Improved duality estimates and applications to reaction-diffusion equations

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Abstract

We present a refined duality estimate for parabolic equations. This estimate entails new results for systems of reaction-diffusion equations, including smoothness and exponential convergence towards equilibrium for equations with quadratic right-hand sides in two dimensions. For general systems in any space dimension, we obtain smooth solutions of reaction-diffusion systems coming out of reversible chemistry under an assumption that the diffusion coefficients are sufficiently close one to another.

1 Introduction

This paper presents a refined duality estimate for reaction-diffusion equations arising in the context of reversible chemistry, of the form

$$\partial_t a_i - d_i \, \Delta_x a_i = (\beta_i - \alpha_i) \left(l \prod_{j=1}^{n} a_j^{\alpha_j} - k \prod_{j=1}^{n} a_j^{\beta_j} \right), \qquad i = 1..n, \tag{1}$$

with the homogeneous Neumann boundary conditions

$$\nabla_x a_i(t,x) \cdot \nu(x) = 0 \quad \text{for } x \in \partial\Omega, \ t \geq 0 \tag{2}$$

corresponding to the diffusion of n species $\mathcal{A}_i$ with concentration $a_i := a_i(t,x) \geq 0$, $i = 1..n$ at time $t \geq 0$ and point $x \in \Omega \subset \mathbb{R}^N$, each with its own diffusion coefficient $d_i \geq 0$, and to the reversible chemical reaction

$$\alpha_1 \mathcal{A}_1 + \cdots + \alpha_n \mathcal{A}_n \rightleftharpoons \beta_1 \mathcal{A}_1 + \cdots + \beta_n \mathcal{A}_n, \quad \alpha_i, \beta_i \in \mathbb{N}, \tag{3}$$

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where the above reaction is modelled according to the *mass action law* with the stoichiometric coefficients $\alpha_i, \beta_i \in \mathbb{N}$ and with the (constant) reaction rates $l, k > 0$. The mixture is assumed to be confined in a domain $\Omega \subset \mathbb{R}^N$ as implied by the homogeneous Neumann condition (2), where $\nu(x)$ denotes the outward normal vector to Ω at point $x \in \partial\Omega$.

Moreover, we systematically denote by $\Omega_T = [0,T] \times \Omega$ (for any given $T > 0$) and by p' the Hölder conjugate exponent of p, i.e. $\frac{1}{p} + \frac{1}{p'} = 1$.

The mathematical difficulties in proving existence, smoothness and large-time behaviour theories for systems like (1), (2) increase with the degree of the polynomials terms (appearing in the r.h.s. of (1)) as well as with the number n of equations or the space dimension N.

The refined duality estimate that we shall derive in this paper depends yet on another parameter of (1), namely the maximal distance between the diffusion rates appearing in (1), that is

$$\delta := \sup_{i=1..n} \{d_i\} - \inf_{i=1..n} \{d_i\}. \tag{4}$$

It is easy to see that when $\delta = 0$, the system (1), (2) can be rewritten as the coupled system between $n-1$ heat equations (for the sums of two concentrations) and a single reaction-diffusion equation, which greatly simplifies the analysis compared to the general case when $\delta > 0$. In particular, the dynamics of system (1) for $\delta = 0$ satisfies a maximum principle, which fails to be true for general diffusion systems with $\delta > 0$ (except for special systems where the structure of the reaction terms enforces a maximum principle).

Perturbation methods can sometimes be used in order to transfer at least partly the properties of the system with $\delta = 0$ to the case when $\delta > 0$ is small, see e.g. [HM].

Our new estimate is also particularly efficient in the case when $\delta > 0$ is small, but it still gives results for all parameters δ in some situations, and even when for example one of the diffusion rates is 0. Moreover, when the smallness of δ is required, it can be explicitly estimated.

In order to obtain this estimate, we use a combination of ideas coming from maximal elliptic regularity, a Meyers-type estimate which provides an explicit perturbation argument, and duality methods in the line of e.g. [HMP, HM, PSch]. We end up with the following Proposition for parabolic equations with variable coefficients:

Proposition 1.1. *Let Ω be a bounded domain of $\mathbb{R}^N$ with smooth (e.g. $C^{2+\alpha}$, $\alpha > 0$) boundary $\partial\Omega$, $T > 0$, and $p \in]2, +\infty[$. We consider a coefficient function $M := M(t,x)$ satisfying*

$$0 < a \leq M(t,x) \leq b < +\infty \quad \text{for } (t,x) \in \Omega_T, \tag{5}$$

for some $0 < a < b < +\infty$, and an initial datum $u_0 \in L^p(\Omega)$.

Then, any weak solution u of the parabolic system:

$$\begin{cases} \partial_t u - \Delta_x (Mu) = 0 & \text{on} \quad \Omega_T, \\ u(0,x) = u_0(x) & \text{for} \quad x \in \Omega, \\ \nabla_x u \cdot \nu(x) = 0 & \text{on} \quad [0,T] \times \partial\Omega, \end{cases} \tag{6}$$

satisfies the estimate (where $p' < 2$ denotes the Hölder conjugate exponent of p)

$$\|u\|_{L^p(\Omega_T)} \leq (1 + b\, D_{a,b,p'})\, T^{1/p}\, \|u_0\|_{L^p(\Omega)}, \tag{7}$$

and where for any $a, b > 0$, $q \in]1,2[$

$$D_{a,b,q} := \frac{C_{\frac{a+b}{2},q}}{1 - C_{\frac{a+b}{2},q} \frac{b-a}{2}}, \tag{8}$$

provided that the following condition holds

$$C_{\frac{a+b}{2},p'} \, \frac{b-a}{2} < 1. \tag{9}$$

Here, the constant $C_{m,q} > 0$ is defined for $m > 0$, $q \in]1,2[$ as the best (that is, smallest) constant in the parabolic regularity estimate

$$\|\Delta_x v\|_{L^q(\Omega_T)} \leq C_{m,q}\, \|f\|_{L^q(\Omega_T)}, \tag{10}$$

where $v : [0,T] \times \Omega \to \mathbb{R}$ is the solution of the backward heat equation with homogeneous Neumann boundary conditions:

$$\begin{cases} \partial_t v + m\, \Delta_x v = f & \text{on} \quad \Omega_T, \\ v(T,x) = 0 & \text{for} \quad x \in \Omega, \\ \nabla_x v \cdot \nu(x) = 0 & \text{on} \quad [0,T] \times \partial\Omega. \end{cases} \tag{11}$$

We recall that one has $C_{m,q} < \infty$ for $m > 0$, $q \in]1,2[$ and in particular $C_{m,2} \leq \frac{1}{m}$ (cf. Lemma 2.1 below). Note that the constant $C_{m,q}$ may depend (besides on m and q) also on the domain Ω and the space dimension N, but does not depend on the time T.

The consequences of Proposition 1.1 can be best understood in the case of the most standard reversible chemical reaction, that is when (3) writes

$$\mathcal{A}_1 + \mathcal{A}_3 \rightleftharpoons \mathcal{A}_2 + \mathcal{A}_4 \tag{12}$$

and (1) becomes (after the rescaling of the nonessential constants k and l to unity)

$$\begin{cases} \partial_t a_1 - d_1\, \Delta_x a_1 = (a_4\, a_2 - a_1\, a_3), \\ \partial_t a_2 - d_2\, \Delta_x a_2 = -(a_4\, a_2 - a_1\, a_3), \\ \partial_t a_3 - d_3\, \Delta_x a_3 = (a_4\, a_2 - a_1\, a_3), \\ \partial_t a_4 - d_4\, \Delta_x a_4 = -(a_4\, a_2 - a_1\, a_3). \end{cases} \tag{13}$$

We recall that for this system (with the boundary conditions (2) and provided that $d_i > 0$ for $i = 1..4$), existence of weak solutions in $L^2(\log L)^2$ was obtained in [DFPV], together with the existence of strong (smooth) solutions when $N = 1$. This result was improved by Th. Goudon and A. Vasseur in [GV] thanks to a careful use of De Giorgi's method, see e.g. [DeG]. They showed that strong solutions also exist when $N = 2$. We also refer to [CV], where smooth solutions were shown to exist in any dimension for systems with a nonlinearity of power law type which is strictly subquadratic, see also e.g. [Ama].

