u_1)] \frac{a'_{12}(u_2)}{u_2} |\nabla u_2|^2 \\ &\quad + 2 a'_{21}(u_1) a'_{12}(u_2) \langle \nabla u_1, \nabla u_2 \rangle. \end{aligned}$$

The concavity property $a_{ji}(u_i) \geq a'_{ji}(u_i) u_i$ used before can be written (here for $i = 1, j = 2$)

$$\frac{a'_{21}(u_1)}{u_1} \geq \frac{a'_{21}(u_1)^2}{a_{21}(u_1)},$$

and together with the assumption **H3**, we get

$$\begin{aligned} \mathbf{Q}(\mathbf{u})(\nabla \mathbf{w}) &\geq a'_{11}(u_1) \frac{a'_{21}(u_1)^2}{a_{21}(u_1)} |\nabla u_1|^2 + a'_{22}(u_2) \frac{a'_{12}(u_2)^2}{a_{12}(u_2)} |\nabla u_2|^2 \\ &\quad + a_{12}(u_2) \frac{a'_{21}(u_1)^2}{a_{21}(u_1)} |\nabla u_1|^2 + a_{21}(u_1) \frac{a'_{12}(u_2)^2}{a_{12}(u_2)} |\nabla u_2|^2 + 2 a'_{21}(u_1) a'_{12}(u_2) \langle \nabla u_1, \nabla u_2 \rangle \\ &= 4 |\nabla \sqrt{a_{21}(u_1)}|^2 + 4 |\nabla \sqrt{a_{12}(u_2)}|^2 \\ &\quad + 4 |\sqrt{a_{12}(u_2)} \nabla \sqrt{a_{21}(u_1)}|^2 + 4 |\sqrt{a_{21}(u_1)} \nabla \sqrt{a_{12}(u_2)}|^2 \\ &\quad + 4 \times 2 \sqrt{a_{21}(u_1)} \sqrt{a_{12}(u_2)} \langle \nabla \sqrt{a_{21}(u_1)}, \nabla \sqrt{a_{12}(u_2)} \rangle, \end{aligned}$$

and we thus end up with eq. (13). \square

2.3 Properties of the functions φ_i and ψ_i

We shall need in the sequel the following elementary result:

Lemma 2.3. *Take $h, \ell \in \mathcal{C}^0(\mathbb{R}_+) \cap \mathcal{C}^1(]0, +\infty[)$ with h concave and ℓ nonnegative and convex, with $\ell'(x) > 0$ for all x large enough.*

Then there exists a constant $A_{h,\ell} > 0$ such that $h(x) \leq A_{h,\ell} (1 + \ell(x))$ for all $x \in \mathbb{R}_+$.

Proof. If h is bounded from above, then $A_{h,\ell} = \sup h$ works. Otherwise, $h' > 0$ on \mathbb{R}_+ . Say that $0 < \ell'(x)$ for $x > M$. Then for all $x \in \mathbb{R}_+$, $x - M \leq \frac{\ell(x) - \ell(M)}{\ell'(M)}$. Then since $h' > 0$ and h is concave, we can write

$$h(x) \leq h(M) + h'(M)(x - M) \leq h(M) + \frac{h'(M)}{\ell'(M)} (\ell(x) - \ell(M)) \leq h(M) + \frac{h'(M)}{\ell'(M)} \ell(x),$$

so that the constant $A_{h,\ell} = \max\left(h(M), \frac{h'(M)}{\ell'(M)}\right)$ works. That concludes the Proof of Lemma 2.3. \square

Using the previous assumptions on the a_{ij} , one can see that the φ_i, ψ_i at least belong to $\mathcal{C}^2(]0, +\infty[)$. The ψ_i are convex functions and the φ_i strictly nondecreasing. Moreover:

Lemma 2.4. *We assume **H2** on the coefficients a_{ij} ($i \neq j$). Then the ψ_i are convex functions and the φ_i are strictly nondecreasing. Moreover (for $i \neq j$):*

- (i) *For all $x, y \in \mathbb{R}_+$, $\psi'_i(x)(x - y) \geq \psi_i(x) - \psi_i(y)$.*
- (ii) *$\psi'_i(x) = o_{x \rightarrow 0^+}(1/x)$ and hence $x\psi'_i(x) \geq B$, for some constant $B < 0$ for all $x \in]0, +\infty[$.*
- (iii) *ψ_i has a limit at point 0^+ ($\psi_1(0) = a_{21}(1)$), furthermore ψ_i is strictly positive on \mathbb{R}_+ .*
- (iv) *There exists a constant $D > 0$ such that, for all $x \in \mathbb{R}_+$*

$$\begin{aligned} \forall \alpha \in [0, 1], \quad x^\alpha + a_{ji}(x) &\leq D(1 + \psi_i(x)), \\ x\psi'_i(x) &\leq D(1 + \psi_i(x)). \end{aligned}$$

Proof. Let us treat only the case $i = 1$, the other one being similar.

- (i) ψ_1 is convex.
- (ii) $\psi'_1(x) = \varphi_1(x)$ and

$$\varphi_1(x) = \int_1^x \frac{a'_{21}(t)}{t} dt = \left[\frac{a_{21}(t)}{t} \right]_1^x + \int_1^x \frac{a_{21}(t)}{t^2} dt = o_{x \rightarrow 0^+}(1/x),$$

since $a_{21} \in \mathcal{C}^0(\mathbb{R}_+)$ and $a_{21}(0) = 0$. The function $x \mapsto x\varphi_1(x)$ is strictly nondecreasing after $x = 1$ and bounded near 0, hence lower bounded.

(iii) Note that $\psi_1''(t)t = a_{21}'(t)$. We have

$$a_{21}(1) - a_{21}(x) = \int_x^1 t\psi_1''(t)dt = \left[t\psi_1'(t)\right]_x^1 - \int_x^1 \psi_1'(t)dt,$$

then noticing that $\psi_i(1) = \varphi_i(1) = 0$ by construction, we have

$$a_{21}(1) - a_{21}(x) = \psi_1(x) - x\varphi_1(x).$$

hence the previous point (ii) gives the limit near 0. For the positiveness, just notice that $\varphi_i = \psi_i'$ is negative on $[0, 1]$.

(iv) We use Lemma 2.3 with $h(x) = x + a_{ji}(x)$, $\ell(x) = \psi_1(x)$ and $x^\alpha \leq 1 + x$ in order to obtain the first inequality. For the second inequality, we use the same Lemma with $h(x) = x\psi_1'(x) - 2\psi_1(x)$, and $\ell(x) = \psi_1(x)$. \square

2.4 A small perturbation

Since φ_i may not be one to one, we use a small perturbation of this function and consider the following definition:

Definition 2.5. Let us assume **H2** (and recall Definition 2.1) on the coefficients a_{ij} ($i \neq j$), and **H3** on the coefficients a_{ii} .

For all $\varepsilon > 0$ (small enough), we introduce

$$\varphi_i^\varepsilon(x) := \varphi_i(x) + \varepsilon \ln(x),$$

and, equivalently, (for $i \neq j$) $a_{ij}^\varepsilon := a_{ij} + \varepsilon x$, and $\psi_i^\varepsilon(x) := \psi_i(x) + \varepsilon x \ln(x) - \varepsilon x$.

We also introduce $a_{ii}^\varepsilon(x) = x d_{ii}^\varepsilon(x)$, with $d_{ii}^\varepsilon = \gamma_\varepsilon(d_{ii})$, where $(\gamma_\varepsilon)_{\varepsilon>0}$ is an increasing family of smooth nonnegative and nondecreasing functions, such that $\gamma_\varepsilon \leq \varepsilon^{-1}$ on \mathbb{R}_+ , $\gamma_\varepsilon(1) = 1$ and $(\gamma_\varepsilon)_{\varepsilon>0}$ is uniformly converging to the identity on compact sets.

Finally, we denote by A^ε the matrix A (defined by (11)), where the coefficients a_{ij} are replaced by a_{ij}^ε (for all $i, j \in \{1, 2\}$).

The crucial point is the following: Proposition 2.2 is still true when one replaces the functions a_{ij} and A by their ε -approximations, a_{ij}^ε and A^ε , (assumption **H2** and **H3** hold for the a_{ij}^ε), and if one tries to reproduce Lemma 2.4 with these new functions, all the inequalities remain the same, the constants being a little bit changed but *not depending on ε* . We write the following lemma, which summarizes the situation:

Lemma 2.6. We assume that **H2** and **H3** holds on the coefficients a_{ij} (for all $i, j \in \{1, 2\}$), and use the notations of Definition 2.5.

Then, the coefficients a_{ij}^ε also satisfy **H2** and **H3** (with constants independent of ε for $\varepsilon > 0$ small enough). Also B and D (the constants of Lemma 2.4) may be changed in order to have, for $i = 1, 2$ and $0 < \varepsilon < 1$:

- (i) For all $x, y \in \mathbb{R}_+$, $\psi_i^{\varepsilon'}(x)(x - y) \geq \psi_i^\varepsilon(x) - \psi_i^\varepsilon(y)$.
- (ii) $\psi_i^{\varepsilon'}(x) = o_{x \rightarrow 0+}(1/x)$ and furthermore, for all $x \in]0, +\infty[$, $x\psi_i^{\varepsilon'}(x) \geq B - \varepsilon e^{-1}$.
- (iii) ψ_i^ε has a limit at point 0^+ . For ε small enough, ψ_i^ε is strictly positive on \mathbb{R}_+ .
- (iv) For all $x \in \mathbb{R}_+$

$$\begin{aligned} \forall \alpha \in [0, 1], \quad x^\alpha + a_{ji}^\varepsilon(x) &\leq D(1 + \varepsilon)(1 + \psi_i^\varepsilon(x)), \\ x\psi_i^{\varepsilon'}(x) &\leq D(1 + \varepsilon)(1 + \psi_i^\varepsilon(x)), \end{aligned}$$

where D is the constant defined above in lemma 2.4.

Proof. As before, we treat only the case $i = 1$.

- (i) ψ_1^ε is still convex.
- (ii) $\ln(x) = o_{x \rightarrow 0+}(1/x)$ and $\varepsilon x \ln(x) \geq -\varepsilon e^{-1} \geq -e^{-1}$.
- (iii) $x \ln(x) - x$ goes to 0 with x and $\varepsilon(x \ln(x) - x) \geq -\varepsilon$.
- (iv) Firstly, we notice that $D \geq 1 + a_{ji}(1) > 1$. For the first inequality, is sufficient to notice that since $\psi_i^\varepsilon \leq \psi_i$, we have

$$\varepsilon x \leq \varepsilon(x + a_{ij}(x)) \leq \varepsilon D(1 + \psi_i),$$

and we conclude with

$$x^\alpha + a_{ij}^\varepsilon(x) \leq D(1 + \varepsilon)(1 + \psi_i(x)) \leq D(1 + \varepsilon)(1 + \psi_i^\varepsilon(x)).$$

For the second inequality, we use

$$\varepsilon x \ln(x) \leq \varepsilon(x \ln(x) - x + 1) + \varepsilon x \leq D\varepsilon(x \ln(x) - x + 1) + D(1 + \psi_i(x)),$$

so that

$$x\psi_i^{\varepsilon'}(x) \leq x\psi_i'(x) + \varepsilon x \ln(x) \leq D(1 + \psi_i(x)) + D\varepsilon(x \ln(x) - x + 1) + D(1 + \psi_i(x)) \leq D(1 + \varepsilon)(1 + \psi_i^\varepsilon(x))$$

for some constant D (obviously not depending on ε). Then we add these inequalities to those of point (iv) of Lemma 2.4, that we already proved. \square

2.5 Two standard results

The preliminaries are concluded by the statement of two results taken from the existing literature, and that will be used in the sequel.

Lemma 2.7 (Discrete Grönwall). *Consider two nonnegative sequences $(v_n, w_n)_{n \in \mathbb{N}}$, satisfying for some positive constants $C > 0, \theta \in]0, 1[$,*

$$\forall n \in \mathbb{N}^*, \quad v_n \leq v_{n-1} + \theta v_n + w_n.$$

Then, for all $n \in \mathbb{N}^*$:

$$v_n \leq e^{n\lambda_\theta} v_0 + \sum_{k=0}^{n-1} e^{k\lambda_\theta} w_{n-k} \leq e^{\lambda_\theta} \left[v_0 + \sum_{k=1}^n e^{(-k+1)\lambda_\theta} w_k \right],$$

with $\lambda_\theta := \theta/(1-\theta)$. If $w_n = C$ is constant, we have

$$v_n \leq e^{n\lambda_\theta} \left[v_0 + \frac{C}{\theta} \right].$$

Proof. Notice first that $\frac{1}{1-\theta} \leq e^{\lambda_\theta}$ by the convexity inequality of the exponential. We hence have

$$v_n \leq e^{\lambda_\theta} v_{n-1} + \frac{w_n}{1-\theta}.$$

By straightforward induction, we get

$$v_n \leq \frac{1}{(1-\theta)^n} v_0 + \sum_{k=1}^n \frac{w_k}{(1-\theta)^{n-k+1}} \leq \frac{1}{(1-\theta)^n} \left[v_0 + \sum_{k=1}^n \frac{1}{(1-\theta)^{1-k}} w_k \right] \leq e^{n\lambda_\theta} \left[v_0 + \sum_{k=1}^n w_k e^{(1-k)\lambda_\theta} \right].$$

In case of constant w_n , we use the following fact

$$\frac{1}{(1-\theta)^n} \left[v_0 + \sum_{k=1}^n \frac{1}{(1-\theta)^{1-k}} C \right] \leq e^{n\lambda_\theta} \leq \frac{1}{(1-\theta)^n} \left[v_0 + \frac{C}{\theta} \right] \leq e^{n\lambda_\theta} \left[v_0 + \frac{C}{\theta} \right].$$

□

The following Theorem can be found in [14] (in the more general case of an infinite dimensional Banach space) where it is presented as the “Leray-Schauder Theorem” (p.286, Theorem 11.6):

Theorem 2.8. *Let $(E, \|\cdot\|)$ be a Euclidian vector space and $T : [0, 1] \times E \rightarrow E$ a continuous function satisfying $T(0, \cdot) \equiv 0$. Suppose furthermore the existence of $R > 0$ such that for any $s \in [0, 1]$, the following a priori estimate holds for the fixed points of $T(s, \cdot)$:*

$$T(s, x) = x \implies \|x\| < R.$$

Then $T(1, \cdot) : E \rightarrow E$ has at least one fixed point in $B(0, R)$.

3 Approximate system of finite dimension

3.1 Notations

We start with a definition related to the discretization w.r.t. time, as in [7]:

Definition 3.1. *We decompose the time interval, $]0, T] = \bigcup_{k=1}^N](k-1)\tau, k\tau]$, where $N \in \mathbb{N}^*$ and $\tau := T/N$, and introduce*

$$\text{the finite difference operator : } \partial_\tau u^k := \frac{u^k - u^{k-1}}{\tau}.$$

We also introduce new definitions related to the reaction terms:

Definition 3.2. *Let us denote by R_{12}^\pm and R_{21}^\pm the positive/negative parts of the source terms, and use the same notation for \mathbf{R}^\pm . To write things more precisely, we have*

$$\mathbf{R}^+(\mathbf{u}) := \begin{pmatrix} r_1 & 0 \\ 0 & r_2 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}, \quad \mathbf{R}^-(\mathbf{u}) := \begin{pmatrix} u_1 & 0 \\ 0 & u_2 \end{pmatrix} \begin{pmatrix} s_{11}(u_1) + s_{12}(u_2) \\ s_{22}(u_2) + s_{21}(u_1) \end{pmatrix},$$

and obviously

$$\mathbf{R}(\mathbf{u}) = \mathbf{R}^+(\mathbf{u}) - \mathbf{R}^-(\mathbf{u}).$$

We introduce the following approximation for this reaction term

$$\mathbf{R}^\varepsilon(\mathbf{u}) = \mathbf{R}^+(\mathbf{u}) - \mathbf{R}^{-,\varepsilon}(\mathbf{u}),$$

with

$$\mathbf{R}^{-,\varepsilon}(\mathbf{u}) := \begin{pmatrix} u_1 & 0 \\ 0 & u_2 \end{pmatrix} \begin{pmatrix} \gamma_\varepsilon(s_{11}(u_1) + s_{12}(u_2)) \\ \gamma_\varepsilon(s_{22}(u_2) + s_{21}(u_1)) \end{pmatrix},$$

where γ_ε is the truncation function that we used in Definition 2.5.

Let us consider a sequence $(V_n)_{n \in \mathbb{N}}$ of subspaces of $\mathcal{C}_\nu^\infty(\overline{\Omega})$, such that for all n , V_n is n -dimensional, $V_n \subset V_{n+1}$, and $\bigcup_{n \in \mathbb{N}} V_n$ is dense in $L^2(\Omega)$. We assume that $V_1 = \langle \mathbf{1}_\Omega \rangle_{\mathbb{R}}$, so that the constant function $\mathbf{1}_\Omega$ lies in all subspaces V_n .

For the sake of clarity, let us denote in boldface the two component vectors:

$$\boldsymbol{\chi} := \begin{pmatrix} \chi_1 \\ \chi_2 \end{pmatrix}, \mathbf{u} := \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}, \mathbf{w} := \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} \quad \text{and} \quad \mathbf{R}(\mathbf{u}) := \begin{pmatrix} R_{12}(u_1, u_2) \\ R_{21}(u_2, u_1) \end{pmatrix}.$$

As previously, all linear operators (such as $\nabla, \partial_\tau \dots$) have to be understood line by line in the previous expressions. Any dot product $\langle \cdot, \cdot \rangle_{\mathbb{E}}$ on some space \mathbb{E} of functions defined on Ω has to be understood as $\langle \mathbf{u}, \boldsymbol{\chi} \rangle_{\mathbb{E}} = \langle u_1, \chi_1 \rangle_{\mathbb{E}} + \langle u_2, \chi_2 \rangle_{\mathbb{E}}$. The nonnegative symmetric matrix $A^\varepsilon(u_1, u_2)$ will be denoted $A^\varepsilon(\mathbf{u})$ and we also will often denote by $\mathbf{Q}^\varepsilon(\mathbf{u})$ the associated quadratic form. Finally we will use the obvious notation

$$\varphi_\varepsilon(\mathbf{u}) = \begin{pmatrix} \varphi_1^\varepsilon(u_1) \\ \varphi_2^\varepsilon(u_2) \end{pmatrix},$$

in such a way that according to the previous notations, $\mathbf{w} = \varphi_\varepsilon(\mathbf{u})$ and $\varphi_\varepsilon^{-1}(\mathbf{w}) = \mathbf{u}$.

We are going to define recursively solutions to a discrete-time problem, all lying in V_n (n is fixed for the moment). Recall that the initial condition \mathbf{u}^0 belongs to $L^2(\Omega)^2$. Given $\mathbf{u}^{k-1} \in L^2(\Omega)^2$, we study the following problem (for fixed $\varepsilon > 0$, $\sigma \in [0, 1]$, $k \geq 1$):

Problem $P_\sigma^\varepsilon(k, \mathbf{u}^{k-1})$:

Find $\mathbf{w}^k \in V_n^2$ such that [denoting $\mathbf{u}^k = \varphi_\varepsilon^{-1}(\mathbf{w}^k)$] we have for all $\boldsymbol{\chi} \in V_n^2$,

$$\sigma \left[\langle \boldsymbol{\chi}, \partial_\tau \mathbf{u}^k \rangle_{L^2(\Omega)} + \langle \nabla \boldsymbol{\chi}, A^\varepsilon(\mathbf{u}^k) \nabla \mathbf{w}^k \rangle_{L^2(\Omega)} - \langle \boldsymbol{\chi}, \mathbf{R}^\varepsilon(\mathbf{u}^k) \rangle_{L^2(\Omega)} \right] = -\varepsilon \langle \boldsymbol{\chi}, \mathbf{w}^k \rangle_{H^1(\Omega)}.$$

Since $\mathbf{w}^k \in V_n \subset L^\infty(\Omega)$, \mathbf{u}^k takes its values in some compact set of $]0, +\infty[^2$, hence the coefficients of $A^\varepsilon(\mathbf{u}^k)$ all belong to $L^\infty(\Omega)$, and this ensures that all the previous brackets are well-defined. Furthermore, \mathbf{u}^k clearly belongs to $L^\infty(\Omega)^2 \subset L^2(\Omega)^2$, so that the previous inductive definition is consistent.

The collection of problems $(P_\sigma^\varepsilon(k, \mathbf{u}^{k-1}))_{k=1 \dots N}$ is hence a discrete-in-time version of the V_n -weak form of (9) in which an ε -perturbation and a parameter σ have been added.

We end up this subsection by introducing (bearing in mind the notations of Definition 2.5) the:

Definition 3.3. We introduce the entropy (for some vector function $\mathbf{u} : \Omega^2 \rightarrow]0, +\infty[^2$):

$$\mathcal{E}_\varepsilon(\mathbf{u}) := \sum_{i=1}^2 \int_{\Omega} \psi_i^\varepsilon(u_i)(x) \, dx.$$

3.2 A priori estimate

Proposition 3.4. Under the assumptions of Theorem 1.2 and keeping in mind the definitions of Subsection 3.1 (and Definition 2.5), there exists a constant K_T depending only on T and the data of the equation (a_{ij} , r_i , s_{ij}) such that for any $\tau > 0$, $\varepsilon > 0$ (small enough), and $\sigma \in [0, 1]$ and any sequence (of length less than N) $(\mathbf{w}^j)_{1 \leq j \leq k}$, with \mathbf{w}^j solving $P_\sigma^\varepsilon(j, \mathbf{u}^{j-1})$, the following entropy inequality holds:

$$\sigma \mathcal{E}_\varepsilon(\mathbf{u}^k) + \tau \sigma \sum_{j=1}^k \int_{\Omega} \mathbf{Q}^\varepsilon(\mathbf{u}^j)(\nabla \mathbf{w}^j) \, dx + \varepsilon \tau \sum_{j=1}^k \|\mathbf{w}^j\|_{H^1(\Omega)}^2 \leq K_T (\mathcal{E}_\varepsilon(\mathbf{u}^0) + 1). \quad (14)$$

Proof. Plug \mathbf{w}^j in $P_\sigma^\varepsilon(j, \mathbf{u}^{j-1})$ to obtain

$$\sigma \int_{\Omega} \mathbf{w}^j \cdot \partial_\tau \mathbf{u}^j \, dx + \sigma \int_{\Omega} \mathbf{Q}^\varepsilon(\mathbf{u}^j)(\nabla \mathbf{w}^j) \, dx + \varepsilon \|\mathbf{w}^j\|_{H^1(\Omega)}^2 = \sigma \int_{\Omega} \mathbf{w}^j \cdot \mathbf{R}^\varepsilon(\mathbf{u}^j) \, dx.$$

Using (i) of Lemma 2.6, one gets, since $w_i = \psi_i^{\varepsilon'}(u_i)$,

$$\int_{\Omega} \mathbf{w}^j \cdot \partial_\tau \mathbf{u}^j \, dx = \frac{1}{\tau} \int_{\Omega} \mathbf{w}^j \cdot [\mathbf{u}^j - \mathbf{u}^{j-1}] \, dx \geq \frac{1}{\tau} (\mathcal{E}(\mathbf{u}^j) - \mathcal{E}(\mathbf{u}^{j-1})).$$

For the reaction term, we have

$$\int_{\Omega} \mathbf{w}^j \cdot \mathbf{R}^\varepsilon(\mathbf{u}^j) \, dx = \int_{\Omega} w_1^j R_{12}^\varepsilon(u_1^j, u_2^j) \, dx + \int_{\Omega} w_2^j R_{21}^\varepsilon(u_2^j, u_1^j) \, dx.$$

For the sake of clarity, we avoid writing the superscript j for a few lines. Let us focus on the first term of the right-hand side (the second one will be similar):

$$\begin{aligned} \int_{\Omega} w_1 R_{12}^\varepsilon(u_1, u_2) \, dx &= \int_{\Omega} \psi_1^{\varepsilon'}(u_1) u_1 (r_1 - \gamma_\varepsilon [s_{11}(u_1) + s_{12}(u_2)]) \, dx \\ &= r_1 \int_{\Omega} \psi_1^{\varepsilon'}(u_1) u_1 \, dx - \int_{\Omega} \psi_1^{\varepsilon'}(u_1) u_1 \gamma_\varepsilon [s_{11}(u_1) + s_{12}(u_2)] \, dx. \end{aligned}$$

From (iv) of Lemma 2.6, we get the existence of a constant D such that

$$x\psi_1^{\varepsilon'}(x) \leq D(1+\varepsilon)(1+\psi_1^\varepsilon(x)),$$

which implies

$$r_1 \int_{\Omega} \psi_1^{\varepsilon'}(u_1) u_1 dx \leq r_1 D(1+\varepsilon) (\mu(\Omega) + \mathcal{E}_\varepsilon(u)).$$

For the two other terms, we use the fact that $\psi_i^{\varepsilon'} \geq 0$ on $[1, +\infty[$, so that

$$\begin{aligned} - \int_{\Omega} \psi_1^{\varepsilon'}(u_1) u_1 \gamma_\varepsilon [s_{11}(u_1) + s_{12}(u_2)] dx &\leq - \int_{u_1 \leq 1} \psi_1^{\varepsilon'}(u_1) u_1 \gamma_\varepsilon [s_{11}(u_1) + s_{12}(u_2)], \\ - \int_{\Omega} \psi_1^{\varepsilon'}(u_1) u_1 \gamma_\varepsilon [s_{11}(u_1) + s_{12}(u_2)] dx &\leq (-B + \frac{1}{e}) \int_{\Omega} s_{11}(u_1) dx + (-B + \frac{1}{e}) \int_{\Omega} s_{12}(u_2) dx. \end{aligned}$$

Finally, we use $s_{ij}(u) \leq K(1+u)$ to get

$$- \int_{\Omega} \psi_1^{\varepsilon'}(u_1) u_1 \gamma_\varepsilon [s_{11}(u_1) + s_{12}(u_2)] dx \leq C \left(\mu(\Omega) + \int_{\Omega} [u_1 + u_2] \right).$$

By lemma 2.6 (iv), we have

$$\int_{\Omega} [u_1 + u_2] \leq D(1+\varepsilon)(\mu(\Omega) + \mathcal{E}_\varepsilon(u)).$$

Finally we have, recovering the superscripts that we omitted before:

$$\frac{\sigma}{\tau} (\mathcal{E}_\varepsilon(\mathbf{u}^j) - \mathcal{E}_\varepsilon(\mathbf{u}^{j-1})) + \sigma \int_{\Omega} \mathbf{Q}^\varepsilon(\mathbf{u}^j) (\nabla \mathbf{w}^j) dx + \varepsilon \|\mathbf{w}^j\|_{H^1(\Omega)}^2 \leq \sigma K(1 + \mathcal{E}_\varepsilon(\mathbf{u}^j)), \quad (15)$$

for some constant K independent of $j, n, \varepsilon, \tau, \sigma$. Hence if one takes τ small enough (such that $1 - \tau K \geq 1/2$), one has in particular :

$$\mathcal{E}_\varepsilon(\mathbf{u}^j) \leq \mathcal{E}_\varepsilon(\mathbf{u}^{j-1}) + \tau K \mathcal{E}_\varepsilon(\mathbf{u}^j) + \tau K,$$

which implies by Lemma 2.7, with $\theta = C := \tau K$,

$$\mathcal{E}_\varepsilon(\mathbf{u}^j) \leq e^{2j\theta} [\mathcal{E}_\varepsilon(\mathbf{u}^0) + 1] \leq e^{2j\tau K} (\mathcal{E}_\varepsilon(\mathbf{u}^0) + 1) \leq e^{2TK} (\mathcal{E}_\varepsilon(\mathbf{u}^0) + 1).$$

Plugging this last inequality in (15), we get for some constant $C_T > 0$,

$$\sigma (\mathcal{E}_\varepsilon(\mathbf{u}^j) - \mathcal{E}_\varepsilon(\mathbf{u}^{j-1})) + \tau \sigma \int_{\Omega} \mathbf{Q}^\varepsilon(\mathbf{u}^j) (\nabla \mathbf{w}^j) dx + \tau \varepsilon \|\mathbf{w}^j\|_{H^1(\Omega)}^2 \leq C_T \tau (1 + \mathcal{E}_\varepsilon(\mathbf{u}^0)),$$

which, after summation over $j \in \{1, \dots, k\}$ and using $k\tau \leq T$, ends the Proof of Proposition 3.4. \square

3.3 Existence and estimates

The a priori estimate proven in the previous subsection leads to the following proposition about existence:

Proposition 3.5. *Consider the assumptions of Theorem 1.2 and recall the Definitions of Subsection 3.1 (and Definition 2.5).*

For fixed $\tau (= T/N)$ (small enough), $n \in \mathbb{N}^$, $\mathbf{w}^0 \in V_n^2$ and $\varepsilon > 0$, there exists a sequence $(\mathbf{w}^k)_{1 \leq k \leq N}$ such that \mathbf{w}^k solves $P_1^\varepsilon(k, \mathbf{u}^{k-1})$ for all $k \in \{1, \dots, N\}$, and which furthermore satisfies the estimate (with K_T depending only on T and the data of the equation (a_{ij}, r_i, s_{ij})):*

$$\mathcal{E}_\varepsilon(\mathbf{u}^k) + \tau \sum_{j=1}^k \int_{\Omega} \mathbf{Q}^\varepsilon(\mathbf{u}^k) (\nabla \mathbf{w}^k) dx + \varepsilon \tau \sum_{j=1}^k \|\mathbf{w}^j\|_{H^1(\Omega)}^2 \leq K_T (\mathcal{E}_\varepsilon(\mathbf{u}^0) + 1), \quad (16)$$

$$\|\mathbf{u}^k\|_{L^1(\Omega)} \leq [\|\mathbf{u}^0\|_{L^1(\Omega)} + O(\sqrt{\varepsilon})] e^{r^\tau T}, \quad (17)$$

$$\tau \sum_{j=1}^k \|\mathbf{R}^{-, \varepsilon}(\mathbf{u}^j)\|_{L^1(\Omega)} \leq [\|\mathbf{u}^0\|_{L^1(\Omega)} + O(\sqrt{\varepsilon})] (1 + T \mathbf{r} e^{r^\tau T}), \quad (18)$$

where the sequence of positive scalars $(r^\tau)_\tau$ goes to $\mathbf{r} = \max(r_1, r_2)$ when $\tau \rightarrow 0$.

Proof. Step 1: Estimates

Let us notice that the two last estimates (17)–(18) are essentially a discrete version of the classical formal L^1 estimate obtained by integrating our system on Ω . In order to recover this rigorously, we notice that since $\mathbf{1}_\Omega \in V_n$, it is an admissible test function for the problem $P_1^\varepsilon(j, \mathbf{u}^{j-1})$ for all $j \in \llbracket 1, k \rrbracket$, $k \in \llbracket 1, N \rrbracket$, and we hence get

$$\|\mathbf{u}^j\|_{L^1(\Omega)} - \|\mathbf{u}^{j-1}\|_{L^1(\Omega)} + \varepsilon \tau \int_{\Omega} \mathbf{w}^j + \tau \|\mathbf{R}^{-, \varepsilon}(\mathbf{u}^j)\|_{L^1(\Omega)} = \tau \|\mathbf{R}^+(\mathbf{u}^j)\|_{L^1(\Omega)}. \quad (19)$$

It follows easily from the definition of \mathbf{R}^+ that the right-hand side of the previous inequality is not larger than $\tau \mathbf{r} \|\mathbf{u}^j\|_{L^1(\Omega)}$. Thus,

$$\|\mathbf{u}^j\|_{L^1(\Omega)} \leq \|\mathbf{u}^{j-1}\|_{L^1(\Omega)} + \tau \mathbf{r} \|\mathbf{u}^j\|_{L^1(\Omega)} - \varepsilon \tau \int_{\Omega} [\mathbf{w}_1^j + \mathbf{w}_2^j].$$

We notice then that,

$$-\varepsilon\tau \int_{\Omega} \mathbf{w}_1^j + \mathbf{w}_2^j = \sqrt{\varepsilon}\tau \left(|\Omega| + \varepsilon \|\mathbf{w}^j\|_{L^2(\Omega)}^2 \right).$$

This leads to

$$\|\mathbf{u}^j\|_{L^1(\Omega)} \leq \|\mathbf{u}^{j-1}\|_{L^1(\Omega)} + \tau r \|\mathbf{u}^j\|_{L^1(\Omega)} + \sqrt{\varepsilon}\tau \left(|\Omega| + \varepsilon \|\mathbf{w}^j\|_{L^2(\Omega)}^2 \right),$$

which implies by Lemma 2.7

$$\|\mathbf{u}^j\|_{L^1(\Omega)} \leq \left[\|\mathbf{u}^0\|_{L^1(\Omega)} + \sqrt{\varepsilon} \left(j\tau|\Omega| + \varepsilon\tau \sum_{k=1}^j \|\mathbf{w}^k\|_{L^2(\Omega)}^2 \right) \right] \exp \left\{ j\tau \frac{r}{1-\tau r} \right\}.$$

Using the entropy estimate, one has

$$\varepsilon\tau \sum_{k=1}^j \|\mathbf{w}^k\|_{L^2(\Omega)}^2 \leq K_T(\mathcal{E}_\varepsilon(\mathbf{u}^0) + 1) \quad \text{and} \quad j\tau|\Omega| \leq |\Omega|T.$$

Therefore, we obtain

$$\|\mathbf{u}^j\|_{L^1(\Omega)} \leq \left[\|\mathbf{u}^0\|_{L^1(\Omega)} + \sqrt{\varepsilon} (T|\Omega| + K_T(\mathcal{E}_\varepsilon(\mathbf{u}^0) + 1)) \right] e^{r^\tau T},$$

with $r(\tau)$ having the mentioned property, that is precisely (17).