We also recall two results on exponential convergence to the equilibrium: First, exponential convergence in any H^p norm in the one-dimensional case $N = 1$ was obtained for the system (13) with boundary conditions (2) in [DF08]. This result is based on the use of the entropy/entropy dissipation method with slowly growing a priori L^∞-bounds (cf. [TV], [DM]). It required "at most polynomially growing w.r.t. T" bounds for the quantities $\sup_{t\in[0,T]} \|a_i(t,\cdot)\|_{L^\infty(\Omega)}$. A related, yet non-explicit approach to entropy methods for reaction-diffusion type systems can be found e.g. in [Grö, GGH].

In a later improvement [DFEqua], the authors showed exponential convergence to equilibrium in relative entropy avoiding any L^∞-bounds on the solution. Thus, interpolating the weak global L^2 solutions constructed in [DFPV], one obtains exponential convergence towards equilibrium in any L^p norm with $p < 2$ for all space dimension $N > 1$.

It is interesting to point out that for space dimensions $N \geq 3$ the existence of global classical solutions for general (even constant) diffusion coefficients and initial data is an open problem despite the fact that all L^2 solutions converge exponentially towards the constant equilibrium in L^p with $p < 2$. Up to our knowledge, it is only known that if solutions to (13) with boundary conditions (2) would blow-up in the L^∞ norm, then such a concentration phenomenon would need to occur in at least two densities a_i at the same position $x_0 \in \Omega$ at the same time $t_0 > 0$ [HM]. Moreover, an upper bound on the Hausdorff dimension of singularities was given in [GV].

Thanks to Proposition 1.1, this paper is able to provide a direct proof of the result in [GV] (that is, without use of De Giorgi's method) when $N = 2$. Moreover, we can obtain immediately the exponential convergence of the solution of (13), (2) towards equilibrium in L^∞, which is a significant improvement on the above mentioned L^{2-0}-convergence of [DFEqua]. It is remarkable that in this specific case, *no smallness requirement for* δ appears in the assumptions. More precisely, we prove the

Proposition 1.2. *Let Ω be a bounded domain of $\mathbb{R}^2$ with smooth (e.g. $C^{2+\alpha}$, $\alpha > 0$) boundary $\partial\Omega$. For all $[i = 1..4]$ assume positive diffusion coefficients $d_i > 0$ and nonnegative initial data $a_{i0} \in L^\infty(\Omega)$.*

Then, there exists a weak nonnegative solution $a_i \in L^\infty([0, +\infty[\times\Omega)$ to the system (13) with homogenenous Neumann boundary conditions (2) subject to the initial data a_{i0} for all $[i = 1..4]$.

Moreover, we denote for $[i = 1..4]$ by $a_{i\infty} > 0$ the equilibrium values of the concentrations a_i: Thus, $\{a_{i\infty}\}_{i=1..4}$ is the unique vector of positive constants balancing the reaction rate

$$a_{1\infty}\, a_{3\infty} = a_{2\infty}\, a_{4\infty}$$

and satisfying the three (linear independent) mass-conservation laws

$$\begin{aligned} a_{1\infty} + a_{2\infty} &= \frac{1}{|\Omega|}\int_\Omega (a_{10} + a_{20})\, dx, \\ a_{1\infty} + a_{4\infty} &= \frac{1}{|\Omega|}\int_\Omega (a_{10} + a_{40})\, dx, \\ a_{2\infty} + a_{3\infty} &= \frac{1}{|\Omega|}\int_\Omega (a_{20} + a_{30})\, dx. \end{aligned}$$

Then, there exist two constants $\kappa_1, \kappa_2 > 0$ such that

$$\forall t \geq 0, \qquad \sum_{i=1}^{4} \|a_i(t,\cdot) - a_{i\infty}\|_{L^\infty(\Omega)} \leq \kappa_1\, e^{-\kappa_2\, t}. \tag{14}$$

Moreover, the norm $\|a_i\|_{L^\infty([0,+\infty[\times\Omega)}$ and the constants κ_1, κ_2 can be explicitly bounded in terms of the domain Ω, space dimension N, the norm $\|a_{i0}\|_{L^\infty(\Omega)}$ of the initial data and the diffusion coefficients d_i, $[i = 1..4]$.

Proposition 1.1 also entails that a similar result to Proposition 1.2 holds in any space dimension provided that $\delta > 0$ is small enough. More precisely, we prove the:

Proposition 1.3. *Let Ω be a bounded domain of $\mathbb{R}^N$ with smooth (e.g. $C^{2+\alpha}$, $\alpha > 0$) boundary $\partial\Omega$. For all $[i = 1..4]$ assume positive diffusion coefficients $d_i > 0$ such that $0 < a = \inf\{d_i\}_{i=1..4}$, $0 < b = \sup\{d_i\}_{i=1..4}$ and nonnegative initial data $a_{i0} \in L^\infty(\Omega)$ $(i = 1..4)$.*

Then, if $\delta = b - a < 2\,(C_{\frac{a+b}{2},1+2/N})^{-1}$, there exists a nonnegative weak solution $a_i \in L^\infty([0,+\infty[\times\Omega)$ to the system (13) with homogeneous Neumann boundary conditions (2) subject to the initial data a_{i0}.

Moreover (with the notation of the previous Proposition 1.2), there exist two constants $\kappa_1, \kappa_2 > 0$ such that

$$\forall t \geq 0, \qquad \sum_{i=1}^{4} \|a_i(t,\cdot) - a_{i\infty}\|_{L^\infty(\Omega)} \leq \kappa_1\, e^{-\kappa_2\, t}. \tag{15}$$

Here, the norm $\|a_i\|_{L^\infty([0,+\infty[\times\Omega)}$ and the constants κ_1, κ_2 can be explicitly bounded in terms of the domain Ω, space dimension N, the norm $\|a_{i0}\|_{L^\infty(\Omega)}$ of the initial data, and the diffusion coefficients d_i, $(i = 1..4)$.

A further result based on Proposition 1.1 states an existence theorem for weak and (with more stringent assumption) bounded weak solutions of (1), (2) when the r.h.s. of (1) is not necessarily quadratic anymore. An assumption about the smallness of $\delta > 0$ is still needed here.

Proposition 1.4. *Let Ω be a bounded domain of $\mathbb{R}^N$ with smooth (e.g. $C^{2+\alpha}$, $\alpha > 0$) boundary $\partial\Omega$. For all $i = 1..n$ assume positive diffusion coefficients $d_i > 0$ such that $0 < a = \inf\{d_i\}_{i=1..n}$, $0 < b = \sup\{d_i\}_{i=1..n}$ and nonnegative initial data $a_{i0} \in L^\infty(\Omega)$. Moreover, let $k, l > 0$, $\alpha_i, \beta_i \in \mathbb{N}$ be such that at least two coefficients $\beta_i - \alpha_i$ are different from 0 and have opposite signs. We define $Q = \sup\{\sum_{i=1}^n \alpha_i, \sum_{i=1}^n \beta_i\}$ and assume that $Q \geq 3$.*

Then, if $\delta = b - a < 2\,(C_{\frac{a+b}{2},Q'})^{-1}$ (where Q' is the Hölder conjugate of Q), there exists a nonnegative weak solution $a_i \in L^Q(\Omega_T)$ for all $T > 0$ to the system (1), (2) with the initial data a_{i0}.