Quite similarly, we sum up in (19) and obtain after using the bound on $\|\mathbf{w}\|_1$:

$$\|\mathbf{u}^k\|_{L^1(\Omega)} + \tau \sum_{j=1}^k \|\mathbf{R}^{-,\varepsilon}(\mathbf{u}^j)\|_{L^1(\Omega)} = \tau \sum_{j=1}^k \|\mathbf{R}^+(\mathbf{u}^j)\|_{L^1(\Omega)} + \sqrt{\varepsilon} (T|\Omega| + K_T(\mathcal{E}_\varepsilon(\mathbf{u}^0) + 1)),$$

and we conclude the estimate using the previous bound.

Step 2: Existence

In the rest of the Proof, we will work in the finite dimensional Hilbert space $E = V_n^2$ with the dot product $\langle \cdot, \cdot \rangle_{H^1(\Omega)}$ and the associated norm. We will proceed by induction and only prove the first iteration $\mathbf{u}^0 \in L^2(\Omega)^2 \Rightarrow \exists \mathbf{w}^1$ solving $P_\sigma^\varepsilon(1, \mathbf{u}^0)$, all the other induction steps will be similar.

First notice that the problem $P_\sigma^\varepsilon(1, \mathbf{u}^0)$ can be seen in the following way

$$\text{Find } \mathbf{w}^1 \in E \text{ such that } \forall \chi \in E, \sigma L_{\mathbf{u}^0, \mathbf{w}^1}(\chi) = -\varepsilon \langle \chi, \mathbf{w}^1 \rangle_{H^1(\Omega)},$$

where, for $\mathbf{w}, \chi \in E$, $L_{\mathbf{u}^0, \mathbf{w}}$ is defined by

$$L_{\mathbf{u}^0, \mathbf{w}}(\chi) := \frac{1}{\tau} \langle \chi, \varphi_\varepsilon^{-1}(\mathbf{w}) - \mathbf{u}^0 \rangle_{L^2(\Omega)} + \langle \nabla \chi, A^\varepsilon(\varphi_\varepsilon^{-1}(\mathbf{w})) \nabla \mathbf{w} \rangle_{L^2(\Omega)} - \langle \chi, \mathbf{R}^\varepsilon(\varphi_\varepsilon^{-1}(\mathbf{w})) \rangle_{L^2(\Omega)}.$$

As noticed before, for $\mathbf{w} \in E \subset L^\infty(\Omega)^2$, $\varphi_\varepsilon^{-1}(\mathbf{w})$ takes its values in some compact set of $]0, +\infty[^2$, so that the coefficients of $A^\varepsilon(\varphi_\varepsilon^{-1}(\mathbf{w}))$ lie in $L^\infty(\Omega)$, whence $L_{\mathbf{u}^0, \mathbf{w}^1} \in V_n^*$.

Let us now define a map

$$T : [0, 1] \times E \longrightarrow E \\ (\sigma, \mathbf{v}) \longmapsto \mathbf{w},$$

such that:

$$\forall \chi \in E \quad \sigma L_{\mathbf{u}^0, \mathbf{v}}(\chi) = -\varepsilon \langle \chi, \mathbf{w} \rangle_{H^1(\Omega)}.$$

Such a map is well-defined because of the usual representation theorem for finite dimensional spaces.

Lemma 3.6. *The map T is continuous.*

Proof. Let $v^1 \in E$. We want to prove that T is continuous at v^1 . Since all norms are equivalent on E , one can use the metric $|\cdot| + \|\cdot\|_{L^\infty(\Omega)}$ on $[0, 1] \times E$ and $\|\cdot\|_{H^1(\Omega)}$ on E . In all what follows, we assume that v^2 belongs to the following open set of E

$$\|v^1 - v^2\|_\infty + \|v^1 - v^2\|_{H^1} < 1.$$

Not that this means that v^2 belongs to a compact set of E (just consider the inequalities as large) that we denote K . In fact for $\|\chi\|_{H^1} \leq 1$, one has

$$\begin{aligned} \left| [L_{\mathbf{u}^0, v^1} - L_{\mathbf{u}^0, v^2}](\chi) \right| &\leq \frac{1}{\tau} \|\varphi_\varepsilon^{-1}(v^1) - \varphi_\varepsilon^{-1}(v^2)\|_{L^2(\Omega)} \\ &\quad + \|A^\varepsilon(\varphi_\varepsilon^{-1}(v^1)) \nabla v^1 - A^\varepsilon(\varphi_\varepsilon^{-1}(v^2)) \nabla v^2\|_{L^2(\Omega)}, \\ &\quad + \|\mathbf{R}^\varepsilon(\varphi_\varepsilon^{-1}(v^1)) - \mathbf{R}^\varepsilon(\varphi_\varepsilon^{-1}(v^2))\|_{L^2(\Omega)}, \end{aligned}$$

and one can write for the second term of the right-hand side:

$$\begin{aligned} \|A^\varepsilon(\varphi_\varepsilon^{-1}(v^1)) \nabla v^1 - A^\varepsilon(\varphi_\varepsilon^{-1}(v^2)) \nabla v^2\|_{L^2(\Omega)} &\leq \|\{A^\varepsilon(\varphi_\varepsilon^{-1}(v^1)) - A^\varepsilon(\varphi_\varepsilon^{-1}(v^2))\} \nabla v^1\|_{L^2(\Omega)} \\ &\quad + \|A^\varepsilon(\varphi_\varepsilon^{-1}(v^2)) \{\nabla v^1 - \nabla v^2\}\|_{L^2(\Omega)}. \end{aligned}$$

Since φ_ε^{-1} , $A^\varepsilon \circ \varphi_\varepsilon^{-1}$ and $\mathbf{R}^\varepsilon \circ \varphi_\varepsilon^{-1}$ are all continuous, they are uniformly continuous (and bounded) on every compact of \mathbb{R}^2 , and particularly on K . Let ω_K be a continuity modulus for all the previous functions on this compact. Hence, if $\mathbf{v}_1, \mathbf{v}_2$ take their values in K and $\|\mathbf{v}_1 - \mathbf{v}_2\|_{L^\infty(\Omega)} \leq \eta$, we have

$$\begin{aligned} \left| [\mathbf{L}_{\mathbf{u}^0, \mathbf{v}_1} - \mathbf{L}_{\mathbf{u}^0, \mathbf{v}_2}](\chi) \right| &\leq \frac{\omega_K(\eta)}{\tau} + \omega_K(\eta) \mu(\Omega)^{1/2} \|\nabla \mathbf{v}_1\|_{L^\infty(\Omega)} \\ &\quad + \|A^\varepsilon(\varphi_\varepsilon^{-1}(\mathbf{v}_2))\|_{L^\infty(\Omega)} \|\mathbf{v}_1 - \mathbf{v}_2\|_{H^1(\Omega)} + \omega_K(\eta), \end{aligned}$$

where we used the continuous injection $L^\infty(\Omega) \hookrightarrow L^2(\Omega)$. Since all norms are equivalent on E ,

$$\left| [\mathbf{L}_{\mathbf{u}^0, \mathbf{v}_1} - \mathbf{L}_{\mathbf{u}^0, \mathbf{v}_2}](\chi) \right| \leq C_{K, \Omega}(\omega_K(\eta) + \eta),$$

hence using the very definition of T :

$$\|T(\sigma, \mathbf{v}_1) - T(\sigma, \mathbf{v}_2)\|_{H^1(\Omega)} = \frac{\sigma}{\varepsilon} \sup_{\|\chi\|_{H^1} \leq 1} \left| [\mathbf{L}_{\mathbf{u}^0, \mathbf{v}_1} - \mathbf{L}_{\mathbf{u}^0, \mathbf{v}_2}](\chi) \right| \leq \frac{\sigma}{\varepsilon} C_{K, \Omega}(\omega_K(\eta) + \eta),$$

which gives the continuity w.r.t. the second variable. The continuity w.r.t. both variables is then straightforward. \square

It is now time to use Proposition 3.4 to get an *a priori* estimate on any fixed point of $T(\sigma, \cdot)$, for any $\sigma \in [0, 1]$. In fact, the case $k = 1$ of this Proposition exactly tells us that all this fixed points are in the ball of center 0 and radius $K_T(\mathcal{E}(\mathbf{u}^0) + 1)/\varepsilon\tau$. Since clearly $T(0, \cdot) \equiv 0$, we now can apply Theorem 2.8 to see that $T(1, \cdot)$ has a fixed point, which is exactly the existence of \mathbf{w}^1 and hence the first step of our induction machinery. Inequality (16) is then a direct consequence of (14) (with $\sigma = 1$). \square

The previous Proposition shows that (for fixed $\tau(=T/N)$ small enough, $n \in \mathbb{N}^*$, $\mathbf{u}^0 \in L^2(\Omega)^2$, and $\varepsilon > 0$) there exists a sequence $(\mathbf{w}^k)_{1 \leq k \leq N}$ such that, for all $k \in \{1, \dots, N\}$, denoting $\mathbf{u}^k := \varphi_\varepsilon^{-1}(\mathbf{w}^k)$, we have for all $\chi \in V_n^2$:

$$\left\langle \chi, \partial_\tau \mathbf{u}^k \right\rangle_{L^2(\Omega)} + \left\langle \nabla \chi, A^\varepsilon(\mathbf{u}^k) \nabla \mathbf{w}^k \right\rangle_{L^2(\Omega)} + \varepsilon \left\langle \chi, \mathbf{w}^k \right\rangle_{H^1(\Omega)} = \left\langle \chi, \mathbf{R}^\varepsilon(\mathbf{u}^k) \right\rangle_{L^2(\Omega)}, \quad (20)$$

that we also can write ($i \neq j = 1, 2$)

$$\partial_\tau \mathbb{P}_n u_i^k - \mathbb{P}_n \Delta \left[a_{ii}^\varepsilon(u_i^k) + u_i^k a_{ij}(u_j^k) + \varepsilon u_i^k u_j^k \right] + \varepsilon w_i^k - \varepsilon \Delta w_i^k = \mathbb{P}_n \left[R_{ij}^\varepsilon(u_i^k, u_j^k) \right],$$

where \mathbb{P}_n is the L^2 -orthonormal projection on V_n .

4 Asymptotic with respect to n

This Section is devoted to the passage to the limit in the space discretization ($n \rightarrow +\infty$), which can be summarized by the following Proposition:

Proposition 4.1. *Consider the assumptions of Theorem 1.2 and recall the notations of Definitions 2.5, 3.3, 3.2 and 3.1. For fixed $\tau(=T/N)$ (small enough), and $\varepsilon > 0$, there exists a sequence $(\mathbf{u}^k)_{1 \leq k \leq N}$ of $L^p(\Omega)^2$ for some $p > 1$ (and a corresponding sequence $(\mathbf{w}^k)_{1 \leq k \leq N}$) such that for all $\chi \in \mathcal{C}_c^\infty(\overline{\Omega})^2$ (and for all $k \in \{1, \dots, N\}$),*

$$\begin{aligned} \frac{1}{\tau} \langle \chi_i, u_i^k \rangle_{L^2(\Omega)} - \frac{1}{\tau} \langle \chi_i, u_i^{k-1} \rangle_{L^2(\Omega)} - \langle \Delta \chi_i, [a_{ii}^\varepsilon(u_i^k) + u_i^k a_{ij}(u_j^k) + \varepsilon u_i^k u_j^k] \rangle_{L^2(\Omega)} \\ + \varepsilon \langle \chi_i - \Delta \chi_i, w_i^k \rangle_{L^2(\Omega)} = \langle \chi_i, R_{ij}^\varepsilon(u_i^k, u_j^k) \rangle_{L^2(\Omega)}, \end{aligned} \quad (21)$$

and which furthermore satisfies the estimates (with K_T depending only on T and the data of the equation (a_{ij}, r_i, s_{ij})):

$$\mathcal{E}_\varepsilon(\mathbf{u}^k) + \varepsilon \tau \sum_{j=1}^k \|\mathbf{w}^j\|_{H^1(\Omega)}^2 \leq K_T(\mathcal{E}_\varepsilon(\mathbf{u}^0) + 1); \quad (22)$$

$$\tau \sum_{j=1}^k \int_\Omega |\nabla \beta_\alpha(\mathbf{u}^j)|^2 dx \leq K_T(\mathcal{E}_\varepsilon(\mathbf{u}^0) + 1), \quad (23)$$

where $\beta_\alpha(x) = x^{\frac{1-\alpha}{2}}$;

$$\|\mathbf{u}^k\|_{L^1(\Omega)} \leq \|\mathbf{u}^0\|_{L^1(\Omega)} e^{T r \tau}; \quad (24)$$

$$\varepsilon \tau \sum_{j=1}^k \|\mathbf{w}^j\|_{L^1(\Omega)} + \tau \sum_{j=1}^k \|\mathbf{R}^{\varepsilon, j}(\mathbf{u}^j)\|_{L^1(\Omega)} \leq \|\mathbf{u}^0\|_{L^1(\Omega)} \left[1 + T r e^{\tau(\tau)T} \right]. \quad (25)$$

Proof. We first recall the bounds that hold on \mathbf{u}_n^k (and \mathbf{w}_n^k), whose existence is given by Proposition 3.5.

Using (12), for all $k \in \{1, \dots, N\}$ and $n \in \mathbb{N}^*$, we have

$$\mathbf{Q}^\varepsilon(\mathbf{u}_n^k)(\nabla \mathbf{w}_n^k) \geq \frac{1}{u_1^{k,n}} a_{21}^\varepsilon(u_1^{k,n}) |\nabla u_1^{k,n}|^2 + \frac{1}{u_2^{k,n}} a_{12}^\varepsilon(u_2^{k,n}) |\nabla u_2^{k,n}|^2.$$

Since $a_{ij}^\varepsilon(x) = a_{ij}(x) + \varepsilon x$, assumption **H2** leads to

$$\mathbf{Q}^\varepsilon(\mathbf{u}_n^k)(\nabla \mathbf{w}_n^k) \geq \frac{4}{(1-\alpha)^2} \left| \nabla \beta_\alpha(u_1^{k,n}) \right|^2 + \frac{4}{(1-\alpha)^2} \left| \nabla \beta_\alpha(u_2^{k,n}) \right|^2 + 4\varepsilon \left| \nabla \sqrt{u_1^{k,n}} \right|^2 + 4\varepsilon \left| \nabla \sqrt{u_2^{k,n}} \right|^2, \quad (26)$$

(where we recall that $\beta_\alpha(x) := x^{\frac{1-\alpha}{2}}$). We also have, because of (17), the boundedness of $(\mathbf{u}_n^k)_{n \in \mathbb{N}^*}$ in $L^1(\Omega)$. Since the asymptotics that we are studying is only w.r.t. n (that is, ε and τ are fixed), we see finally that $\left(\sqrt{u_1^{k,n}}\right)_{n \in \mathbb{N}^*}$ and $\left(\sqrt{u_2^{k,n}}\right)_{n \in \mathbb{N}^*}$ are bounded in $H^1(\Omega) \hookrightarrow L^{2^*}(\Omega)$, with $2^* = 2d/(d-2) > 2$, and hence $(\mathbf{u}_n^k)_{n \in \mathbb{N}^*}$ is eventually bounded in some $L^p(\Omega)$ space with $1 < p < \infty$.

On the other hand, estimate (13) gives us

$$\mathcal{Q}^\varepsilon(\mathbf{u}_n^k)(\mathbf{w}_n^k) \geq 4 \left| \nabla \sqrt{a_{21}^\varepsilon(u_1^{k,n}) a_{12}^\varepsilon(u_2^{k,n})} \right|^2,$$

which together with (16) leads to

$$\left\| \nabla \sqrt{a_{21}^\varepsilon(u_1^{k,n}) a_{12}^\varepsilon(u_2^{k,n})} \right\|_{L^2(\Omega)}^2 \leq \frac{1}{4\tau} K_T(\mathcal{E}_\varepsilon(\mathbf{u}^0) + 1).$$

Using Poincaré-Wirtinger inequality, we get, for some constant C_Ω depending only on Ω

$$\begin{aligned} \left\| \sqrt{a_{21}^\varepsilon(u_1^{k,n}) a_{12}^\varepsilon(u_2^{k,n})} \right\|_{L^2(\Omega)} &\leq C_\Omega \left\| \sqrt{a_{21}^\varepsilon(u_1^{k,n}) a_{12}^\varepsilon(u_2^{k,n})} \right\|_{L^1(\Omega)} + \frac{C_\Omega}{2\sqrt{\tau}} \sqrt{K_T(\mathcal{E}_\varepsilon(\mathbf{u}^0) + 1)} \\ &\leq C_\Omega \sqrt{\|a_{21}^\varepsilon(u_1^{k,n})\|_{L^1(\Omega)} \|a_{12}^\varepsilon(u_2^{k,n})\|_{L^1(\Omega)}} + \frac{C_\Omega}{2\sqrt{\tau}} \sqrt{K_T(\mathcal{E}_\varepsilon(\mathbf{u}^0) + 1)} \\ &\leq C_\Omega \left[\|a_{21}^\varepsilon(u_1^{k,n})\|_{L^1(\Omega)} + \|a_{12}^\varepsilon(u_2^{k,n})\|_{L^1(\Omega)} \right] + \frac{C_\Omega}{2\sqrt{\tau}} \sqrt{K_T(\mathcal{E}_\varepsilon(\mathbf{u}^0) + 1)}, \end{aligned}$$

and, again because point (iv) of Lemma 2.6, for some constant D ($\varepsilon < 1$),

$$a_{ji}^\varepsilon(u_i^{k,n}) \leq D(2 + \psi_i^\varepsilon(u_i^{k,n})),$$

so we finally have

$$\left\| \sqrt{a_{21}^\varepsilon(u_1^{k,n}) a_{12}^\varepsilon(u_2^{k,n})} \right\|_{L^2(\Omega)} \leq C_\Omega \left[8D\mu(\Omega) + 2D\mathcal{E}_\varepsilon(\mathbf{u}_n^k) \right] + \frac{1}{2\sqrt{\tau}} \sqrt{K_T(\mathcal{E}_\varepsilon(\mathbf{u}^0) + 1)},$$

and hence thanks to (16), the sequence $\left(\sqrt{a_{21}^\varepsilon(u_1^{k,n}) a_{12}^\varepsilon(u_2^{k,n})}\right)_{n \in \mathbb{N}^*}$ is bounded in $H^1(\Omega) \hookrightarrow L^{2^*}(\Omega)$, with $2^* = 2d/(d-2) > 2$. The previous continuous injection implies then that $(a_{21}^\varepsilon(u_1^{k,n}) a_{12}^\varepsilon(u_2^{k,n}))_{n \in \mathbb{N}^*}$ is bounded in some $L^p(\Omega)$ space with $1 < p < \infty$ and since $a_{ji}^\varepsilon(u_i^{k,n}) = a_{ji}(u_i^{k,n}) + \varepsilon u_i^{k,n}$, we eventually get that $(u_1^{k,n} u_2^{k,n})_{n \in \mathbb{N}^*}$, $(u_1^{k,n} a_{12}^\varepsilon(u_2^{k,n}))_{n \in \mathbb{N}^*}$ and $(u_2^{k,n} a_{21}^\varepsilon(u_1^{k,n}))_{n \in \mathbb{N}^*}$ are bounded in the same $L^p(\Omega)$.

Summing the bounds already obtained [we recall that they hold for a given ε and τ], we see that (for all $k \in \{1, \dots, N\}$):

- $\left(\sqrt{u_1^{k,n}}\right)_{n \in \mathbb{N}^*}$ and $\left(\sqrt{u_2^{k,n}}\right)_{n \in \mathbb{N}^*}$ are bounded in $H^1(\Omega)$,
- $\left(\sqrt{a_{21}^\varepsilon(u_1^{k,n}) a_{12}^\varepsilon(u_2^{k,n})}\right)_{n \in \mathbb{N}^*}$ is bounded in $H^1(\Omega)$,
- $(u_1^{k,n})_{n \in \mathbb{N}^*}$, $(u_2^{k,n})_{n \in \mathbb{N}^*}$, $(u_1^{k,n} u_2^{k,n})_{n \in \mathbb{N}^*}$, $(u_1^{k,n} a_{12}^\varepsilon(u_2^{k,n}))_{n \in \mathbb{N}^*}$ and $(u_2^{k,n} a_{21}^\varepsilon(u_1^{k,n}))_{n \in \mathbb{N}^*}$ are bounded in some $L^p(\Omega)$ space, with $1 < p < \infty$,
- and obviously $(\mathbf{w}_n^k)_{n \in \mathbb{N}^*}$ is bounded in $H^1(\Omega)$, because of estimate (16).

Since we are only dealing with a finite number of values for $k \in \{1, \dots, N\}$ (at this point, the functions are not time-depending), we shall only detail the study of $(\mathbf{u}_n^1)_{n \in \mathbb{N}^*}$, the other values of k being similar (one only has to extract a finite number of subsequences). For every test function $\chi = (\chi_1, \chi_2) \in V_n^2$, the weak formulation (20) may be written ($i \neq j \in \{1, 2\}$)

$$\begin{aligned} \frac{1}{\tau} \langle \chi_i, u_i^{1,n} \rangle_{L^2(\Omega)} - \frac{1}{\tau} \langle \chi_i, u_i^{0,n} \rangle_{L^2(\Omega)} - \langle \Delta \chi_i, [a_{ii}^\varepsilon(u_i^{1,n}) + u_i^{1,n} a_{ij}(u_j^{1,n}) + \varepsilon u_i^{1,n} u_j^{1,n}] \rangle_{L^2(\Omega)} \\ + \varepsilon \langle \chi_i - \Delta \chi_i, w_i^{1,n} \rangle_{L^2(\Omega)} = \langle \chi_i, R_{ij}^\varepsilon(u_i^{1,n}, u_j^{1,n}) \rangle. \end{aligned}$$

First we extract (but do not change the indexes) a subsequence of $(\mathbf{u}_n^1)_{n \in \mathbb{N}^*}$ converging in $L^2(\Omega)$ and almost everywhere to some element $\mathbf{w}^1 \in H^1(\Omega)^2$. The almost everywhere convergence is transmitted to $(\mathbf{u}_n^1)_{n \in \mathbb{N}^*}$, since $\mathbf{u}_n^1 = \varphi_\varepsilon^{-1}(\mathbf{w}_n^1)$, the φ_i^ε functions being homeomorphisms. Because of the $L^p(\Omega)$ ($1 < p < \infty$) bound for $(\mathbf{u}_n^1)_{n \in \mathbb{N}^*}$, we can extract a subsequence (of the previous subsequence) converging weakly in $L^p(\Omega)$ to some function $\mathbf{v}^1 \in L^1(\Omega)^2$. It follows by a classical argument that \mathbf{v}^1 is almost everywhere equal to $\mathbf{u}^1 := \varphi_\varepsilon^{-1}(\mathbf{w}^1)$, that is the almost everywhere limit of $(\mathbf{u}_n^1)_{n \in \mathbb{N}^*}$. The same argument holds for $(u_1^{1,n} u_2^{1,n})_{n \in \mathbb{N}^*}$ and $(u_i^{1,n} a_{ij}(u_j^{1,n}))_{n \in \mathbb{N}^*}$, $i \neq j = 1, 2$, since we have the same $L^p(\Omega)$ bound. We may also assume¹ that $(\mathbf{u}^{0,n})_{n \in \mathbb{N}^*}$ converges weakly in $L^p(\Omega)$ to some function $\mathbf{u}^0 \in L^p(\Omega)$. Eventually, the cutoff perturbation introduced on a_{ii}^ε and the (superlinear) reaction terms ensures the weak convergence, in $L^p(\Omega)$, of $(a_{ii}^\varepsilon(u_i^{1,n}))_{n \in \mathbb{N}^*}$ and $(\mathbf{R}^\varepsilon(\mathbf{u}^{1,n}))_{n \in \mathbb{N}^*}$, respectively to $a_{ii}^\varepsilon(u_i^1)$ and $\mathbf{R}^\varepsilon(\mathbf{u}^1)$.

¹In fact we have $\mathbf{u}^{0,n} = \mathbf{u}^0$! This line is just to justify the handling of the same term in the other time steps.

We now fix a test function χ in $\cup_{n \in \mathbb{N}^*} V_n^2$, say $\chi \in V_m^2$, $m \in \mathbb{N}^*$. Then, for n greater than m , since the sequence of spaces V_n is increasing, we have the previous weak formulation that passes to the limit thanks to all the extractions that we made, and get eq. (21).

It remains to prove the bounds announced in the Proposition.

Since we have almost everywhere convergence of $(u_n^k)_{n \in \mathbb{N}}$ and weak $H^1(\Omega)$ convergence of $(w_n^k)_{n \in \mathbb{N}}$, Fatou's Lemma and the classical estimate for the weak limits give directly, for all $k \in \llbracket 1, N \rrbracket$, estimate (22).

We use then (26) that we rewrite here:

$$Q^\varepsilon(u_n^k)(\nabla w_n^k) \geq \frac{4}{(1-\alpha)^2} \left| \nabla \beta_\alpha(u_1^{k,n}) \right|^2 + \frac{4}{(1-\alpha)^2} \left| \nabla \beta_\alpha(u_2^{k,n}) \right|^2 + 4\varepsilon \left| \nabla \sqrt{u_1^{k,n}} \right|^2 + 4\varepsilon \left| \nabla \sqrt{u_2^{k,n}} \right|^2,$$

where we recall that $\beta_\alpha(x) = x^{\frac{1-\alpha}{2}}$. We notice that for all $j \in \llbracket 1, k \rrbracket$, the sequence $(\beta_\alpha(u_n^j))_{n \in \mathbb{N}}$ was, precisely in view of the previous inequality, bounded in $H^1(\Omega)$, so that we can (adding another extraction) assume that we also had the convergence $(\beta_\alpha(u_n^j))_{n \in \mathbb{N}} \rightharpoonup \beta_\alpha(u^j)$ in $H^1(\Omega)$ (using the uniqueness of the weak limit) and we hence have estimate (23) using Fatou's Lemma.

As for (17) and (18), we notice that we had a $L^p(\Omega)$ ($p > 1$) bound for both $(u^{j,n})_{n \in \mathbb{N}^*}$ and $(R^{-,\varepsilon}(u^{j,n}))_{n \in \mathbb{N}^*}$, associated with almost everywhere convergence. This is sufficient (using Egoroff's theorem) to get the (strong) convergence of these two sequences in $L^1(\Omega)$. Estimates (17) and (18) give hence (24), and (25) after taking the limit in n . \square

5 Duality Estimate

5.1 Notations

Definition 5.1. For a given family $h := (h^k)_{1 \leq k \leq N}$ of functions defined on Ω , we denote by \underline{h}^τ the step (in time) function defined on $]0, T] \times \Omega$ by

$$\underline{h}^\tau(t, x) := \sum_{k=1}^N h^k(x) \mathbf{1}_{[(k-1)\tau, k\tau]}(t).$$

We then have by definition, for all $p, q \in [1, \infty[$,

$$\|\underline{h}^\tau\|_{L^q([0, T]; L^p(\Omega))} = \left(\sum_{k=1}^N \tau \|h^k\|_{L^p(\Omega)}^q \right)^{1/q},$$

and in particular

$$\|\underline{h}^\tau\|_{L^p(Q_T)} = \left(\sum_{k=1}^N \tau \int_\Omega |h^k(x)|^p dx \right)^{1/p}.$$

5.2 Duality estimate: abstract result

This subsection is devoted to the establishment of a discretized version of the duality estimates devised for singular parabolic equations in [23], [21]. We start with the following Lemma:

Lemma 5.2. Consider a smooth function $b \in \mathcal{C}^\infty(\Omega)$ such that $b \geq \gamma > 0$ for some constant γ . For a given $\Psi \in L^2(\Omega)$, the variational problem associated to the equation

$$-\Phi + b\Delta\Phi = \Psi \tag{27}$$

is well-posed in $H_\nu^2(\Omega)$ (with $L^2(\Omega)$ test functions). If Ψ is assumed to be nonpositive, then the solution of the previous equation is nonnegative.

Proof. Since b is bounded from below (by γ) and above, we can first (uniquely) solve the (variational formulation of the) problem

$$-b^{-1}\Phi + \Delta\Phi = b^{-1}\Psi, \tag{28}$$

in the Hilbert space $H^1(\Omega)$, using Lax-Milgram Theorem.

Elliptic regularity ensures that the constructed solution lies in fact in $H^2(\Omega)$, and even in $H_\nu^2(\Omega)$ because of the variational formulation, so that (28) is satisfied almost everywhere and we recover a solution of (27) after a multiplication by b . Uniqueness is straightforward. All $L^2(\Omega)$ test functions are obtained by density. The last part of the Lemma is obtained by maximum principle. \square

We then turn to the:

Lemma 5.3. Consider a real number $r > 0$ such that $1 - 2r\tau > 0$ and two families $b := (b^k)_{1 \leq k \leq N}$, $F := (F^k)_{1 \leq k \leq N}$ of $\mathcal{C}^\infty(\Omega)$ functions. Assume that for all $k \in \llbracket 1, N \rrbracket$, $b^k \geq 1$ and $F^k \leq 0$ (pointwise).

Then there exists a family $\Phi := (\Phi^k)_{1 \leq k \leq N} \in H_\nu^2(\Omega)^N$ of nonnegative functions such that, (defining $\Phi^{N+1} := 0$),

$$\forall k \in \llbracket 1, N \rrbracket, \quad \frac{\Phi^{k+1} - \Phi^k}{\tau} + b^k \Delta \Phi^k = \sqrt{b^k} F^k - r \Phi^k, \tag{29}$$

where eq. (29) has to be understood weakly, against $L^2(\Omega)$ test functions.

This family satisfies furthermore

$$\forall j \in \llbracket 1, N \rrbracket, \quad \|\nabla \Phi^j\|_{L^2(\Omega)} \leq e^{r(\tau)T} \|\underline{F}^\tau\|_{L^2(Q_T)}, \quad (30)$$

$$\|\sqrt{b^\tau} \Delta \Phi^\tau\|_{L^2(Q_T)} \leq e^{r(\tau)T} \|\underline{F}^\tau\|_{L^2(Q_T)}, \quad (31)$$

$$\forall j \in \llbracket 1, N \rrbracket, \quad \|\Phi^j\|_{H^1(\Omega)} \leq e^{r(\tau)T} C_\Omega \left[e^{r(\tau)T} + \|\sqrt{b^\tau}\|_{L^2(Q_T)} \right] \|\underline{F}^\tau\|_{L^2(Q_T)}, \quad (32)$$

where the sequence of positive scalars $(r(\tau))_\tau$ goes to r when $\tau \rightarrow 0$.

Proof. A straightforward induction using Lemma 5.2 gives us step by step the existence of the family $(\Phi^k)_{1 \leq k \leq N}$ and the nonnegativity of its elements. Now in the k -th equation, $\tau \Delta \Phi^k \in L^2(\Omega)$ is an admissible test function and we have hence after integration by parts:

$$\int_\Omega |\nabla \Phi^k|^2 dx - \int_\Omega \nabla \Phi^{k+1} \cdot \nabla \Phi^k dx + \tau \int_\Omega b^k |\Delta \Phi^k|^2 dx = \tau \int_\Omega \sqrt{b^k} F^k \Delta \Phi^k dx + r\tau \int_\Omega |\nabla \Phi^k|^2 dx.$$

Using

$$\sqrt{b^k} F^k \Delta \Phi^k \leq \frac{|F^k|^2}{2} + \frac{b^k |\Delta \Phi^k|^2}{2}$$

and

$$|\nabla \Phi^k|^2 - \nabla \Phi^{k+1} \cdot \nabla \Phi^k \geq \frac{1}{2} \left(|\nabla \Phi^k|^2 - |\nabla \Phi^{k+1}|^2 \right),$$

we get, for all $k \in \llbracket 1, N \rrbracket$

$$\int_\Omega |\nabla \Phi^k|^2 dx - \int_\Omega |\nabla \Phi^{k+1}|^2 dx + \tau \int_\Omega b^k |\Delta \Phi^k|^2 dx \leq \tau \int_\Omega |F^k|^2 dx + 2r\tau \int_\Omega |\nabla \Phi^k|^2 dx.$$

Let us introduce the auxiliary sequences $(1 \leq k \leq N)$:

$$\Psi^k := \frac{\Phi^k}{(1 - 2r\tau)^{k/2}},$$

$$G^k := \frac{F^k}{(1 - 2r\tau)^{k/2}},$$

we have then, dividing the previous inequality by $(1 - 2r\tau)^{k+1}$,

$$\frac{1}{1 - 2r\tau} \int_\Omega |\nabla \Psi^k|^2 dx - \int_\Omega |\nabla \Psi^{k+1}|^2 dx + \frac{\tau}{1 - 2r\tau} \int_\Omega b^k |\Delta \Psi^k|^2 dx \leq \frac{\tau}{1 - 2r\tau} \int_\Omega |G^k|^2 dx + \frac{2r\tau}{1 - 2r\tau} \int_\Omega |\nabla \Psi^k|^2 dx,$$

hence

$$\int_\Omega |\nabla \Psi^k|^2 dx - \int_\Omega |\nabla \Psi^{k+1}|^2 dx + \frac{\tau}{1 - 2r\tau} \int_\Omega b^k |\Delta \Psi^k|^2 dx \leq \frac{\tau}{1 - 2r\tau} \int_\Omega |G^k|^2 dx.$$

Now for $j \in \llbracket 1, N \rrbracket$, summing up over $j \leq k \leq N$ and using $\Phi^{N+1} = 0$, we get

$$\int_\Omega |\nabla \Psi^j|^2 dx + \frac{\tau}{1 - 2r\tau} \sum_{k=j}^N \int_\Omega b^k |\Delta \Psi^k|^2 dx \leq \frac{\tau}{1 - 2r\tau} \sum_{k=1}^N \int_\Omega |G^k|^2 dx,$$

and hence, in terms of the Φ^k, F^k ,

$$\begin{aligned} \int_\Omega |\nabla \Phi^j|^2 dx + \tau \sum_{k=j}^N \int_\Omega b^k |\Delta \Phi^k|^2 dx &\leq \frac{\tau}{(1 - 2r\tau)^{N+1}} \sum_{k=1}^N \int_\Omega |F^k|^2 dx \\ &\leq \exp \left\{ \frac{(N+1)2r\tau}{1 - 2r\tau} \right\} \sum_{k=1}^N \int_\Omega \tau |F^k|^2 dx, \end{aligned}$$

which gives immediately (30) and (31) in the particular case $j = 1$.