Moreover, if $\delta = b - a < 2\,(C_{\frac{a+b}{2},\frac{(Q-1)\,(N+2)}{(Q-1)\,(N+2)-2}})^{-1}$, then the solution a_i lies in $L^\infty(\Omega_T)$ for all $T > 0$.

Finally, we show that there exist bounded weak solutions to the system (13) with homogeneous Neumann boundary conditions (2) in space dimension 2 even when one of the diffusion rates (say d_4 w.l.o.g.) is equal to 0. We remark that existence of global weak solutions in the case of degenerate diffusion was also shown in [DFEqua].

Proposition 1.5. *Let Ω be a bounded domain of $\mathbb{R}^2$ with smooth (e.g. $C^{2+\alpha}$, $\alpha > 0$) boundary $\partial\Omega$. For all $[i = 1..4]$ assume nonnegative initial data $a_{i0} \in L^\infty(\Omega)$ and nonnegative diffusion coefficients $d_i \geq 0$ with e.g. $d_i > 0$, $[i = 1..3]$ and $d_4 = 0$.*

Then, there exists a nonnegative weak solution a_i lying in $L^\infty(\Omega_T)$ for all $T > 0$ to the system (13), (2) (without the Neumann boundary condition on a_4) subject to the initial data a_{i0}.

Remark 1.6. The assumption that the initial data lie in L^∞ enables to give a simple formulation of the Propositions above, but it is not optimal (if one is only interested in the bounds of the solutions after a given positive time $t_0 > 0$). For example, in the case of Proposition 1.2, it is easy to see that if the initial data lie in $L^p(\Omega)$ for some $p > 2$, then the conclusion remains true with the time interval $[0, +\infty[$ changed into $[t_0, +\infty[$ (for any $t_0 > 0$) for the bounds. A more careful analysis (cf. Remark 2.3) shows that the assumption on the initial data can even be relaxed to $L^p(\Omega)$ for some $p > 1$.

Remark 1.7. Classical bootstrap arguments also show that all the above weak solutions, once they belong to $L^\infty(Q_T)$, are in fact strong and smooth provided that the set Ω has a smooth enough boundary (and provided that the initial datum is also smooth enough, if one wishes to get smoothness even at point $t = 0$). Moreover in this case, those solutions are unique (in the set of smooth enough solutions) and (in the case of Propositions 1.2 and 1.3) converge towards equilibrium exponentially fast (with explicitly computable constants) in any H^p norm ($p \in \mathbb{N}$), thanks to interpolation arguments similar to those exposed in [DF08].

Remark 1.8. The smoothness assumption $C^{2+\alpha}$, $\alpha > 0$ on the boundary $\partial\Omega$ is likely not optimal. One could conjecture that the above results hold true also for $C^{1+\alpha}$ boundaries or even Lipschitz boundaries. However, for more general boundaries, we lack, for instance, a reference which states explicitly the time-independence of the constant $C_{m,q}$ in (10).

The paper is organized as follows: In Section 2, we present the Proof of Proposition 1.1. Section 3 is devoted to the applications of Proposition 1.1 to the "four species" system (13), first in dimension 2 (Proof of Proposition 1.2) and then in any dimension (Proof of Proposition 1.3). Finally, we present in Section 4 the extensions to more general reaction-diffusion systems (Proof of Proposition 1.4) and to the case when one diffusion rate is 0 (Proof of Proposition 1.5).

2 An estimate for singular parabolic problems

We first recall a well-known result for the heat equation, which ensures that for $m > 0$, $p \in]1, 2[$ the constants $C_{m,p}$ stated in eq. (10) are well-defined, finite and time-independent.

Lemma 2.1. *Let Ω be a bounded domain of $\mathbb{R}^N$ with smooth (e.g. $C^{2+\alpha}$, $\alpha > 0$) boundary $\partial\Omega$, $m > 0$, and $p \in]1, 2]$. Then, there exists a constant $C_{m,p} > 0$ depending on m, p, the domain Ω and the space dimension N, but not on T, such that the solution $v : [0, T] \times \Omega \to \mathbb{R}$ of the backward heat equation with homogeneous Neumann boundary conditions (11) satisfies*

$$\|\Delta_x v\|_{L^p(\Omega_T)} \leq C_{m,p} \, \|f\|_{L^p(\Omega_T)}, \tag{16}$$

where $f \in L^p(\Omega_T)$ is the r.h.s. of (11). Moreover, $C_{m,2} \leq 1/m$.

Proof of Lemma 2.1. After introducing the time variable $\tau = T - t \in [0, T]$, the backward heat equation (11) with Neumann boundary conditions and zero end data transforms into the forward heat equation with zero initial data:

$$\begin{cases} \partial_\tau v - m\,\Delta_x v = -f & \text{on} \quad \Omega_T, \\ v(0, x) = 0 & \text{for} \quad x \in \Omega, \\ \nabla_x v \cdot \nu(x) = 0 & \text{on} \quad [0, T] \times \partial\Omega. \end{cases}$$

Moreover, the semigroup of the forward heat equation with homogeneous Neumann boundary condition satisfies the contraction property $\|e^{t\Delta_x} v(0, \cdot)\|_p \leq$

$\|v(0,\cdot)\|_p$ for all $p \in [1,\infty]$ and for all $t \geq 0$. Thus, the statement of the Lemma follows from [L], where it is explicitly stated that $C_{m,p}$ can be taken as time-independent. In particular, the Hilbert space case $p = 2$ allows explicit calculations by testing the above forward heat equation with $-\Delta_x$, which shows that $C_{m,2} \leq \frac{1}{m}$, see eq. (32) below. □

We now consider the corresponding problem with variable diffusion rate and obtain similar estimates:

Lemma 2.2. *Let Ω be a bounded domain of $\mathbb{R}^N$ with smooth (e.g. $C^{2+\alpha}$, $\alpha > 0$) boundary $\partial\Omega$, $T > 0$, $p \in]1,2]$, and $M := M(t,x)$ be bounded above and below; i.-e. for some $0 < a \leq b$,*

$$0 < a \leq M(t,x) \leq b < +\infty \quad \text{for } (t,x) \in \Omega_T. \tag{17}$$

We assume (using the notation of Lemma 2.1) that

$$C_{\frac{a+b}{2},p}\, \frac{b-a}{2} < 1. \tag{18}$$

We consider $f \in L^p(\Omega_T)$, and a solution v of the parabolic equation with variable diffusion rate given by M:

$$\begin{cases} \partial_t v + M\,\Delta_x v = f & \text{on} \quad \Omega_T, \\ v(T,x) = 0 & \text{for} \quad x \in \Omega, \\ \nabla_x v \cdot \nu(x) = 0 & \text{on} \quad [0,T] \times \partial\Omega. \end{cases} \tag{19}$$

Then,

$$\|\Delta_x v\|_{L^p(\Omega_T)} \leq D_{a,b,p}\, \|f\|_{L^p(\Omega_T)}, \tag{20}$$

and

$$\|v(0,\cdot)\|_{L^p(\Omega)} \leq (1 + b\, D_{a,b,p})\, T^{1/p'}\, \|f\|_{L^p(\Omega_T)}, \tag{21}$$

where $D_{a,b,p}$ is given by (8), i.e.

$$D_{a,b,p} := \frac{C_{\frac{a+b}{2},p}}{1 - C_{\frac{a+b}{2},p} \frac{b-a}{2}},$$

Proof of Lemma 2.2. In order to find estimates analogous to (16) for our variable coefficients parabolic equation, we take $m := (a+b)/2$ and rewrite (19) as the perturbative problem

$$\partial_t v + m\,\Delta_x v = (m - M)\ \Delta_x v + f. \tag{22}$$

Then from (16), we get

$$\begin{aligned} \|\Delta_x v\|_{L^p(\Omega_T)} &\leq C_{m,p}\ \|(m-M)\,\Delta_x v + f\|_{L^p(\Omega_T)} \\ &\leq C_{m,p} \left(\frac{b-a}{2}\, \|\Delta_x v\|_{L^p(\Omega_T)} + \|f\|_{L^p(\Omega_T)} \right). \end{aligned} \tag{23}$$

Provided that (18) holds, this directly implies (20).