Now for the last bound, notice that after integrating the k -th equation, we have

$$(1 - r\tau) \int_\Omega \Phi^k dx = \int_\Omega \Phi^{k+1} dx + \tau \int_\Omega \{b^k \Delta \Phi^k - \sqrt{b^k} F^k\} dx.$$

Defining this time

$$\Psi^k := \frac{\Phi^k}{(1 - r\tau)^k},$$

$$G^k := \frac{F^k}{(1 - r\tau)^k},$$

we get, dividing the previous equality by $(1 - r\tau)^{k+1}$,

$$\int_\Omega \Psi^k dx = \int_\Omega \Psi^{k+1} dx + \frac{\tau}{1 - r\tau} \int_\Omega \{b^k \Delta \Psi^k - \sqrt{b^k} G^k\} dx,$$

which, after summing up all over $j \leq k \leq N$, leads to

$$\begin{aligned} \int_{\Omega} \Phi^j dx &\leq \int_{\Omega} \Psi^j dx = \frac{1}{1-r\tau} \sum_{k=j}^N \tau \int_{\Omega} \{b^k \Delta \Psi^k - \sqrt{b^k} G^k\} dx \\ &\leq \frac{1}{(1-r\tau)^{N+1}} \sum_{k=j}^N \tau \int_{\Omega} \{b^k |\Delta \Phi^k| + \sqrt{b^k} |F^k|\} dx, \end{aligned}$$

and we have hence, using (31),

$$\left| \int_{\Omega} \Phi^j dx \right| \leq \left[\|\sqrt{b^{\tau}}\|_{L^2(Q_T)} + e^{r(\tau)^T} \right] \|F^{\tau}\|_{L^2(Q_T)} e^{r(\tau)^T}, \quad (33)$$

so that (32) follows from (30) and Poincaré-Wirtinger inequality. \square

If we want to use functions of the previous family Φ as test functions in the weak formulation (21), we have to show that they actually belong to $W^{2,p'}(\Omega)$. This can be done using the following lemma

Lemma 5.4. *Let r be a strictly positive real number. If $w \in H_{\nu}^2(\Omega)$ satisfies $w \geq 0$ and $-\Delta w \leq f + rw$ almost everywhere, for some f in $L^q(\Omega)$, $q > \max(d/2, 2)$, then*

$$\|w\|_{L^{\infty}(\Omega)} \leq C_{\Omega,q,r} \left(\|f\|_{L^q(\Omega)} + \|w\|_{L^1(\Omega)} \right).$$

Proof. $C_{\Omega,q}$ and C_q denotes constants that may vary from line to line. Lemma 5.2 gives us the existence of $g \in H_{\nu}^2(\Omega)$, unique solution of

$$g - \Delta g = (r+1)w + f \in L^{\min(2^{\bullet}, q)}(\Omega),$$

where p^{\bullet} is defined by the Sobolev embedding $W^{2,p}(\Omega) \hookrightarrow L^{p^{\bullet}}(\Omega)$. If $2^{\bullet} < q$, then the elliptic regularity of $(\text{Id} - \Delta)$ and a Sobolev embedding (cf. [14]) ensures

$$\|g\|_{L^{2^{\bullet\bullet}}(\Omega)} \leq C_{\Omega} \|g\|_{W^{2,2^{\bullet}}(\Omega)} \leq C_{\Omega} \|(r+1)w + f\|_{L^{2^{\bullet}}(\Omega)}.$$

But we also have

$$w - \Delta w \leq (r+1)w + f,$$

that we can write

$$(w - g) - \Delta(w - g) \leq 0,$$

so that by the weak maximum principle in $H_{\nu}^1(\Omega)$, we have

$$0 \leq w \leq g \in L^{2^{\bullet\bullet}}(\Omega).$$

The previous argument can be iterated in order to finally get $w \in L^q(\Omega)$, and thus using the elliptic regularity mentioned before,

$$\|g\|_{W^{2,q}(\Omega)} \leq C_{\Omega} \|(r+1)w + f\|_{L^q(\Omega)}.$$

Since $q > d/2$, we get the L^{∞} estimate

$$\|g\|_{L^{\infty}(\Omega)} \leq C_{\Omega,q} \|(r+1)w + f\|_{L^q(\Omega)},$$

that is again transmitted to w , thanks to the weak maximum principle :

$$\|w\|_{L^{\infty}(\Omega)} \leq C_{\Omega,q} \|f\|_{L^q(\Omega)} + C_{\Omega,q} (r+1) \|w\|_{L^q(\Omega)}.$$

Since

$$\|w\|_{L^q(\Omega)} \leq \|w\|_{L^1(\Omega)}^{1/q} \|w\|_{L^{\infty}(\Omega)}^{1/q'},$$

we get by Young's inequality

$$C_{\Omega,q} (r+1) \|w\|_{L^q(\Omega)} \leq C_{q,r} \|w\|_{L^1(\Omega)} + \frac{1}{2} \|w\|_{L^{\infty}(\Omega)},$$

and we are eventually able to conclude that

$$\|w\|_{L^{\infty}(\Omega)} \leq C_{\Omega,q,r} \left(\|f\|_{L^q(\Omega)} + \|w\|_{L^1(\Omega)} \right).$$

\square

We now can prove the following:

Lemma 5.5. *Under the assumptions of Lemma 5.3, the sequence of functions Φ satisfies in fact, for all $k \in \llbracket 1, N \rrbracket$*

$$\|\Phi^k\|_{L^{\infty}(\Omega)} + \|\Delta \Phi^k\|_{L^{\infty}(\Omega)} \leq C_{\tau,\Omega} \left[1 + \|\sqrt{b^{\tau}}\|_{L^2(Q_T)} \right] \|F^{\tau}\|_{L^{\infty}(Q_T)},$$

the constant $C_{\tau,\Omega}$ depending only on τ and Ω , but (severely) blowing up as $\tau \rightarrow 0$. In particular we get that $\Phi^k \in W^{2,q}(\Omega)$ for all $q \in [1, \infty[$.

Proof. Since $\Phi^N \geq 0$, $\Phi^{N+1} = 0$, $F^N \leq 0$ and $b^N \geq 1$, the N -th equation of (29) gives us

$$-\Delta \Phi^N \leq -F^N + r\Phi^N,$$

and since $F^N \in L^\infty(\Omega) \hookrightarrow L^s(\Omega)$ for all $s \geq 1$, the previous Lemma applies:

$$\|\Phi^N\|_{L^\infty(\Omega)} \leq C_{\Omega,r} \left[\|\underline{F}^\tau\|_{L^\infty(Q_T)} + \|\Phi^N\|_{L^1(\Omega)} \right],$$

and we use then (33) to get

$$\|\Phi^N\|_{L^\infty(\Omega)} \leq C_{\tau,\Omega,T} \left[1 + \|\sqrt{b^\tau}\|_{L^2(Q_T)} \right] \|\underline{F}^\tau\|_{L^\infty(Q_T)}.$$

For all $k \in \llbracket 1, N \rrbracket$, using the k -th equation of (29), we have in the same way

$$-\Delta \Phi^k \leq \Phi^{k+1} - F^k + r\Phi^k,$$

and hence a descending induction gives eventually ($C_{\tau,\Omega}$ varies from line to line)

$$\|\Phi^k\|_{L^\infty(\Omega)} \leq C_{\tau,\Omega} \left[1 + \|\sqrt{b^\tau}\|_{L^2(Q_T)} \right] \|\underline{F}^\tau\|_{L^\infty(Q_T)}.$$

To handle the laplacian terms, we write for all $k \in \llbracket 1, N \rrbracket$, using again (29),

$$-\Delta \Phi^k = \frac{\Phi^{k+1} - \Phi^k}{b^k \tau} - \frac{F^k}{\sqrt{b^k}} + r \frac{\Phi^k}{b^k},$$

which directly gives, using the previous estimate,

$$\|\Delta \Phi^k\|_{L^\infty(\Omega)} \leq C_{\tau,\Omega} \left[1 + \|\sqrt{b^\tau}\|_{L^2(Q_T)} \right] \|\underline{F}^\tau\|_{L^\infty(Q_T)}.$$

In particular, the norms $\|\Phi^k - \Delta \Phi^k\|_{L^q(\Omega)}$ are all finite for $q \in]1, \infty[$. Thereby, using again the aforementioned (see [14]) result on elliptic regularity, we get the finiteness of all the norms $\|\Phi^k\|_{W^{2,q}(\Omega)}$ for all $q < \infty$. \square

5.3 Duality estimate: application to the system

Proposition 5.6. *Under the assumptions of Theorem 1.2 and using the Definitions 2.5, 3.1 and 5.1, any weak solution of eq. (21) satisfies the following estimate:*

$$\left\| \underline{u}_1^\tau \sqrt{d_{11}^\varepsilon(\underline{u}_1^\tau) + a_{12}^\varepsilon(\underline{u}_2^\tau)} \right\|_{L^2(Q_T)} \leq D, \quad (34)$$

where D is a constant depending only on T, Ω and \mathbf{u}^0 . The same estimate holds when the subscripts 1 and 2 are exchanged.

Proof. We shall denote by \underline{u}_1^τ and \underline{u}_2^τ the step (in time) functions (see Definition 5.1) associated to the two families $(u_1^k)_{1 \leq k \leq N}$ and $(u_2^k)_{1 \leq k \leq N}$ (excluding hence u_1^0 and u_2^0). Since the original system is symmetric in u_1 and u_2 , let us focus on u_1 for now. As in Lemma 5.3, we consider two families $b := (b^k)_{1 \leq k \leq N}$ and $F := (F^k)_{1 \leq k \leq N}$ of $\mathcal{C}^\infty(\Omega)$ functions such that $b^k \geq 1$ and $F^k \leq 0$, and the associated sequence $\Phi := (\Phi^k)_{1 \leq k \leq N}$, with $r := r_1$. As shown in Lemma 5.5, the functions Φ^k belong to $W^{2,p'}(\Omega)$ (for all $p > 1$) and are hence admissible in the weak formulation (21). We may therefore write, for all $k \in \llbracket 1, N \rrbracket$, taking Φ^k for test function in the weak formulation on u_1^k :

$$\begin{aligned} \frac{1}{\tau} \langle \Phi^k, u_1^k \rangle_{L^2(\Omega)} - \frac{1}{\tau} \langle \Phi^k, u_1^{k-1} \rangle_{L^2(\Omega)} - \left\langle \Delta \Phi^k, \left[a_{11}^\varepsilon(u_1^k) + u_1^k a_{12}^\varepsilon(u_2^k) \right] \right\rangle_{L^2(\Omega)} \\ + \varepsilon \langle \Phi^k - \Delta \Phi^k, w_1^k \rangle_{L^2(\Omega)} = \left\langle \Phi^k, R_{12}^\varepsilon(u_1^k, u_2^k) \right\rangle. \end{aligned}$$

Thanks to the regularity of Φ^k , one can take u_1^k as a test function in the k -th equation (29), so that

$$\frac{1}{\tau} \langle \Phi^{k+1}, u_1^k \rangle_{L^2(\Omega)} - \frac{1}{\tau} \langle \Phi^k, u_1^k \rangle_{L^2(\Omega)} + \langle \Delta \Phi^k, b^k u_1^k \rangle_{L^2(\Omega)} = \langle \sqrt{b^k} F^k, u_1^k \rangle_{L^2(\Omega)} - r_1 \langle \Phi^k, u_1^k \rangle_{L^2(\Omega)}.$$

We hence have, for all $k \in \llbracket 1, N \rrbracket$, adding the two previous equations,

$$\begin{aligned} \frac{1}{\tau} \langle \Phi^{k+1}, u_1^k \rangle_{L^2(\Omega)} - \frac{1}{\tau} \langle \Phi^k, u_1^{k-1} \rangle_{L^2(\Omega)} + \langle \Delta \Phi^k, [b^k - d_{11}^\varepsilon(u_1^k) - a_{12}^\varepsilon(u_2^k)] u_1^k \rangle_{L^2(\Omega)} \\ + \varepsilon \langle \Phi^k - \Delta \Phi^k, w_1^k \rangle_{L^2(\Omega)} = \langle \sqrt{b^k} F^k, u_1^k \rangle_{L^2(\Omega)} - \left\langle \Phi^k, R_{12}^{-,\varepsilon}(u_1^k, u_2^k) \right\rangle, \end{aligned}$$

recalling that $a_{12}^\varepsilon(x) := a_{12}(x) + \varepsilon x$, $a_{11}^\varepsilon(x) := x d_{11}^\varepsilon(x)$, $d_{11}^\varepsilon := \gamma_\varepsilon(d_{11})$, and the decomposition $R_{12}^\varepsilon = R_{12}^+ - R_{12}^{-,\varepsilon}$. Since Φ^k is nonnegative, we get, if we denote $c^k := b^k - d_{11}^\varepsilon(u_1^k) - a_{12}^\varepsilon(u_2^k)$, after summing over $k \in \llbracket 1, N \rrbracket$ (and recalling that $\Phi^{N+1} = 0$),

$$-\frac{1}{\tau} \langle \Phi^1, u_1^0 \rangle_{L^2(\Omega)} + \sum_{k=1}^N \langle \Delta \Phi^k, c^k u_1^k \rangle_{L^2(\Omega)} + \sum_{k=1}^N \varepsilon \langle \Phi^k, w_1^k \rangle_{H^1(\Omega)} \leq \sum_{k=1}^N \langle \sqrt{b^k} F^k, u_1^k \rangle_{L^2(\Omega)}. \quad (35)$$

The following approximation Lemma will help us to handle the sequence $c := (c^k)_{1 \leq k \leq N}$:

Lemma 5.7. *There exists a sequence of families $(b_m)_{m \in \mathbb{N}} := (b_m^1, b_m^2, \dots, b_m^N)_{m \in \mathbb{N}}$ of $\mathcal{C}^\infty(\Omega)$ functions such that*

$$\forall (k, m) \in \llbracket 1, N \rrbracket \times \mathbb{N}, \quad b_m^k \geq 1, \quad (36)$$

$$\forall k \in \llbracket 1, N \rrbracket, \quad \int_{\Omega} |b_m^k - d_{11}^\varepsilon(u_1^k) - a_{12}^\varepsilon(u_2^k)| (1 + u_1^k) dx \xrightarrow{m \rightarrow +\infty} 0, \quad (37)$$

$$\forall k \in \llbracket 1, N \rrbracket, \quad \int_{\Omega} \left| \sqrt{b_m^k} - \sqrt{d_{11}^\varepsilon(u_1^k) + a_{12}^\varepsilon(u_2^k)} \right| (1 + u_1^k) dx \xrightarrow{m \rightarrow +\infty} 0, \quad (38)$$

$$\left\| \sqrt{b_m^\tau} \right\|_{L^2(Q_T)}^2 \xrightarrow{m \rightarrow +\infty} \sum_{k=1}^N \tau \int_{\Omega} |d_{11}^\varepsilon(u_1^k) + a_{12}^\varepsilon(u_2^k)| dx = \left\| \sqrt{d_{11}^\varepsilon(\underline{u}_1^\tau) + a_{12}^\varepsilon(\underline{u}_2^\tau)} \right\|_{L^2(Q_T)}^2. \quad (39)$$

Proof. This Lemma is a rather direct consequence of the regularity of the borelian measure $(1 + u_1^k) dx$: we know that $d_{11}^\varepsilon(u_1^k) + a_{12}^\varepsilon(u_2^k) \in L^1(\Omega, (1 + u_1^k) dx)$ and is lower bounded by a strictly positive constant, we may hence approximate it in this space by smooth $\mathcal{C}^\infty(\Omega)$ functions, which are still lower bounded by a strictly positive constant. This gives (36)–(37). Using the mentioned lower bound, we have

$$\begin{aligned} \left| \sqrt{b_m^k} - \sqrt{d_{11}^\varepsilon(u_1^k) + a_{12}^\varepsilon(u_2^k)} \right| &\leq \left| \sqrt{b_m^k} - \sqrt{d_{11}^\varepsilon(u_1^k) + a_{12}^\varepsilon(u_2^k)} \right| \left[\sqrt{b_m^k} + \sqrt{d_{11}^\varepsilon(u_1^k) + a_{12}^\varepsilon(u_2^k)} \right] \\ &= |b_m^k - d_{11}^\varepsilon(u_1^k) - a_{12}^\varepsilon(u_2^k)|, \end{aligned}$$

so that we also get (38). Noticing that

$$\left\| \sqrt{b_m^\tau} \right\|_{L^2(Q_T)}^2 = \|b_m^\tau\|_{L^1(Q_T)} = \sum_{k=1}^N \tau \int_{\Omega} b_m^k dx,$$

we see that (39) is a simple consequence of the convergence (37). \square

We now fix a family $F := (F^k)_{1 \leq k \leq N}$ of $\mathcal{C}^\infty(\Omega)$ nonpositive functions, and for each family b_m of the sequence $(b_m)_{m \in \mathbb{N}}$ defined in Lemma 5.7, we define the corresponding family $\Phi_m := (\Phi_m^1, \Phi_m^2, \dots, \Phi_m^N)$, using Lemma 5.3. The previous estimate (35) can now be written

$$-\langle \Phi_m^1, u_1^0 \rangle_{L^2(\Omega)} + \sum_{k=1}^N \tau \langle \Delta \Phi_m^k, c_m^k u_1^k \rangle_{L^2(\Omega)} + \sum_{k=1}^N \tau \varepsilon \langle \Phi_m^k, w_1^k \rangle_{H^1(\Omega)} \leq \sum_{k=1}^N \tau \langle \sqrt{b_m^k} F^k, u_1^k \rangle_{L^2(\Omega)},$$

where $c_m^k := b_m^k - d_{11}^\varepsilon(u_1^k) - a_{12}^\varepsilon(u_2^k)$. Since the right-hand side is nonpositive, we may write

$$\begin{aligned} \boxed{1} &:= \left| \sum_{k=1}^N \tau \langle \sqrt{b_m^k} F^k, u_1^k \rangle_{L^2(\Omega)} \right| \leq \overbrace{\langle \Phi_m^1, u_1^0 \rangle_{L^2(\Omega)}}^{\boxed{2}} + \overbrace{\sum_{k=1}^N \tau \langle \Delta \Phi_m^k, c_m^k u_1^k \rangle_{L^2(\Omega)}}^{\boxed{3}} \\ &\quad + \overbrace{\sum_{k=1}^N \tau \varepsilon \langle \Phi_m^k, w_1^k \rangle_{H^1(\Omega)}}^{\boxed{4}}. \end{aligned}$$

Using (32) of Lemma 5.3 and (39), we get (for a given τ, ε),

$$\begin{aligned} \boxed{2} &\leq C_\Omega \|u^0\|_{L^2(\Omega)} \left[e^{r_1(\tau)T} + \left\| \sqrt{b_m^\tau} \right\|_{L^2(Q_T)} \right] \|F^\tau\|_{L^2(Q_T)} e^{r_1(\tau)T} \\ &\xrightarrow{m \rightarrow +\infty} C_\Omega \|u^0\|_{L^2(\Omega)} \left[e^{r_1(\tau)T} + \left\| \sqrt{d_{11}^\varepsilon(\underline{u}_1^\tau) + a_{12}^\varepsilon(\underline{u}_2^\tau)} \right\|_{L^2(Q_T)} \right] \|F^\tau\|_{L^2(Q_T)} e^{r_1(\tau)T}. \end{aligned}$$

Using Lemma 5.5, we get on the other hand

$$\begin{aligned} \boxed{3} &\leq \tau N \sup_{1 \leq k \leq N} \left\{ \|\Delta \Phi_m^k\|_{L^\infty(\Omega)} \|c_m^k u_1^k\|_{L^1(\Omega)} \right\} \\ &\leq \tau N C_{\tau, \Omega} \left[1 + \left\| \sqrt{b_m^\tau} \right\|_{L^2(Q_T)} \right] \|F^\tau\|_{L^\infty(Q_T)} \sup_{1 \leq k \leq N} \|c_m^k u_1^k\|_{L^1(\Omega)} \xrightarrow{m \rightarrow +\infty} 0, \end{aligned}$$

using (37) and (39) for the convergence. In order to treat $\boxed{4}$, notice that we know, from inequality (22):

$$\varepsilon \tau \sum_{k=1}^N \|w_1^k\|_{H^1(\Omega)}^2 \leq K_T(\mathcal{E}_\varepsilon(u^0) + 1),$$

hence, using again (32) of Lemma 5.3, we get ($\tau N = T$)

$$\begin{aligned} \boxed{4} &\leq e^{r_1(\tau)T} C_\Omega \left[e^{r_1(\tau)T} + \left\| \sqrt{b_m^\tau} \right\|_{L^2(Q_T)} \right] \left\| \underline{F}^\tau \right\|_{L^\infty(Q_T)} \sum_{k=1}^N \tau \varepsilon \|w_1^k\|_{H^1(\Omega)} \\ &\leq e^{r_1(\tau)T} C_\Omega \left[e^{r_1(\tau)T} + \left\| \sqrt{b_m^\tau} \right\|_{L^2(Q_T)} \right] \left\| \underline{F}^\tau \right\|_{L^\infty(Q_T)} \tau \varepsilon \sqrt{N} \left\{ \sum_{k=1}^N \|w_1^k\|_{H^1(\Omega)}^2 \right\}^{1/2} \\ &\leq e^{r_1(\tau)T} C_\Omega \left[e^{r_1(\tau)T} + \left\| \sqrt{b_m^\tau} \right\|_{L^2(Q_T)} \right] \left\| \underline{F}^\tau \right\|_{L^\infty(Q_T)} \sqrt{\varepsilon T} \sqrt{K_T(\mathcal{E}_\varepsilon(\mathbf{u}^0) + 1)}, \end{aligned}$$

and we may use again (39) to get that the last upper bound is converging, as m goes to $+\infty$, to

$$e^{r_1(\tau)T} C_\Omega \left[e^{r_1(\tau)T} + \left\| \sqrt{d_{11}^\varepsilon(\underline{u}_1^\tau) + a_{12}^\varepsilon(\underline{u}_2^\tau)} \right\|_{L^2(Q_T)} \right] \left\| \underline{F}^\tau \right\|_{L^2(Q_T)} \sqrt{\varepsilon T} \sqrt{K_T(\mathcal{E}_\varepsilon(\mathbf{u}^0) + 1)}.$$

Finally, because of (38), we can see that $\boxed{1}$ converges, as m goes to $+\infty$, to

$$\left| \sum_{k=1}^N \tau \int_{\Omega} u_1^k \sqrt{d_{11}^\varepsilon(u_1^k) + a_{12}^\varepsilon(u_2^k)} F^k dx \right| = \left| \int_{Q_T} \underline{u}_1^\tau \sqrt{d_{11}^\varepsilon(\underline{u}_1^\tau) + a_{12}^\varepsilon(\underline{u}_2^\tau)} \underline{F}^\tau dx dt \right|.$$

All the previous estimates give hence, denoting $\underline{h}^\tau := \underline{u}_1^\tau \sqrt{d_{11}^\varepsilon(\underline{u}_1^\tau) + a_{12}^\varepsilon(\underline{u}_2^\tau)}$,

$$\begin{aligned} \left| \int_{Q_T} \underline{h}^\tau \underline{F}^\tau dx dt \right| &\leq e^{r_1(\tau)T} C_\Omega \left[e^{r_1(\tau)T} + \left\| \sqrt{d_{11}^\varepsilon(\underline{u}_1^\tau) + a_{12}^\varepsilon(\underline{u}_2^\tau)} \right\|_{L^2(Q_T)} \right] \\ &\quad \times \left[\sqrt{\varepsilon T} \sqrt{K_T(\mathcal{E}_\varepsilon(\mathbf{u}^0) + 1)} + \|\mathbf{u}^0\|_{L^2(\Omega)} \right] \left\| \underline{F}^\tau \right\|_{L^2(Q_T)}. \end{aligned}$$

Since \underline{h}^τ is a step (in time) nonnegative function and the previous holds true for all non-positive smooth (in x) step (in time) functions, we have then by duality

$$\begin{aligned} \left\| \underline{u}_1^\tau \sqrt{d_{11}^\varepsilon(\underline{u}_1^\tau) + a_{12}^\varepsilon(\underline{u}_2^\tau)} \right\|_{L^2(Q_T)} &\leq e^{2r_1(\tau)T} C_\Omega \left[1 + \left\| \sqrt{d_{11}^\varepsilon(\underline{u}_1^\tau) + a_{12}^\varepsilon(\underline{u}_2^\tau)} \right\|_{L^2(Q_T)} \right] \\ &\quad \times \left[\sqrt{\varepsilon T} \sqrt{K_T(\mathcal{E}_\varepsilon(\mathbf{u}^0) + 1)} + \|\mathbf{u}^0\|_{L^2(\Omega)} \right]. \end{aligned}$$

Hence, in L^1 norms, we get

$$\begin{aligned} \left\| \underline{u}_1^{\tau^2} \left[d_{11}^\varepsilon(\underline{u}_1^\tau) + a_{12}^\varepsilon(\underline{u}_2^\tau) \right] \right\|_{L^1(Q_T)} &\leq e^{4r_1(\tau)T} C_\Omega \left[1 + \left\| d_{11}^\varepsilon(\underline{u}_1^\tau) + a_{12}^\varepsilon(\underline{u}_2^\tau) \right\|_{L^1(Q_T)} \right] \\ &\quad \times \left[\varepsilon T K_T(\mathcal{E}_\varepsilon(\mathbf{u}^0) + 1) + \|\mathbf{u}^0\|_{L^2(\Omega)}^2 \right]. \end{aligned}$$

Since $\varepsilon < 1$ and $(r_1(\tau))_\tau \xrightarrow{\tau \rightarrow 0} r_1$, the previous inequality can be written in a simpler form :

$$\left\| \underline{u}_1^{\tau^2} \left[d_{11}^\varepsilon(\underline{u}_1^\tau) + a_{12}^\varepsilon(\underline{u}_2^\tau) \right] \right\|_{L^1(Q_T)} \leq D \left[1 + \left\| d_{11}^\varepsilon(\underline{u}_1^\tau) \right\|_{L^1(\Omega)} + \left\| a_{12}^\varepsilon(\underline{u}_2^\tau) \right\|_{L^1(Q_T)} \right],$$

where the constant D only depends on T, Ω and \mathbf{u}^0 . Because of the concavity of a_{12}^ε (and since it vanishes in 0), we have, with some constant C_1, C_2 **not depending** on ε ,

$$a_{12}^\varepsilon(x) \leq C_1 + C_2 \psi_2^\varepsilon(x),$$

so that, from (22), we have

$$\left\| \underline{u}_1^{\tau^2} \left[d_{11}^\varepsilon(\underline{u}_1^\tau) + a_{12}^\varepsilon(\underline{u}_2^\tau) \right] \right\|_{L^1(Q_T)} \leq D \left[1 + \left\| d_{11}^\varepsilon(\underline{u}_1^\tau) \right\|_{L^1(\Omega)} + K_T(\mathcal{E}_\varepsilon(\mathbf{u}^0) + 1) \right],$$

that we may write (changing the definition of D)

$$\left\| \underline{u}_1^{\tau^2} \left[d_{11}^\varepsilon(\underline{u}_1^\tau) + a_{12}^\varepsilon(\underline{u}_2^\tau) \right] \right\|_{L^1(Q_T)} \leq D \left[1 + \left\| d_{11}^\varepsilon(\underline{u}_1^\tau) \right\|_{L^1(\Omega)} \right].$$

But we have then, since $d_{11}^\varepsilon \leq d_{11}$ which is a non-decreasing function,

$$\begin{aligned} \left\| \underline{u}_1^{\tau^2} \left[d_{11}^\varepsilon(\underline{u}_1^\tau) + a_{12}^\varepsilon(\underline{u}_2^\tau) \right] \right\|_{L^1(Q_T)} &\leq D \left[1 + \left\| \mathbf{1}_{\underline{u}_1^\tau \geq \sqrt{2D}} d_{11}^\varepsilon(\underline{u}_1^\tau) \right\|_{L^1(\Omega)} + \left\| \mathbf{1}_{\underline{u}_1^\tau < \sqrt{2D}} d_{11}^\varepsilon(\underline{u}_1^\tau) \right\|_{L^1(\Omega)} \right] \\ &\leq D \left[1 + \frac{1}{2D} \left\| \underline{u}_1^{\tau^2} d_{11}^\varepsilon(\underline{u}_1^\tau) \right\|_{L^1(\Omega)} + d_{11}(\sqrt{2D})\mu(\Omega) \right], \end{aligned}$$

so that

$$\left\| \underline{u}_1^{\tau^2} \left[\frac{1}{2} d_{11}^\varepsilon(\underline{u}_1^\tau) + a_{12}^\varepsilon(\underline{u}_2^\tau) \right] \right\|_{L^1(Q_T)} \leq D \left[1 + d_{11}(\sqrt{2D})\mu(\Omega) \right],$$

that is to say, eq. (34). \square

6 Proof of the main Theorem

This section is dedicated to the proof of our main Theorem.

Proof. Thanks to Proposition 4.1, we can consider a solution $\underline{u}_1^{\tau^\varepsilon}, \underline{u}_2^{\tau^\varepsilon}$ to system (21). For the sake of clarity, we shall denote this solution by $\underline{u}_1^\tau, \underline{u}_2^\tau$. Let us keep in mind that since we are dealing with the limit $(\tau, \varepsilon) \rightarrow (0, 0)$, we shall have to use bounds that are uniform w.r.t. these two parameters. Hence in the sequel, if not mentioned, the term “bounded” will always have to be understood as “uniformly w.r.t. τ and ε ”. We shall only work with \underline{u}_1^τ , the study of \underline{u}_2^τ being exactly identical.

We start with a simple computation. Take some real number $\eta > 0$ and write

$$\begin{aligned} \underline{u}_1^\tau(t + \eta, x) - \underline{u}_1^\tau(t, x) &= \sum_{k=1}^N [\mathbf{1}_{(k-1)\tau, k\tau}(t + \eta) - \mathbf{1}_{(k-1)\tau, k\tau}(t)] u_1^k(x) \\ &= \sum_{k=1}^N [\mathbf{1}_{(k-1)\tau - \eta, k\tau - \eta}(t) - \mathbf{1}_{(k-1)\tau, k\tau}(t)] u_1^k(x). \end{aligned}$$

Hence, using

$$\mathbf{1}_{(k-1)\tau - \eta, k\tau - \eta}(t) - \mathbf{1}_{(k-1)\tau, k\tau}(t) = \mathbf{1}_{(k-1)\tau - \eta, (k-1)\tau}(t) - \mathbf{1}_{k\tau - \eta, k\tau}(t),$$

we get

$$\begin{aligned} \underline{u}_1^\tau(t + \eta, x) - \underline{u}_1^\tau(t, x) &= \sum_{k=0}^{N-1} \mathbf{1}_{k\tau - \eta, k\tau}(t) u_1^{k+1}(x) - \sum_{k=1}^N \mathbf{1}_{k\tau - \eta, k\tau}(t) u_1^k(x) \\ &= \sum_{k=1}^{N-1} \mathbf{1}_{k\tau - \eta, k\tau}(t) [u_1^{k+1}(x) - u_1^k(x)] + \mathbf{1}_{-\eta, 0](t) u_1^1(x) - \mathbf{1}_{T-\eta, T](t) u_1^N(x). \end{aligned}$$

If we denote by σ_η the translation operator $g(\cdot, x) \mapsto g(\cdot + \eta, x)$, we have, introducing the space $E_\eta := L^1([0, T - \eta]; W^{2,\infty}(\Omega)')$,

$$\|\sigma_\eta \underline{u}_1^\tau - \underline{u}_1^\tau\|_{E_\eta} \leq \eta \sum_{k=1}^{N-1} \|u_1^{k+1} - u_1^k\|_{W^{2,\infty}(\Omega)'}$$

Now recall that, for all $j \in \llbracket 1, N \rrbracket$,

$$\partial_\tau u_1^j - \Delta \left([d_{11}^\varepsilon(u_1^j) + a_{12}^\varepsilon(u_2^j)] u_1^j \right) + \varepsilon(w_1^j - \Delta w_1^j) = R_{12}^\varepsilon(\mathbf{u}^j),$$

weakly. We then have (taking $j = k + 1$), since $H^1(\Omega) \hookrightarrow H^{-1}(\Omega) \hookrightarrow W^{2,\infty}(\Omega)'$, and $L^1(\Omega) \hookrightarrow W^{2,\infty}(\Omega)'$

$$\begin{aligned} \frac{1}{\tau} \|u_1^{k+1} - u_1^k\|_{W^{2,\infty}(\Omega)'} &\leq \left\| [d_{11}^\varepsilon(u_1^{k+1}) + a_{12}^\varepsilon(u_2^{k+1})] u_1^{k+1} \right\|_{L^1(\Omega)} + \varepsilon C_\Omega \|w_1^{k+1}\|_{H^1(\Omega)} + \|R_{12}^\varepsilon(\mathbf{u}^{k+1})\|_{L^1(\Omega)} \\ &= \left\| [d_{11}^\varepsilon(u_1^{k+1}) + a_{12}^\varepsilon(u_2^{k+1})] u_1^{k+1} \mathbf{1}_{u_1^{k+1} \geq 1} \right\|_{L^1(\Omega)} \\ &\quad + \left\| [d_{11}^\varepsilon(u_1^{k+1}) + a_{12}^\varepsilon(u_2^{k+1})] u_1^{k+1} \mathbf{1}_{u_1^{k+1} < 1} \right\|_{L^1(\Omega)} \\ &\quad + \varepsilon C_\Omega \|w_1^{k+1}\|_{H^1(\Omega)} + \|R_{12}^{\varepsilon,-}(\mathbf{u}^{k+1})\|_{L^1(\Omega)} + r_1 \|u_1^{k+1}\|_{L^1(\Omega)}. \end{aligned}$$