Using now estimate (20) in eq. (19), we get

$$\|\partial_t v\|_{L^p(\Omega_T)} \leq (1 + b\, D_{a,b,p})\, \|f\|_{L^p(\Omega_T)}. \tag{24}$$

Taking into account that $v(T, x) = 0$ for $x \in \Omega$,

$$\begin{aligned} \|v(0,\cdot)\|_{L^p(\Omega)} &= \Big\| \int_0^T \partial_t v(t,\cdot)\, dt \Big\|_{L^p(\Omega)} \\ &\leq \int_0^T \|\partial_t v(t,\cdot)\|_{L^p(\Omega)}\, dt \leq \|\partial_t v(t,\cdot)\|_{L^p(\Omega_T)} T^{1/p'}, \end{aligned} \tag{25}$$

using Hölder's inequality in the last step. Together with (24), this proves Lemma 2.2. □

From the previous Lemmas, we obtain by duality Proposition 1.1.

Proof of Proposition 1.1. Take any $f \in L^{p'}(\Omega_T)$, and consider v the solution of the backward heat equation (19). Testing (19) with the solution u of (6), one easily checks that

$$\frac{d}{dt}\left(\int_\Omega u(t,x)\, v(t,x)\, dx \right) = \int_\Omega u(t,x)\, f(t,x)\, dx,$$

which implies that

$$\begin{aligned} \left| \int_{\Omega_T} u f \right| = \left| -\int_\Omega u_0(x) v(0,x)\, dx \right| &\leq \|u_0\|_{L^p(\Omega)} \|v(0,\cdot)\|_{L^{p'}(\Omega)} \\ &\leq (1 + b\, D_{a,b,p'})\, T^{1/p}\, \|u_0\|_{L^p(\Omega)}\, \|f\|_{L^{p'}(\Omega_T)}, \end{aligned}$$

where we have used (21) for the last inequality, with p replaced by p'. As this holds for an arbitrary $f \in L^{p'}(\Omega_T)$, we conclude that (7) holds, and Proposition 1.1 is proven. □

Remark 2.3. In fact, one can observe that estimate (25) is not optimal. One can show using the properties of the heat equation that

$$\|v(0,\cdot)\|_{L^r(\Omega)} \leq C_T\, \left(\|\partial_t v\|_{L^p(\Omega_T)} + \|\Delta_x v\|_{L^p(\Omega_T)} \right),$$

for any $r < \frac{p\,N}{N+2-2p}$, and any $T > 0$ [r can be taken as large as wanted if $2p > N + 2$].

As a consequence, in the proof by duality of Proposition 1.1, the norm $\|u_0\|_{L^p(\Omega)}$ can be replaced by the weaker norm $\|u_0\|_{L^q(\Omega)}$, for any $q > p/(1 + 2/N)$. This improvement allows one to consider more singular initial data in the reaction-diffusion problems studied in the sequel.

3 The "four species" equation

We now turn to the application of Proposition 1.1 to the "four species" system (13).

3.1 A general *a priori* estimate

We begin with the following *a priori* estimate for the "four species" equation, which is a direct consequence of Proposition 1.1:

Lemma 3.1. *Let Ω be a bounded domain of $\mathbb{R}^N$ with smooth ($C^{2+\alpha}$, $\alpha > 0$) boundary $\partial\Omega$ and $T > 0$. Consider a weak solution $\{a_i\}_{i=1,\dots,4}$ to system (13) with homogeneous Neumann boundary conditions (2) on $[0,T]$ and initial condition $a_{i0} \in L^p(\Omega)$ ($i = 1..4$) for some $p > 2$, and diffusion rates $d_i > 0$ ($i = 1..4$). We denote*

$$a := \min_{i=1,\dots,4}\{d_i\}, \quad b := \max_{i=1,\dots,4}\{d_i\}, \tag{26}$$

and assume that

$$C_{\frac{a+b}{2},p'} \, \frac{b-a}{2} < 1. \tag{27}$$

Then

$$\|a_i\|_{L^p(\Omega_T)} \le (1 + b\, D_{a,b,p'})\, T^{1/p} \left\| \sum_{j=1}^{4} a_{j0} \right\|_{L^p(\Omega)}, \tag{28}$$

for $i = 1..4$.

Here, $C_{m,p}$ and $D_{a,b,q}$ are the constants defined in Proposition 1.1.

Proof of Lemma 3.1. We call $u := \sum_{i=1}^4 a_i$ the total local mass of the system. Then u satisfies the forward heat equation (6) with

$$M(t,x) := \frac{\sum_{i=1}^4 d_i\, a_i(t,x)}{\sum_{i=1}^4 a_i(t,x)}\,. \tag{29}$$

We observe that the bound (5) holds. Moreover, assumption (27) is identical to assumption (9). As a consequence, thanks to Proposition 1.1, we end up with estimate (28). □

Next we turn to a Lemma which is specially devised for the two-dimensional case. Note that it does not depend on the size of $b - a$:

Lemma 3.2. *Let Ω be a bounded domain of $\mathbb{R}^2$ with smooth ($C^{2+\alpha}$) boundary $\partial\Omega$, $T > 0$ and diffusion rates $d_i > 0$ ($i = 1..4$). We still use the notation (26).*

Then, one can find two constants $K_1 > 0$ and $p > 2$ depending on a, b and Ω, such that any weak solution $(a_i)_{i=1..4}$ of system (13) with the homogenenous Neumann boundary conditions (2) and initial conditions in $L^p(\Omega)$ satisfies

$$\|a_i\|_{L^p(\Omega_T)} \leq K_1 \,(1+T)^{1/2} \left\| \sum_{i=1}^{4} a_{i0} \right\|_{L^p(\Omega)}. \tag{30}$$

Proof of Lemma 3.2. At first we shall deduce that for any $m > 0$ and for $3/2 \leq r \leq 2$ the following esimtate holds:

$$C_{m,r} \leq m^{-\frac{4}{r}\left(r-\frac{3}{2}\right)} \,(C_{m,3/2})^{\frac{3}{r}(2-r)}. \tag{31}$$

Indeed, multiplying (11) by $\Delta_x v$ and integrating on Ω_T, we easily obtain

$$\|\Delta_x v\|_{L^2(\Omega_T)} \leq \frac{1}{m}\, \|f\|_{L^2(\Omega_T)}, \tag{32}$$

so that $C_{m,2} \leq m^{-1}$ for $m > 0$.

Then, an interpolation with (16) for $p = \frac{3}{2}$ gives for all $r \in [3/2, 2]$

$$\|\Delta_x v\|_{L^r(\Omega_T)} \leq m^{-\theta}\,(C_{m,3/2})^{1-\theta}\, \|f\|_{L^r(\Omega_T)}, \tag{33}$$

with the interpolation exponent $\theta \in [0,1]$ satisfying

$$\frac{\theta}{2} + \frac{1-\theta}{3/2} = \frac{1}{r} \quad \Rightarrow \quad \theta = \frac{4r-6}{r},$$

which yields (31).