Using $\mathbf{1}_\Omega = \mathbf{1}_{u_1^{k+1} \geq 1} + \mathbf{1}_{u_1^{k+1} < 1}$, we have

$$\begin{aligned} \left\| [d_{11}^\varepsilon(u_1^{k+1}) + a_{12}^\varepsilon(u_2^{k+1})] u_1^{k+1} \right\|_{L^1(\Omega)} &\leq \left\| u_1^{k+1} \sqrt{d_{11}^\varepsilon(u_1^{k+1}) + a_{12}^\varepsilon(u_2^{k+1})} \right\|_{L^2(\Omega)}^2 \\ &\quad + \mu(\Omega) \sup_{x \in [0,1]} d_{11}^\varepsilon(x) + \|a_{12}^\varepsilon(u_2^{k+1})\|_{L^1(\Omega)}. \end{aligned}$$

Since for ε small enough, d_{11}^ε coincides with d_{11} on $[0, 1]$, using also the last point of Lemma 2.6 to get $a_{12}^\varepsilon \leq D(2 + \psi_2^\varepsilon)$, we have eventually (recalling (22)) for some constant C not depending on ε ,

$$\begin{aligned} \left\| [d_{11}^\varepsilon(u_1^{k+1}) + a_{12}^\varepsilon(u_2^{k+1})]u_1^{k+1} \right\|_{L^1(\Omega)} &\leq \left\| u_1^{k+1} \sqrt{d_{11}^\varepsilon(u_1^{k+1}) + a_{12}^\varepsilon(u_2^{k+1})} \right\|_{L^2(\Omega)}^2 + CK_T(\mathcal{E}_\varepsilon(\mathbf{u}^0) + 1) \\ &\leq \left\| u_1^{k+1} \sqrt{d_{11}^\varepsilon(u_1^{k+1}) + a_{12}^\varepsilon(u_2^{k+1})} \right\|_{L^2(\Omega)}^2 + C, \end{aligned}$$

where we change the definition of C , so that finally

$$\begin{aligned} \|\sigma_\eta \underline{u}_1^\tau - \underline{u}_1^\tau\|_{E_\eta} &\leq \eta \sum_{k=1}^{N-1} \tau \left\{ \left\| u_1^{k+1} \sqrt{d_{11}^\varepsilon(u_1^{k+1}) + a_{12}^\varepsilon(u_2^{k+1})} \right\|_{L^2(\Omega)}^2 + C + \varepsilon C_\Omega \|w_1^{k+1}\|_{H^1(\Omega)} \right\} \\ &\quad + \eta \sum_{k=1}^{N-1} \tau \left\{ \|R_{12}^{\varepsilon,-}(\mathbf{u}^{k+1})\|_{L^1(\Omega)} + r_1 \|u_1^{k+1}\|_{L^1(\Omega)} \right\} \\ &\leq 4\eta \left\| \underline{u}_1^\tau \sqrt{d_{11}^\varepsilon(\underline{u}_1^\tau) + a_{12}^\varepsilon(\underline{u}_2^\tau)} \right\|_{L^2(Q_T)}^2 + \eta TC + \eta \varepsilon \tau C_\Omega \sum_{k=1}^N \|\mathbf{w}^k\|_{H^1(\Omega)} \\ &\quad + \eta \sum_{k=1}^{N-1} \tau \left\{ \|R_{12}^{\varepsilon,-}(\mathbf{u}^{k+1})\|_{L^1(\Omega)} + r_1 \|u_1^{k+1}\|_{L^1(\Omega)} \right\}, \end{aligned}$$

which, using (34), (22), (24) and (25), is going to 0 as $\eta \rightarrow 0$, uniformly w.r.t. τ and ε .

We also have

$$|\nabla \underline{u}_1^\tau|(t, x) = \sum_{k=1}^N \mathbf{1}_{(k-1)\tau, k\tau]}(t) |\nabla u_1^k|(x).$$

Recall the notation $\beta_\alpha(x) := x^{\frac{1-\alpha}{2}}$. Given $v : \Omega \rightarrow \mathbb{R}_+$ such that $\beta_\alpha(v) \in H^1(\Omega)$ (and considering for example the sequence $(v + 1/m)_{m \in \mathbb{N}}$), we get

$$\frac{1-\alpha}{2} v^{-\frac{\alpha+1}{2}} \nabla v = \nabla \beta_\alpha(v)$$

weakly. Hence, using (23), we have for all $k \in \llbracket 1, N \rrbracket$:

$$\tau \sum_{j=1}^k \int_\Omega |u_1^j|^{-\alpha-1} |\nabla u_1^j|^2 dx \leq K_T(\mathcal{E}_\varepsilon(\mathbf{u}^0) + 1).$$

We know that

$$|\nabla \underline{u}_1^\tau|(t, x) = \sum_{k=1}^N \mathbf{1}_{(k-1)\tau, k\tau]}(t) |u_1^k|^{\frac{\alpha+1}{2}} |u_1^k|^{-\frac{\alpha+1}{2}} |\nabla u_1^k|.$$

Hence, using the dual estimate (34), we see that $\nabla \underline{u}_1^\tau$ is bounded in $L^{q_\alpha}(Q_T)$, where q_α is defined by the equality $(\alpha \in]0, 1[)$

$$\frac{\alpha+1}{4} + \frac{1}{2} = \frac{1}{q_\alpha},$$

that is $q_\alpha = 8/(2\alpha + 6) > 1$. At this stage we obtained, denoting $E_\eta := L^1([0, T - \eta]; W^{2,\infty}(\Omega)')$,

- $\|\sigma_\eta \underline{u}_1^\tau - \underline{u}_1^\tau\|_{E_\eta}$ goes to 0 with η uniformly in τ and ε ,
- $(\underline{u}_1^\tau)_\tau$ is bounded in $L^{q_\alpha}([0, T]; W^{1,q_\alpha}(\Omega))$.

We know that $W^{1,q_\alpha}(\Omega) \hookrightarrow L^r(\Omega) \hookrightarrow W^{2,\infty}(\Omega)'$, where the first injection is compact, the second one is continuous, and r denotes any real number of the interval $[1, q_\alpha^*]$. We may hence use Theorem 5 of [25] to get the compactness of $(\underline{u}_1^\tau)_\tau$ in $L^1([0, T]; L^r(\Omega))$.

From the definition of \underline{u}_1^τ :

$$\underline{u}_1^\tau(t, x) := \sum_{k=1}^N u_1^k(x) \mathbf{1}_{(k-1)\tau, k\tau]}(t),$$

we get

$$\begin{aligned} \sigma_{-\tau} \underline{u}_1^\tau(t, x) &:= \underline{u}_1^\tau(t - \tau, x) = \sum_{k=1}^N u_1^k(x) \mathbf{1}_{(k-1)\tau, k\tau]}(t - \tau) \\ &= \sum_{k=1}^N u_1^k(x) \mathbf{1}_{k\tau, (k+1)\tau]}(t) \\ &= \sum_{k=2}^{N+1} u_1^{k-1}(x) \mathbf{1}_{(k-1)\tau, k\tau]}(t). \end{aligned}$$

We hence obtain the following expression for the rate of growth

$$\frac{\underline{u}_1^\tau - \sigma_{-\tau} \underline{u}_1^\tau}{\tau}(t, x) = \sum_{k=1}^N \mathbf{1}_{[(k-1)\tau, k\tau]}(t) \partial_\tau u_1^k(x) + \frac{1}{\tau} \mathbf{1}_{[0, \tau]}(t) u_1^1(x) - \frac{1}{\tau} \mathbf{1}_{[T, T+\tau]}(t) u_1^N(x),$$

recalling the definition of the finite difference operator,

$$\partial_\tau u_1^k := \frac{u_1^k - u_1^{k-1}}{\tau}.$$

Now let us fix a test function $\theta \in \mathcal{D}([0, T[; \mathcal{C}_\nu^\infty(\bar{\Omega}))$. We have

$$\int_0^T \int_\Omega \frac{\underline{u}_1^\tau - \sigma_{-\tau} \underline{u}_1^\tau}{\tau} \cdot \theta(t, x) \, dx \, dt = \sum_{k=1}^N \int_{(k-1)\tau}^{k\tau} \langle \partial_\tau u_1^k, \theta(t) \rangle \, dt + \frac{1}{\tau} \int_0^\tau \langle u_1^1, \theta(t) \rangle \, dt,$$

where we denote by $\langle \cdot, \cdot \rangle$ the duality bracket $\mathcal{D}'(\Omega)/\mathcal{D}(\Omega)$ (which is simply the integration on Ω here ...). A change of variable leads to

$$\begin{aligned} \int_0^T \int_\Omega \frac{\underline{u}_1^\tau - \sigma_{-\tau} \underline{u}_1^\tau}{\tau} \cdot \theta(t, x) \, dx \, dt &= \int_0^T \int_\Omega \underline{u}_1^\tau \cdot \frac{\theta}{\tau}(t, x) \, dx \, dt - \int_{-\tau}^{T-\tau} \int_\Omega \underline{u}_1^\tau \cdot \frac{\sigma_\tau \theta}{\tau}(t, x) \, dx \, dt \\ &= \int_0^{T-\tau} \int_\Omega \underline{u}_1^\tau \cdot \frac{\theta - \sigma_\tau \theta}{\tau}(t, x) \, dx \, dt \\ &\quad + \underbrace{\int_{T-\tau}^T \int_\Omega \underline{u}_1^\tau \cdot \frac{\theta}{\tau}(t, x) \, dx \, dt}_{\boxed{1}} - \underbrace{\int_{-\tau}^0 \int_\Omega \underline{u}_1^\tau \cdot \frac{\sigma_\tau \theta}{\tau}(t, x) \, dx \, dt}_{\boxed{2}}. \end{aligned}$$

Then, $\boxed{1}$ equals to 0 for τ small enough because $\theta \in \mathcal{D}([0, T[\times \Omega)$, and $\boxed{2}$ equals to 0 for all τ because of the definition of \underline{u}_1^τ , so that finally

$$\sum_{k=1}^N \int_{(k-1)\tau}^{k\tau} \langle \partial_\tau u_1^k, \theta(t) \rangle \, dt + \frac{1}{\tau} \int_0^\tau \langle u_1^1, \theta(t) \rangle \, dt = \int_0^{T-\tau} \int_\Omega \underline{u}_1^\tau \cdot \frac{\theta - \sigma_\tau \theta}{\tau}(t, x) \, dx \, dt. \quad (40)$$

Since for all $k \in \llbracket 1, N \rrbracket$ we have (in the weak sense)

$$\partial_\tau u_1^k - \Delta \left([d_{11}^\varepsilon(u_1^k) + a_{12}^\varepsilon(u_2^k)] u_1^k \right) + \varepsilon(w_1^k - \Delta w_1^k) = R_{12}^\varepsilon(\mathbf{u}^k),$$

we also can write, for all $t \in [0, T[$,

$$\langle \partial_\tau u_1^k, \theta(t) \rangle - \left\langle a_{11}^\varepsilon(u_1^k) + u_1^k a_{12}(u_2^k) + \varepsilon u_1^k u_2^k, \Delta \theta(t) \right\rangle + \varepsilon \langle w_1^k, \theta(t) - \Delta \theta(t) \rangle = \langle R_{12}^\varepsilon(\mathbf{u}^k), \theta(t) \rangle,$$

so that integrating on $[(k-1)\tau, k\tau]$ and summing over $k \in \llbracket 1, N \rrbracket$, we get

$$\begin{aligned} \sum_{k=1}^N \int_{(k-1)\tau}^{k\tau} \langle \partial_\tau u_1^k, \theta(t) \rangle \, dt - \int_0^T \int_\Omega \left[a_{11}^\varepsilon(\underline{u}_1^\tau) + \underline{u}_1^\tau a_{12}(\underline{u}_2^\tau) + \varepsilon \underline{u}_1^\tau \underline{u}_2^\tau \right] \cdot \Delta \theta(t, x) \, dx \, dt \\ + \varepsilon \sum_{k=1}^N \int_{(k-1)\tau}^{k\tau} \langle w_1^k, \theta(t) - \Delta \theta(t) \rangle \, dt = \int_0^T \int_\Omega R_{12}^\varepsilon(\underline{\mathbf{u}}^\tau)(t, x) \theta(t, x) \, dx \, dt. \end{aligned}$$

Using (40), we eventually get

$$\begin{aligned} \int_0^T \int_\Omega \mathbf{1}_{[0, T-\tau[}(t) \underline{u}_1^\tau \cdot \frac{\theta - \sigma_\tau \theta}{\tau}(t, x) \, dx \, dt - \int_0^T \int_\Omega \left[a_{11}^\varepsilon(\underline{u}_1^\tau) + \underline{u}_1^\tau a_{12}(\underline{u}_2^\tau) + \varepsilon \underline{u}_1^\tau \underline{u}_2^\tau \right] \cdot \Delta \theta(t, x) \, dx \, dt \\ + \varepsilon \sum_{k=1}^N \int_{(k-1)\tau}^{k\tau} \langle w_1^k, \theta(t) - \Delta \theta(t) \rangle \, dt \\ = \frac{1}{\tau} \int_0^\tau \langle u_1^1, \theta(t) \rangle \, dt + \int_0^T \int_\Omega R_{12}^\varepsilon(\underline{\mathbf{u}}^\tau)(t, x) \theta(t, x) \, dx \, dt. \end{aligned}$$

We now can study the limit $(\tau, \varepsilon) \rightarrow (0, 0)$. Thanks to the compactness result that we proved above, we get the existence of $\mathbf{u} := (u_1, u_2) \in L^1([0, T]; L^r(\Omega))$ (with $r < q_\alpha^*$) such that, up to a subsequence, we have

$$\begin{aligned} (\underline{u}_1^\tau)_{\tau, \varepsilon} &\xrightarrow{(\tau, \varepsilon) \rightarrow (0, 0)} u_1, \\ (\underline{u}_2^\tau)_{\tau, \varepsilon} &\xrightarrow{(\tau, \varepsilon) \rightarrow (0, 0)} u_2, \end{aligned}$$

in $L^1([0, T]; L^r(\Omega))$, and also almost everywhere on Q_T . Because of the dual estimate (34) $\left(\underline{u}_1^\tau \sqrt{a_{12}(\underline{u}_2^\tau)} \right)_{\tau, \varepsilon}$ is bounded in $L^2(Q_T)$. Using assumption **H2**, we see that a_{12} is at most linearly growing, so that using again the dual estimate (34) (but inverting the subscripts 1 and 2), $\left(a_{12}(\underline{u}_2^\tau) \right)_{\tau, \varepsilon}$ is bounded in $L^2(Q_T)$. Writing

$$\underline{u}_1^\tau a_{12}(\underline{u}_2^\tau) = \underline{u}_1^\tau \sqrt{a_{12}(\underline{u}_2^\tau)} \sqrt{a_{12}(\underline{u}_2^\tau)},$$

we see eventually that $\left(\underline{u}_1^\tau a_{12}(\underline{u}_2^\tau)\right)_{\tau,\varepsilon}$ is bounded in $L^{4/3}(Q_T)$ and we may thus extract a subsequence converging weakly in this space, and whose limit has to be equal to $u_1 a_{12}(u_2)$ (because of the previous almost everywhere convergence).

As for the self-diffusion, the dual estimate (34) ensures that $\left(\underline{u}_1^\tau \sqrt{d_{11}^\varepsilon(\underline{u}_1^\tau)}\right)_{\tau,\varepsilon}$ is bounded in $L^2(Q_T)$. Since d_{11}^ε is a nondecreasing function, we infer the boundedness of $\left(a_{11}^\varepsilon(\underline{u}_1^\tau)\right)_{\tau,\varepsilon} = \left(\underline{u}_1^\tau d_{11}^\varepsilon(\underline{u}_1^\tau)\right)_{\tau,\varepsilon}$ in $L^1(Q_T)$, so that the weak convergence of this sequence can be deduced from its uniform integrability (Dunford-Pettis theorem). Since $d_{11}^\varepsilon \leq d_{11}$ which is a nondecreasing continuous function, introducing the pseudo-inverse $g_{11}(t) := \inf\{x \in \mathbb{R}_+; x d_{11}(x) \geq t\}$ which is also going to $+\infty$ with t , we get

$$\begin{aligned} \int_0^T \int_\Omega \underline{u}_1^\tau d_{11}^\varepsilon(\underline{u}_1^\tau) \mathbf{1}_{\underline{u}_1^\tau d_{11}^\varepsilon(\underline{u}_1^\tau) \geq M}(t, x) dx dt &\leq \int_0^T \int_\Omega \underline{u}_1^\tau d_{11}^\varepsilon(\underline{u}_1^\tau) \mathbf{1}_{\underline{u}_1^\tau d_{11}(\underline{u}_1^\tau) \geq M}(t, x) dx dt \\ &= \int_0^T \int_\Omega \underline{u}_1^\tau d_{11}^\varepsilon(\underline{u}_1^\tau) \mathbf{1}_{\underline{u}_1^\tau \geq g_{11}(M)}(t, x) dx dt \\ &\leq \frac{1}{g_{11}(M)} \left\| \underline{u}_1^\tau \sqrt{d_{11}^\varepsilon(\underline{u}_1^\tau)} \right\|_{L^2(Q_T)}^2, \end{aligned}$$

which indeed goes to 0 with M , uniformly in ε, τ , thanks to (34).