Using now (31) for $r = p'$ and $m = \frac{a+b}{2}$, we see that eq. (27) is satisfied as soon as the following inequality is satisfied:

$$-\theta(p')\ln(m) + (1-\theta(p'))\ln(C_{m,3/2}) + \ln\left(\frac{b-a}{2}\right) < 0,$$

which yields with $1-\theta(p') = \frac{3}{p'}(2-p')$

$$\frac{3}{p'}\,(2-p')\,\left(\log(C_{\frac{a+b}{2},3/2}) + \log\left(\frac{a+b}{2}\right)\right) < \log\left(\frac{a+b}{b-a}\right). \tag{34}$$

Thus, the condition (27) is satisfied provided that $\frac{a+b}{2}\,C_{\frac{a+b}{2},3/2} > 1$ as soon as

$$2-p' < \frac{p'}{3}\,\frac{\log\left(\frac{a+b}{b-a}\right)}{\log(\frac{a+b}{2}\,C_{\frac{a+b}{2},3/2})},$$

and, therefore, as soon as we choose a $p' \in [3/2, 2]$ satisfying

$$2-p' < \frac{1}{2}\,\frac{\log\left(\frac{a+b}{b-a}\right)}{\log(\frac{a+b}{2}\,C_{\frac{a+b}{2},3/2})}.$$

Note that in the case $\frac{a+b}{2}\,C_{\frac{a+b}{2},3/2} < 1$, the eq. (34) implies that condition (27) is always satisfied for all $p' \in [3/2, 2]$ since $\log(\frac{a+b}{b-a}) > 0$. □

3.2 Polynomial w.r.t. time bootstrap estimates

We prove below a standard estimate for the heat equation (with Neumann boundary condition) which amounts to proving that the corresponding Green function has the same singularity as the Green function in the case of the whole x-space $\mathbb{R}^N$

We put the stress on the dependence of the constants w.r.t. the length T of the time interval (this dependence is only tracked with very great effort in the classical books like [LSU], cf. Remark 3.4 below).

Lemma 3.3. *Let Ω be a bounded domain of $\mathbb{R}^N$ with smooth ($C^{2+\alpha}$) boundary $\partial\Omega$ and $T > 0$. Let u be a solution of the forward heat equation with r.h.s. $f \in L^q(\Omega_T)$ and homogeneous Neumann boundary conditions.*

$$\begin{cases} \partial_t u - d\,\Delta_x u = f & \text{on} \quad \Omega_T, \\ u(0,x) = u_0(x) & \text{for} \quad x \in \Omega, \\ \nabla u \cdot \nu(x) = 0 & \text{on} \quad [0,T] \times \partial\Omega. \end{cases} \tag{35}$$

If $1 < q < \frac{N+2}{2}$, we consider $s = \frac{q\,(N+2)}{N+2-2q} > 0$, and assume further that the initial datum u_0 belong to $L^s(\Omega)$.

Then, for any $0 < \epsilon < s-1$, there is a constant $C_T > 0$ depending on N, Ω, ϵ, d, q, $\|u_0\|_{L^s(\Omega)}$, $\|f\|_{L^q(\Omega_T)}$ and which has an at most polynomial dependence w.r.t. T, such that

$$\|u\|_{L^{s-\epsilon}(\Omega_T)} \le C_T. \tag{36}$$

On the other hand, if $q \ge \frac{N+2}{2}$, we assume that the initial datum u_0 belongs to $L^\infty(\Omega)$. Then for any $r \in [1,+\infty[$, there exists a constant C_T depending on N, Ω, r, d, q, $\|u_0\|_{L^\infty(\Omega)}$, $\|f\|_{L^q(\Omega_T)}$ and which has an at most polynomial dependence w.r.t. T, such that

$$\|u\|_{L^r(\Omega_T)} \le C_T. \tag{37}$$

Remark 3.4. The statement of the above lemma is classical (and non even optimal) except that it crucially shows that the regularising effect of a parabolic equation involves constants which depend polynomially on the time interval $[0,T]$ for all $T > 0$. In 1D, this was already shown in [DF08] using a Fourier representation of the solution. For general domains however, the polynomial dependence of the constants seems to be nowhere in the literature. Moreover, tracking the constants, for instance, in the approach of [LSU] (where the Green function for the half-space problem are used locally along the sufficiently smooth boundary as transformed approximating problem) is much more difficult than the following proof.

Proof of Lemma 3.3. We consider a solution of eq. (35).

Step 1: Setting $p_0 = q$ and testing (35) with $p_0\,|u|^{p_0-1}\mathrm{sgn}(u)$ (more precisely by testing with a smoothed version of the modulus $|u|$ and its derivative $\mathrm{sgn}(u)$

and letting then the smoothing tend to zero) we obtain by integration-by-parts and Hölders inequality with the constant $C_0(p_0) := \frac{4(p_0-1)\,d}{p_0}$:

$$\frac{d}{dt}\left(\|u\|^{p_0}_{L^{p_0}_x}\right) + C_0(p_0,d)\int_{\Omega}\left|\nabla_x(u^{p_0/2})\right|^2 dx \le p_0\,\|f\|_{L^{p_0}_x}\,(\|u\|^{p_0}_{L^{p_0}_x})^{\frac{p_0-1}{p_0}}. \quad (38)$$

Then, the Gronwall lemma $\dot{y} \le \alpha(t)y^{1-\frac{1}{p_0}} \Rightarrow\; y(T) \le \left[y(0)^{\frac{1}{p_0}} + \frac{1}{p_0}\int_0^T \alpha(t)\,dt\right]^{p_0}$ yields for all $T>0$:

$$\begin{aligned}\|u\|^{p_0}_{L^{p_0}_x}(T) &\le \left[\|u_0\|_{L^{p_0}_x} + \int_0^T \|f\|_{L^{p_0}_x}(s)\,ds\right]^{p_0} \le \left[\|u_0\|_{L^{p_0}_x} + \|f\|_{L^{p_0}_{t,x}} T^{\frac{p_0-1}{p_0}}\right]^{p_0}\\ &\le 2^{p_0-1}\left[\|u_0\|^{p_0}_{L^{p_0}_x} + \|f\|^{p_0}_{L^{p_0}_{t,x}} T^{p_0-1}\right] := C_{T,0}.\end{aligned}$$

We thus conclude that there is a constant $C_{T,0}$ depending only on p_0, d, $\|u_0\|_{L^{p_0}_x}$, $\|f\|_{L^{p_0}_{t,x}}$ and polynomially on T such that

$$\sup_{t\in[0,T]}\|u\|^{p_0}_{L^{p_0}_x}(t) \le C_{T,0}. \quad (39)$$

Step 2: Gradient estimate and Sobolev embedding. The integration-in-time of (38) and Hölder's inequality show

$$C_0\int_{\Omega_T}\left|\nabla_x(u^{p_0/2})\right|^2 dx \le \|u_0\|^{p_0}_{L^{p_0}_x} + p_0\,\|f\|_{L^{p_0}_{t,x}}\,\|u\|^{p_0-1}_{L^{p_0}_{t,x}}.$$

The above estimate and Sobolev's embedding for H^1 with constant C_{S} yields for $s_0<\infty$ for $N=2$ and $s_0=\frac{p_0 N}{N-2}$ for $N>2$ (together with Young's inequality)

$$\int_0^T \|u\|^{p_0}_{L^{s_0}_x}\,dt \le \frac{C_{\mathrm{S}}^2}{C_0}\left[\|u_0\|^{p_0}_{L^{p_0}_x} + p_0\,\|f\|_{L^{p_0}_{t,x}}(T\,C_{T,0})^{\frac{p_0-1}{p_0}}\right] := D_{T,0}, \quad (40)$$

where $D_{T,0}$ is a constant depending only on p_0, d, $\|u_0\|_{L^{p_0}_x}$, $\|f\|_{L^{p_0}_{t,x}}$ and polynomially on T.

In the Steps 3 and 4, we construct a sequence of exponents p_n, s_n and bounds

$$\sup_{t\in[0,T]}\|u\|^{p_n}_{L^{p_n}_x}(t) \le C_{T,n}, \quad (41)$$

$$\int_0^T \|u\|^{p_n}_{L^{s_n}_x}\,dt \le D_{T,n}. \quad (42)$$

In particular we set $s_n<\infty$ if $N=2$ and $s_n = p_n\frac{N}{N-2}$ if $N\ge 3$.

Step 3: Iteration of (41): Similar to Step 1 we test (35) with $p_{n+1}\,u^{p_{n+1}-1}$:

$$\frac{d}{dt}\left(\|u\|^{p_{n+1}}_{L^{p_{n+1}}_x}\right) + C_{n+1}\int_{\Omega}\left|\nabla_x(u^{p_{n+1}/2})\right|^2 dx = p_{n+1}\int_{\Omega} f\,u^{p_{n+1}-1}\,dx, \quad (43)$$

where $C_{n+1}(p_{n+1}) := \frac{4(p_{n+1}-1)\,d}{p_{n+1}}$. In order to iterate the bound (41), we fix the exponent p_{n+1} by introducing the n-independent exponent

$$r = \frac{\frac{s_n}{p_0} - 1}{s_n - p_{n+1}} := 1 - \frac{2}{N} + \frac{2}{Np_0} \quad \text{iff} \quad N \geq 3 \quad \text{which satisfy } \frac{1}{p_0} < r < 1, \tag{44}$$

and any r satisfying $\frac{1}{p_0} < r < 1$ if $N = 2$. Then, we estimate with $p_{n+1} - 1 = p_{n+1}(1-r) + s_n(r - 1/p_0)$ the above right-hand side of (43) by Hölder's inequality

$$\int_\Omega f\, u^{s_n(r-1/p_0)}\, u^{p_{n+1}(1-r)}\, dx \leq \|f\|_{L_x^{p_0}}\, \|u\|_{L_x^{s_n}}^{s_n(r-1/p_0)}\, \|u\|_{L_x^{p_{n+1}}}^{p_{n+1}(1-r)}, \tag{45}$$

and a Gronwall estimate for $\dot{y} \leq \alpha(t) y^{1-r}$ yields

$$\begin{aligned}
\|u\|_{L_x^{p_{n+1}}}^{p_{n+1}}(T) &\leq \left[\|u_0\|_{L_x^{p_{n+1}}}^{r\,p_{n+1}} + p_{n+1} r \int_0^T \|f\|_{L_x^{p_0}}\, \|u\|_{L_x^{s_n}}^{s_n(r-1/p_0)}\, dt \right]^{1/r} \\
&\leq \left[\|u_0\|_{L_x^{p_{n+1}}}^{r\,p_{n+1}} + p_{n+1} r \|f\|_{L_{t,x}^{p_0}} \left(\int_0^T \|u\|_{L_x^{s_n}}^{s_n(r-1/p_0)\frac{p_0}{p_0-1}}\, dt \right)^{\frac{p_0-1}{p_0}} \right]^{1/r}.
\end{aligned}$$

Thus, by the definition of r we have $s_n(r - 1/p_0)\frac{p_0}{p_0-1} = s_n \frac{N-2}{N} = p_n$, and we are able to use the bound (42) to obtain

$$\|u\|_{L_x^{p_{n+1}}}^{p_{n+1}}(T) \leq \left[\|u_0\|_{L_x^{p_{n+1}}}^{r\,p_{n+1}} + p_{n+1} r \|f\|_{L_{t,x}^{p_0}} D_{T,n}^{\frac{p_0-1}{p_0}} \right]^{1/r} =: C_{T,n+1}. \tag{46}$$

Step 4: Iteration of (42): Returning to (43) and (45), we collect

$$\begin{aligned}
\frac{1}{r}\frac{d}{dt}\left([\|u\|_{L_x^{p_{n+1}}}^{p_{n+1}}]^r \right) + C_{n+1}\, \|u\|_{L_x^{p_{n+1}}}^{(r-1)\,p_{n+1}} \int_\Omega \left| \nabla_x (u^{\frac{p_{n+1}}{2}}) \right|^2 dx \\
\leq p_{n+1}\, \|f\|_{L_x^{p_0}}\, \|u\|_{L_x^{s_n}}^{s_n(r-\frac{1}{p_0})}.
\end{aligned}$$

Since $r < 1$ we have $\|u\|_{L_x^{p_{n+1}}}^{p_{n+1}(r-1)} \geq (C_{T,n+1})^{r-1}$ and integration-in-time and Hölder's inequality as in (45) yield

$$(C_{T,n+1})^{r-1}\, r\, C_{n+1} \int_{\Omega_T} \left| \nabla_x (u^{p_{n+1}/2}) \right|^2 dx \leq \|u_0\|_{L_x^{p_{n+1}}}^{r\,p_{n+1}} + r\, p_{n+1}\, \|f\|_{L_{t,x}^{p_0}}\, D_{T,n}^{\frac{p_0-1}{p_0}}.$$

Finally, with $s_{n+1} = p_{n+1}\frac{N}{N-2}$ if $N > 2$ and using Sobolev's embedding,

$$\int_0^T \|u\|_{L_x^{s_{n+1}}}^{p_{n+1}}\, dt \leq \frac{C_S^2 C_{T,n+1}^{1-r}}{C_{n+1}\, r} \left[\|u_0\|_{L_x^{p_{n+1}}}^{r\,p_{n+1}} + r\, p_{n+1} \|f\|_{L_{t,x}^{p_0}}\, D_{T,n}^{\frac{p_0-1}{p_0}} \right] =: D_{T,n+1}.$$

Step 5: Iteration in n. From the definition of r in (44) it follows that $p_{n+1} = s_n(1 - \frac{1}{rp_0}) + \frac{1}{r}$ and thus $p_{n+1} < \infty$ if $N = 2$ and for dimensions $N \geq 3$, where $s_n = p_n \frac{N}{N-2}$:

$$p_{n+1} = p_n \frac{N}{N-2}\left(1 - \frac{1}{rp_0}\right) + \frac{1}{r} = p_n \frac{N(p_0-1)}{p_0(N-2)+2} + \frac{N\,p_0}{p_0(N-2)+2}$$

which has the fixed point $p_\infty = \frac{Np_0}{N+2-2p_0}$ and

$$p_\infty < 0 \quad \Longleftrightarrow \quad p_0 > \frac{N+2}{2} \quad \Longleftrightarrow \quad \frac{N(p_0-1)}{p_0(N-2)+2} > 1.$$

Thus, with $p_0 = q$ and $f \in L^q_{t,x}$ we distinguish the cases

$$\begin{cases} q < \frac{N+2}{2} & \text{where} \quad p_n \xrightarrow{n\to\infty} p_\infty > q \quad \text{for} \quad q > 1, \\ q \geq \frac{N+2}{2} & \text{where} \quad p_n \xrightarrow{n\to\infty} +\infty. \end{cases} \tag{47}$$

In dimension $N = 2$ we can always choose $p_\infty < +\infty$ to be arbitrarily large. Note that for any n in the iteration, the constants $C_{T,n}$ and $D_{T,n}$ are polynomial with respect to T!

Step 6: Interpolation of (41) and (42) in the cases $p_\infty < +\infty$ (and thus $N \geq 3$). For any n we use Hölder's inequality

$$\int_{\Omega_T} u^{p_n}\, u^{p_n \frac{2}{N}}\, dxdt \leq \int_0^T \|u\|^{p_n}_{L^{s_n}_x} \|u\|^{\frac{2}{N}p_n}_{L^{p_n}_x}\, dt \leq C^{\frac{2}{N}}_{T,n} D_{T,n}.$$

In the limit $n \to \infty$ we find $p_\infty \frac{N+2}{N} = \frac{(N+2)q}{N+2-2q} > 0$.