We hence get

$$\int_0^T \int_\Omega \left[a_{11}^\varepsilon(\underline{u}_1^\tau) + \underline{u}_1^\tau a_{12}(\underline{u}_2^\tau) \right] \cdot \Delta \theta(t, x) dx dt \xrightarrow{(\tau,\varepsilon) \rightarrow (0,0)} \int_0^T \int_\Omega \left[a_{11}(u_1) + u_1 a_{12}(u_2) \right] \cdot \Delta \theta(t, x) dx dt.$$

Since θ is smooth, we have

$$\frac{\theta - \sigma_\tau \theta}{\tau} \xrightarrow{\tau \rightarrow 0} -\partial_t \theta,$$

uniformly on Q_T , and $\mathbf{1}_{[0, T-\tau]}$ converges to 1 in all $L^s(Q_T)$ ($s < \infty$), hence

$$\int_0^T \int_\Omega \mathbf{1}_{[0, T-\tau]}(t) \underline{u}_1^\tau \cdot \frac{\theta - \sigma_\tau \theta}{\tau}(t, x) dx dt \xrightarrow{(\tau,\varepsilon) \rightarrow (0,0)} - \int_0^T \int_\Omega u_1 \cdot \partial_t \theta(t, x) dx dt.$$

For the reactions terms, since the nonlinearities are always strictly sublinear or dominated by the self diffusion, one easily manages (using the dual estimate) to use Dunford Pettis criterion. Indeed, first, since $(\gamma_\varepsilon)_\varepsilon$ is increasing to the identity, the uniform integrability of $R_{12}^{-,\varepsilon}(\underline{u}^\tau)$ reduces to check this property for both $s_{11}(\underline{u}_1^\tau) \underline{u}_1^\tau$ and $s_{12}(\underline{u}_2^\tau) \underline{u}_1^\tau$. Since d_{11} is nondecreasing, we always have $\lim_{z \rightarrow +\infty} \frac{s_{11}(z)}{z d_{11}(z)} = 0$, so that using again the pseudo-inverse $f_{11}(t) := \inf\{x \in \mathbb{R}_+; x s_{11}(x) \geq t\}$, which is also going to $+\infty$ with t , we get for $M > 0$

$$\begin{aligned} \int_0^T \int_\Omega s_{11}(\underline{u}_1^\tau) \underline{u}_1^\tau \mathbf{1}_{s_{11}(\underline{u}_1^\tau) \underline{u}_1^\tau \geq M}(t, x) dx dt &= \int_0^T \int_\Omega s_{11}(\underline{u}_1^\tau) \underline{u}_1^\tau \mathbf{1}_{\underline{u}_1^\tau \geq f_{11}(M)}(t, x) dx dt \\ &\leq \sup_{z \geq f_{11}(M)} \frac{s_{11}(z)}{z d_{11}(z)} \left\| \underline{u}_1^\tau \sqrt{d_{11}(\underline{u}_1^\tau)} \right\|_{L^2(Q_T)}^2, \end{aligned}$$

which goes to 0 with M^{-1} uniformly in ε, τ thanks to (34). We used here the fact that estimate (34) is still true if one replaces d_{11}^ε by d_{11} (Fatou's lemma). For $s_{12}(\underline{u}_2^\tau) \underline{u}_1^\tau$, using again the pseudo-inverse, it is easy to exhibit some positive function ℓ going to $+\infty$ in $+\infty$ such as $s_{12}(\underline{u}_2^\tau) \underline{u}_1^\tau \geq M$ large enough, $\underline{u}_2^\tau \leq \ell(M) \Rightarrow \underline{u}_1^\tau \geq \ell(M)$. In the case of the previous implication, one may easily use the already noticed uniform integrability of $(\underline{u}_1^\tau)_\tau$ and if $\underline{u}_2^\tau \leq \ell(M)$, then we have by Young's inequality for any small $\delta > 0$,

$$u_1 s_{12}(u_2) \leq \delta u_1^2 a_{12}(u_2) + \frac{s_{12}^2(u_2)}{\delta a_{12}(u_2)},$$

so that we may conclude using (34) and assumption **H1** :

$$\lim_{z \rightarrow +\infty} \frac{s_{12}(z)}{z \sqrt{d_{22}(z) + a_{12}(z)}} = 0.$$

For the other terms, first notice that

$$\begin{aligned} \left| \varepsilon \sum_{k=1}^N \int_{(k-1)\tau}^{k\tau} \langle w_1^k, \theta(t) - \Delta \theta(t) \rangle dt \right| &\leq \varepsilon \|\theta - \Delta \theta\|_{L^\infty([0,T]; L^2(\Omega))} \sum_{k=1}^N \tau \|w_1^k\|_{L^2(\Omega)} \\ &\leq \|\theta - \Delta \theta\|_{L^\infty([0,T]; L^2(\Omega))} \varepsilon \tau \sqrt{N} \left\{ \sum_{k=1}^N \|w_1^k\|_{L^2(\Omega)}^2 \right\}^{1/2} \\ &\leq \|\theta - \Delta \theta\|_{L^\infty([0,T]; L^2(\Omega))} \varepsilon \tau \sqrt{N} \frac{\sqrt{K_T(\mathcal{E}_\varepsilon(\mathbf{u}^0) + 1)}}{\sqrt{\varepsilon \tau}} \\ &= \|\theta - \Delta \theta\|_{L^\infty([0,T]; L^2(\Omega))} \sqrt{\varepsilon} \sqrt{T} \sqrt{K_T(\mathcal{E}_\varepsilon(\mathbf{u}^0) + 1)} \xrightarrow{(\tau,\varepsilon) \rightarrow (0,0)} 0, \end{aligned}$$

where we used (22) for the last but one inequality.

We also have

$$\left| \int_0^T \int_{\Omega} \varepsilon \underline{u}_1^{\tau} \underline{u}_2^{\tau} \Delta \theta(t, x) \, dx \, dt \right| \leq \varepsilon \|\underline{u}_1^{\tau}\|_{L^2(Q_T)} \|\underline{u}_2^{\tau}\|_{L^2(Q_T)} \|\Delta \theta\|_{L^{\infty}(Q_T)},$$

which goes to zero with (τ, ε) thanks to the dual estimate (34).

Finally because of the continuity of θ , we can write

$$\frac{1}{\tau} \int_0^{\tau} \langle u_1^0, \theta(t) \rangle \, dt = \int_0^1 \langle u_1^0, \theta(\tau t) \rangle \, dt \xrightarrow{(\tau, \varepsilon) \rightarrow (0, 0)} \langle u_1^0, \theta(\tau t) \rangle = \int_{\Omega} u_1^0(x) \theta(0, x) \, dx,$$

so that it only remains to show

$$\frac{1}{\tau} \int_0^{\tau} \langle u_1^1 - u_1^0, \theta(t) \rangle \, dt = \int_0^{\tau} \langle \partial_{\tau} u_1^1, \theta(t) \rangle \, dt \xrightarrow{(\tau, \varepsilon) \rightarrow (0, 0)} 0.$$

From equation (21) for $k = 1$, $i = 1$, $j = 2$, we have $(\chi_1 = \theta(t))$

$$\begin{aligned} \langle \partial_{\tau} u_1^1, \theta(t) \rangle - \left\langle \left[a_{11}^{\varepsilon}(u_1^1) + u_1^1 a_{12}(u_2^1) + \varepsilon u_1^1 u_2^1 \right], \Delta \theta(t) \right\rangle + \varepsilon \langle w_1^1, \theta(t) - \Delta \theta(t) \rangle \\ = \langle R_{12}^{\varepsilon}(u_1^1, u_2^1), \theta(t) \rangle, \end{aligned}$$

that gives, since $L^1(\Omega) \hookrightarrow W^{2,\infty}(\Omega)'$, for some positive constant C_{θ} depending on θ ,

$$\begin{aligned} \frac{1}{C_{\theta}} \left| \int_0^{\tau} \langle \partial_{\tau} u_1^1, \theta(t) \rangle \, dt \right| &\leq \tau \|a_{11}^{\varepsilon}(u_1^1) + u_1^1 a_{12}(u_2^1) + \varepsilon u_1^1 u_2^1 + \varepsilon w_1^1 + R_{12}^{\varepsilon}(u_1^1, u_2^1)\|_{L^1(\Omega)} \\ &\leq \|\underline{u}_1^{\tau} d_{11}^{\varepsilon}(\underline{u}_1^{\tau}) \mathbf{1}_{[0, \tau] \times \Omega}\|_{L^1(Q_T)} + \|\underline{u}_1^{\tau} a_{12}(\underline{u}_2^{\tau}) \mathbf{1}_{[0, \tau] \times \Omega}\|_{L^1(Q_T)} \\ &\quad + \varepsilon \|\underline{u}_1^{\tau}\|_{L^2(Q_T)}^2 + \varepsilon \|\underline{u}_2^{\tau}\|_{L^2(Q_T)}^2 + \varepsilon \tau \sqrt{\mu(\Omega)} \|w_1^1\|_{L^2(\Omega)} + \|R_{12}^{\varepsilon}(\underline{u}^{\tau}) \mathbf{1}_{[0, \tau] \times \Omega}\|_{L^1(Q_T)}. \end{aligned}$$

As shown before, all the sequences written in front of the characteristic function $\mathbf{1}_{[0, \tau] \times \Omega}$ are uniformly integrable (we actually showed this for the self diffusion term, and have some $L^p(Q_T)$ with $p > 1$ for the two others), and they hence vanish with (τ, ε) . The three other terms also vanish, because of the duality bound for the $\|\underline{u}_i^{\tau}\|_{L^2(Q_T)}^2$ terms and because estimate (22) for the $\|w_1^1\|_{L^2(\Omega)}$ term.

We get then

$$\int_{Q_T} u_1 \left\{ \partial_t \theta - [d_{11}(u_1) + a_{22}(u_2)] \Delta \theta \right\} \, dx \, dt = \int_{\Omega} u_1^0(x) \theta(0, x) \, dx,$$

that is the weak formulation on Q_T of the equation

$$\partial_t u_1 - \Delta [a_{11}(u_1) + u_1 a_{12}(u_2)] = R_{12}(u_1, u_2),$$

initialized with $u_1(0, x) = u_1^0(x)$. A similar (symmetric) formulation holds for u_2 .

We end up the proof of our Theorem with the passage to the limit in the duality estimates. First notice that the constant D in (34) can be written as a polynomial function (with positive coefficients) of $(\|\mathbf{u}^0\|_{L^1(\mathbb{R})}, \mathcal{E}_{\varepsilon}(\mathbf{u}^0))$. Since $(\underline{u}_1^{\tau} \sqrt{d_{11}^{\varepsilon}(\underline{u}_1^{\tau}) + a_{12}^{\varepsilon}(\underline{u}_2^{\tau})})_{(\varepsilon, \tau)}$ converges almost everywhere to $u_1 \sqrt{d_{11}(u_1) + a_{12}(u_2)}$, and since $\mathcal{E}_{\varepsilon}(\mathbf{u}^0)$ converges to $\mathcal{E}(\mathbf{u}^0)$, the classical weak estimate ensures that

$$\left\| u_1 \sqrt{d_{11}(u_1) + a_{12}(u_2)} \right\|_{L^2(Q_T)} \leq P(\|\mathbf{u}^0\|_{L^1(\Omega)}, \mathcal{E}(\mathbf{u}^0)),$$

for some polynomial function P with positive coefficients. \square

References

- [1] H. Amann. Erratum: “Dynamic theory of quasilinear parabolic systems. III. Global existence” [Math. Z. **202** (1989), no. 2, 219–250; MR1013086 (90i:35125)]. Math. Z., 205(2):331, 1990.
- [2] M. Bendahmane, T. Lepoutre, A. Marrocco, and B. Perthame. Conservative cross diffusions and pattern formation through relaxation. Journal de Mathématiques Pures et Appliquées, 92(6):651 – 667, 2009.
- [3] D. Bothe, M. Pierre, and G. Rolland. Cross-diffusion limit for a reaction-diffusion system with fast reversible reaction. Communications in Partial Differential Equations, 37(11):1940–1966, 2012.
- [4] N. Boudiba and M. Pierre. Global existence for coupled reaction-diffusion systems. J. Math. Anal. Appl., 250:1–12, 2000.
- [5] J. Canizo, L. Desvillettes, K. Fellner, and F. Otto. In preparation.
- [6] L. Chen and A. Jüngel. Analysis of a multidimensional parabolic population model with strong cross-diffusion. SIAM J. Math. Anal., 36(1):301–322 (electronic), 2004.
- [7] L. Chen and A. Jüngel. Analysis of a parabolic cross-diffusion population model without self-diffusion. J. Differential Equations, 224(1):39–59, 2006.
- [8] Y. S. Choi, R. Lui, and Y. Yamada. Existence of global solutions for the Shigesada-Kawasaki-Teramoto model with strongly coupled cross-diffusion. Discrete Contin. Dyn. Syst., 10(3):719–730, 2004.

- [9] P. Degond, S. Génieys, and A. Jüngel. Symmetrization and entropy inequality for general diffusion equations. *C. R. Acad. Sci. Paris Sér. I Math.*, 325(9):963–968, 1997.
- [10] L. Desvillettes and F. Conforto. Rigorous passage to the limit in a system of reaction-diffusion equations towards a system including cross diffusions. *CMLA2009-34*, 2009.
- [11] L. Desvillettes, K. Fellner, M. Pierre, and J. Vovelle. Global existence for quadratic systems of reaction-diffusion. *Adv. Nonlinear Stud.*, 7(3):491–511, 2007.
- [12] P. Deuring. An initial-boundary-value problem for a certain density-dependent diffusion system. *Mathematische Zeitschrift*, 194:375–396, 1987.
- [13] G. Galiano, M. L. Garzón, and A. Jüngel. Semi-discretization in time and numerical convergence of solutions of a nonlinear cross-diffusion population model. *Numer. Math.*, 93(4):655–673, 2003.
- [14] D. Gilbarg and N. S. Trudinger. Elliptic partial differential equations of second order, volume 224 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, second edition, 1983.
- [15] S. Hittmeir and A. Jüngel. Cross diffusion preventing blow-up in the two-dimensional Keller-Segel model. *SIAM J. Math. Anal.*, 43(2):997–1022, 2011.
- [16] M. Iida, M. Mimura, and H. Ninomiya. Diffusion, cross-diffusion and competitive interaction. *J. Math. Biol.*, 53(4):617–641, 2006.
- [17] A. Jüngel and I. V. Stelzer. Entropy structure of a cross-diffusion tumor-growth model. *Math. Models Methods Appl. Sci.*, 22(7):1250009, 26, 2012.
- [18] S. Kawashima and Y. Shizuta. On the normal form of the symmetric hyperbolic-parabolic systems associated with the conservation laws. *Tohoku Math. J. (2)*, 40(3):449–464, 1988.
- [19] Y. Lou and S. Martinez. Evolution of cross-diffusion and self-diffusion. Preprint, pages 1–19, 2007.
- [20] Y. Lou, W.-M. Ni, and Y. Wu. On the global existence of a cross diffusion system. *Discrete Contin. Dynam. Systems*, 4(2):193–203, 1998.
- [21] R. H. Martin, Jr. and M. Pierre. Nonlinear reaction-diffusion systems. In *Nonlinear equations in the applied sciences*, volume 185 of *Math. Sci. Engrg.*, pages 363–398. Academic Press, Boston, MA, 1992.
- [22] J. D. Murray. Mathematical biology. II, volume 18 of *Interdisciplinary Applied Mathematics*. Springer-Verlag, New York, third edition, 2003. Spatial models and biomedical applications.
- [23] M. Pierre and D. Schmitt. Blowup in reaction-diffusion systems with dissipation of mass. *SIAM J. Math. Anal.*, 28(2):259–269, 1997.
- [24] N. Shigesada, K. Kawasaki, and E. Teramoto. Spatial segregation of interacting species. *J. Theoret. Biol.*, 79(1):83–99, 1979.
- [25] J. Simon. Compact sets in the space $L^p(0, T; B)$. *Ann. Mat. Pura Appl. (4)*, 146:65–96, 1987.
- [26] A. Yagi. Global solution to some quasilinear parabolic system in population dynamics. *Nonlinear Anal.*, 21(8):603–630, 1993.