Thus, for all $\varepsilon > 0$ and in all dimensions $N \geq 3$ we obtain after finitely many iterations the following bound

$$\|u\|_{L^{p_\infty \frac{N+2}{N} - \varepsilon}_{t,x}} \leq C_T(\|u_0\|_{L^{p_\infty}_x}, \|f\|_{L^q_{t,x}}, q, d, C_S),$$

where C_T is a constant depending only on $\|u_0\|_{L^{p_\infty}_x}$, $\|f\|_{L^q_{t,x}}$, d, q, the Sobolev constant C_{S}, and T. Moreover, C_T depends polynomially on T. □

Remark 3.5. We remark that

$$\frac{1}{p_\infty \frac{N+2}{N} - \varepsilon} = \frac{1}{q} - 1 + \frac{N}{N+2} + O(\varepsilon),$$

which corresponds to the regularity expected by convolution with the heat kernel being in $L^{\frac{N+2}{N} - \mu}$, for all $\mu > 0$.

When applied to the quadratic "four species" eq. (13), the bootstrap above yields the following lemma:

Lemma 3.6. *Let Ω be a bounded smooth ($C^{2+\alpha}$) open subset of $\mathbb{R}^N$ and $T > 0$. Consider then a weak solution $\{a_i\}_{i=1..4}$ to equation (13), (2) on $[0,T]$ with initial condition $\{a_{i0}\}_{i=1..4} \in L^\infty(\Omega_T)$ and diffusion rates $d_i > 0$, $[i = 1..4]$.*

Assume that $\{a_i\}_{i=1..4}$ lie in $L^{q_0}(\Omega_T)$ for some $q_0 > (N+2)/2$, and that $\|a_i\|_{L^{q_0}(\Omega_T)}$ grows at most polynomially w.r.t. T for $i = 1..4$.

Then, for any $r \in [1, +\infty[$, we have $\{a_i\}_{i=1..4} \in L^r(\Omega_T)$ and $\|a_i\|_{L^r(\Omega_T)}$ grows at most polynomially w.r.t. T.

Proof of Lemma 3.6. We use Lemma 3.3 repeatedly. In general, if $a_i \in L^q(\Omega_T)$ for some $q > 2$, then $a_i a_j \in L^{q/2}(\Omega_T)$ $(i, j = 1, \ldots, 4)$. Hence the right-hand side of eq. (13) is in $L^{q/2}$, so from Lemma 3.3 we have

$$a_i \in L^{r-\delta}(\Omega_T) \quad \text{with } r = \begin{cases} \frac{1}{2}\frac{q(N+2)}{N+2-q} & \text{if } 1 < q < N+2, \\ \infty & \text{if } q \geq N+2, \end{cases}$$

for any $\delta > 0$. If we define the sequence q_n starting with the q_0 given in the Lemma, and satisfying

$$q_{n+1} = \frac{1}{2}\frac{q_n(N+2)}{N+2-q_n}, \quad \text{as long as} \quad q_n < N+2, \tag{48}$$

one can readily check that $q_{n+1} > q_n$ is equivalent to $q_n > \frac{N+2}{2}$ and thus $\frac{q_{n+2}}{q_{n+1}} > \frac{q_{n+1}}{q_n}$ and we obtain within finitely many iterations that

$$a_i \in L^{q_n-\delta}(\Omega_T), \quad \text{with} \quad q_n > N+2 \quad \text{for some } n \geq 0, \quad \text{and any} \quad \delta > 0.$$

Thus, applying once more Lemma 3.3, we end up with $a_i \in L^r(\Omega_T)$ for any $r \in [1, +\infty[$. □

In order to get an L^∞ estimate, we need one more computation:

Lemma 3.7. *Let Ω be a bounded domain of $\mathbb{R}^N$ with smooth ($C^{2+\alpha}$) boundary $\partial\Omega$ and $T > 0$. Consider a solution $\{a_i\}_{i=1..4}$ to the system (13), (2) on $[0, T]$ with initial condition $\{a_{i0}\}_{i=1..4} \in L^\infty(\Omega)$ and diffusion rates $d_i > 0$ $[i = 1..4]$.*

Assume that $\{a_i\}_{i=1..4}$ is a weak solution to the system (13), (2) on $[0, T]$ satisfying $a_i \in L^{q_0}(\Omega_T)$ for $i = 1..4$ and some $q_0 > (N+2)/2$. Also assume that $\|a_i\|_{L^{q_0}(\Omega_T)}$ grows at most polynomially w.r.t. T.

Then,

$$\|a_i\|_{L^\infty(\Omega_T)} \leq C_T, \qquad i = 1..4,$$

where C_T grows at most polynomially w.r.t. T.

Proof of Lemma 3.7. Using Lemma 3.6, we know that $\partial_t a_i - d_i\,\Delta_x a_i$ lies in $L^r(\Omega_T)$ for all $r \in [1, +\infty[$, with a norm in $L^r(\Omega_T)$ which grows at most polynomially w.r.t. T.

Using the result of [L], we obtain that the derivatives $\partial_t a_i$ and $\partial_{x_j x_k} a_i$ lie in $L^r(\Omega_T)$ for all $r \in [1, +\infty[$, with a norm in $L^r(\Omega_T)$ which grows at most polynomially w.r.t. T.

A standard Sobolev inequality (in $\mathbb{R}^{N+1}$) together with the use of an extension/restriction operator implies that a_i lies in $L^\infty(\Omega_T)$, with a norm in $L^\infty(\Omega_T)$ which grows at most polynomially w.r.t. T. □

3.3 Existence of bounded solutions and large time behaviour for the "four species" model

The results of the previous subsections enable us to prove Proposition 1.2 and Proposition 1.3.

Proof of Proposition 1.2. The existence of a weak solution to eq. (13), (2) [with given initial data in $L^\infty(\Omega)$] is already known (cf. [DFPV]). We observe that thanks to Lemma 3.2, we have a $L^p(\Omega_T)$ estimate for a_i ($i = 1..4$) for some $p > 2$. According to Lemma 3.7, the a_i ($i = 1..4$) lie in fact in $L^\infty(\Omega_T)$. Since T can be taken arbitrarily large, we end up with solutions defined on all $\mathbb{R}_+ \times \Omega$.

Moreover, still according to Lemma 3.7, the $L^\infty(\Omega_T)$ bounds of the a_i ($i = 1..4$) are at most polynomially increasing w.r.t. T. Using the entropy/entropy-dissipation estimate proved in [DFEqua] (or [DF08], which used the assumption $N = 1$ only in order to show at most polynomially growing L^∞ bounds), we end up with the exponential decay towards equilibrium (14), and in particular we get a uniform-in-time bound for the a_i in $L^\infty(\mathbb{R}_+ \times \Omega)$. □

Proof of Proposition 1.3. Once again, the existence of a weak solution to eq. (13), (2) [with given initial data in $L^\infty(\Omega)$], is already known (cf. [DFPV]). Under the smallness assumption made on $\delta = b - a$, Lemma 3.1 implies an $L^p(\Omega_T)$ estimate for a_i ($i = 1..4$) for some $p > N/2 + 1$. Then, Lemma 3.7 ensures that the a_i ($i = 1..4$) lie in fact in $L^\infty(\Omega_T)$. The end of the proof (that is, the estimates about the convergence towards equilibrium) is exactly the same as in the Proof of Proposition 1.2. □

4 Extensions: General chemical kinetics and degenerate diffusion rates

Proof of Proposition 1.4. We introduce the approximated system of equations constituted of the approximated equation (for any $r \in \mathbb{N}^*$)

$$\partial_t a_i^r - d_i\, \Delta_x a_i^r = \frac{(\beta_i - \alpha_i)\left(l \prod_{j=1}^n (a_j^r)^{\alpha_j} - k \prod_{j=1}^n (a_j^r)^{\beta_j} \right)}{1 + \frac{1}{r}\left(\sum_{j=1}^n (a_j^r)^2 \right)^{Q/2}}, \qquad i = 1..n, \quad (49)$$

where $Q = \sup\{\sum_{i=1}^n \alpha_i, \sum_{i=1}^n \beta_i\}$ is defined as in the statement of the Proposition 1.4 and assumed to be superquadratic, i.e. we consider $Q \geq 3$. Here, eq. (49) is to be considered together with the homogeneous Neumann boundary condition (2) and a set of smooth approximated initial data $\{a_{i0}^r\}_{i=1,..,n}$ (converging a.e. towards $\{a_{i0}\}_{i=1,..,n}$ as $r \to \infty$ and bounded in $L^\infty(\Omega)$).

The existence of a smooth (strong) solution to this approximated system follows from standard existence results of systems of reaction-diffusion equations with bounded and Lipschitz-continuous r.h.s. (cf. [D, QS], for example).

With the assumption that at least two coefficients $\alpha_i - \beta_i$ are different from zero and have opposite signs, one can find coefficients $\gamma_i > 0$ such that

$$\sum_{i=1}^n \gamma_i\, (\alpha_i - \beta_i) = 0.$$

At first, we observe then that

$$\partial_t\Big(\sum_{i=1}^n \gamma_i\, a_i^r\Big) - \Delta_x(M_r\, a_i^r) = 0,$$

where

$$M_r(t,x) = \frac{\sum_{i=1}^n \gamma_i\, d_i\, a_i^r(t,x)}{\sum_{i=1}^n \gamma_i\, a_i^r(t,x)}$$

is bounded $a \le M_r(t,x) \le b$ a.e. in Ω_T.

We now assume that

$$b - a < 2\,(C_{\frac{a+b}{2},Q'})^{-1}, \qquad \frac{1}{Q} + \frac{1}{Q'} = 1. \tag{50}$$

Using Proposition 1.1, we see that for any $i = 1,..,n$, the sequence $(a_i^r)_{r\in\mathbb{N}}$ is bounded in $L^Q(\Omega_T)$ (for all $T > 0$). Moreover, since (50) is a strict inequality, an interpolation argument similar to the one used in Lemma 3.2 implies that $(a_i^r)_{r\in\mathbb{N}}$ is bounded in $L^{Q+\varepsilon}(\Omega_T)$ for some sufficiently small $\varepsilon > 0$. As a consequence, the quantities $\partial_t a_i^r - d_i\,\Delta_x a_i^r$ are bounded in $L^{1+\varepsilon/Q}(\Omega_T)$.

The sequence $(a_i^r)_{r\in\mathbb{N}}$ converges therefore (up to extraction of a subsequence) a.e. as well as strongly in $L^Q(\Omega_T)$ towards a limit $a_i \in L^{Q+\varepsilon}(\Omega_T)$. Finally, one can pass to the limit $r \to \infty$ without difficulties, which ensures that the limiting concentrations a_i satisfy the original system in the weak sense.

Secondly, if we suppose

$$b - a < 2\,(C_{\frac{a+b}{2},\frac{(Q-1)\,(N+2)}{(Q-1)\,(N+2)-2}})^{-1},$$

Proposition 1.1 (and the same interpolation argument as previously) ensures that the weak solution defined above satisfies the extra estimate $a_i \in L^{z_0}(\Omega_T)$, for some $z_0 > (1+N/2)\,(Q-1)$. Then, $\partial_t a_i - d_i\,\Delta_x a_i \in L^{z_0/Q}(\Omega_T)$, and thanks to the properties of the heat kernel summarised in Lemma 3.3, $a_i \in L^p(\Omega_T)$ for all $p < z_1$, with $z_1 = \frac{z_0}{Q - \frac{2}{N+2}\,z_0}$ (or all $p \in [1,+\infty[$ if $z_0 \ge Q\,(1+N/2)$).

We see that a simple bootstrap ensures that $a_i \in L^p(\Omega_T)$ for all $p < z_k$, with $z_k = \frac{z_{k-1}}{Q - \frac{2}{N+2}\,z_{k-1}}$ (or all $p \in [1,+\infty[$ if $z_{k-1} \ge Q\,(1+N/2)$). The sequence z_k is increasing up to the point when $z_k \ge Q\,(1+N/2)$, therefore we obtain the estimate $a_i \in L^p(\Omega_T)$ for all $p \in [1,+\infty[$. We proceed as in Lemma 3.7 to get the final estimate $a_i \in L^\infty(\Omega_T)$. □

Proof of Proposition 1.5. Existence of weak solutions (in $L^2(\Omega_T)$) for the set of equations considered in this Proposition is shown in [DF07].

By adding the equations satisfied by a_1 and a_2, we see that

$$\partial_t(a_1 + a_2) - \Delta_x(M\,(a_1 + a_2)) = 0, \tag{51}$$

where $M(t,x) \in [\inf\{d_1,d_2\}, \sup\{d_1,d_2\}]$ almost everywhere.

Adding the equations satisfied by a_2 and a_3, we also see that

$$\partial_t(a_2 + a_3) - \Delta_x(M\,(a_2 + a_3)) = 0, \tag{52}$$

where $M(t,x) \in [\inf\{d_2, d_3\}, \sup\{d_2, d_3\}]$ almost everywhere.

As a consequence, we see that thanks to Proposition 1.1 and an interpolation argument similar to the one used in Lemma 3.2, for some $\delta \in]0,2[$ (and any $T > 0$), $a_i \in L^{2+\delta}(\Omega_T)$, when $i = 1,2,3$. Then, $a_1\,a_3 \in L^{1+\delta/2}(\Omega_T)$. Since

$$\partial_t a_2 - d_2\,\Delta_x a_2 \le a_1\,a_3,$$

we see using the properties of the heat kernel in 2D (cf. Lemma 3.3) that $a_2 \in L^{\frac{2+\delta}{1-\delta/2}-0}(\Omega_T)$ (here and in the sequel, the notation L^{p-0} means $\cap_{q\in[1,p[}L^q$).

Using a duality estimate (see e.g. [Pie10, Lemma 3.4], [QS, Lemma 33.3]) for solutions of (51), it follows that if $a_2 \in L^q(\Omega_T)$ for any $1 < q < \infty$ then also a_1 lies in $L^q(\Omega_T)$. Thus, we deduce from the estimate on a_2 that a_1, $a_3 \in L^{\frac{2+\delta}{1-\delta/2}-0}(\Omega_T)$. We build the (finite) increasing sequence $p_n \in]2,4[$ such that $p_0 = 2+\delta$, and $\frac{1}{p_{n+1}} = \frac{2}{p_n} - \frac{1}{2}$. We denote by N_0 the last index such that $p_{N_0} < 4$. The properties of the heat kernel in 2D (once again and in all the sequel, cf. Lemma 3.3 for a precise exposition of those properties) and the duality estimate of [QS] implies that $a_i \in L^{p_{N_0+1}}(\Omega_T)$, when $i = 1,2,3$. A last application of the properties of the heat kernel in 2D and the duality estimate of [Pie10, QS] shows that $a_i \in L^{\infty-0}(\Omega_T)$, when $i = 1,2,3$. Finally, thanks to a computation similar to the one in Lemma 3.7, $a_2 \in L^\infty(\Omega_T)$.

Observing that

$$\partial_t a_4 \le a_1\,a_3, \tag{53}$$

we also see (performing the integration in time) that $a_4 \in L^{\infty-0}(\Omega_T)$. Then, for $i = 1,3$,

$$\partial_t a_i - d_i\,\Delta_x a_i \le a_2\,a_4,$$

so that thanks to the same computation as above (similar to the one in Lemma 3.7),

$$a_1, a_3 \in L^\infty(\Omega_T).$$

Using once again eq. (53), we obtain that $a_4 \in L^\infty(\Omega_T)$. □

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