New results for triangular reaction cross diffusion system

L. Desvillettes & A. Trescases

CMLA, ENS Cachan & CNRS
61 Av. du Pdt. Wilson, 94235 Cachan Cedex, France
E-mails: desville@cmla.ens-cachan.fr, trescase@cmla.ens-cachan.fr

Abstract

We present an approach based on entropy and duality methods for "triangular" reaction cross diffusion systems of two equations, in which cross diffusion terms appear only in one of the equations. This approach enables to recover and extend many existing results on the classical "triangular" Shigesada-Teramoto-Kawasaki model.

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1 Introduction

Reaction cross diffusion equations naturally appear in physics (cf [4] for example) as well as in population dynamics. We are interested here in the study of a class of systems first introduced by Shigesada, Teramoto and Kawasaki (cf. [22]), in order to model the repulsive effect of populations of two different species in competition, possibly leading to the apparition of patterns when stable steady solutions are concerned.

The original model writes

$$\begin{cases} \partial_t u - \Delta_x(d_u\, u + d_{11}\, u^2 + d_{12}\, u\, v) = u\,(r_u - r_a\, u - r_b\, v), \\ \partial_t v - \Delta_x(d_v\, v + d_{21}\, u\, v + d_{22}\, v^2) = v\,(r_v - r_c\, v - r_d\, u), \end{cases} \tag{1}$$

where $u := u(t,x) \geq 0$ and $v := v(t,x) \geq 0$ are the number densities of the two considered species (say, species 1 and species 2), $r_u, r_v > 0$ are the growth rates in absence of other individuals, $r_a, r_b, r_c, r_d > 0$ correspond to the logistic inter- and intra-specific competition effects, $d_u, d_v > 0$ are the diffusion rates. The coefficients $d_{ij} \geq 0$ ($i,j = 1,2$) represent the repulsive effect: individuals of species i increase their diffusion rate in presence of individuals of their own species when $d_{ii} > 0$ (self diffusion) or of the other species when $d_{ij} > 0$ ($i \neq j$).

In the sequel, we shall only consider the case when $d_{21} = 0$ and $d_{12} > 0$, which is sometimes called "triangular", since in that particular case, the two-ways coupling between the two equations is due to the competition (reaction) terms only (the fully coupled system when $d_{21} > 0$ and $d_{12} > 0$ has a quite different mathematical structure, cf [6] and [11] for example). We shall also only focus on the case when no self diffusion appears (that is $d_{11} = d_{22} = 0$) since this case is the most studied one: note however that the mathematical theory when self diffusion is present (that is, $d_{11} > 0$ and/or $d_{22} > 0$) is probably not fundamentally different.

Under those assumptions, the Shigesada-Teramoto-Kawasaki system writes

$$\begin{cases} \partial_t u - \Delta_x(d_u\, u + d_{12}\, u\, v) = u\,(r_u - r_a\, u - r_b\, v), \\ \partial_t v - d_v\, \Delta_x v = v\,(r_v - r_c\, v - r_d\, u). \end{cases} \tag{2}$$

Following [17], this system can be seen as the formal singular limit of a reaction diffusion system which writes

$$\begin{cases} \partial_t u_A^\varepsilon - d_u\, \Delta_x u_A^\varepsilon = [r_u - r_a\,(u_A^\varepsilon + u_B^\varepsilon) - r_b\, v^\varepsilon]\, u_A^\varepsilon + \dfrac{1}{\varepsilon}[k(v^\varepsilon)\, u_B^\varepsilon - h(v^\varepsilon)\, u_A^\varepsilon], \\ \partial_t u_B^\varepsilon - (d_u + d_B)\, \Delta_x u_B^\varepsilon = [r_u - r_a\,(u_A^\varepsilon + u_B^\varepsilon) - r_b\, v^\varepsilon]\, u_B^\varepsilon - \dfrac{1}{\varepsilon}[k(v^\varepsilon)\, u_B^\varepsilon - h(v^\varepsilon)\, u_A^\varepsilon], \\ \partial_t v^\varepsilon - d_v\, \Delta_x v^\varepsilon = [r_v - r_c\, v^\varepsilon - r_d\,(u_A^\varepsilon + u_B^\varepsilon)]\, v^\varepsilon, \end{cases} \tag{3}$$

where $d_B > 0$, and h, k are two (continuous) functions from $\mathbb{R}_+$ to $\mathbb{R}_+$.

This limit holds in the following sense: if u_A^ε, u_B^ε, and v^ε are solutions to system (3) (with ε-independent initial data and suitable boundary condition), and provided that $d_B \, \frac{h(v)}{h(v)+k(v)} = d_{12}\, v$, the quantity $u_A^\varepsilon + u_B^\varepsilon$ converges (at the formal level) towards u, and the quantity v^ε converges (at the formal level) towards v, where u and v are solutions to system (2). Note that this asymptotics can be biologically meaningful: when $\varepsilon > 0$, the system represents a microscopic model in which the species u can be found in two states (the quiet state u_A and the stressed state u_B) and the individuals of this species switch from one state to the other with a "large" rate (proportional to $1/\varepsilon$).

We present in this paper results for the existence, uniqueness and stability of a large class of systems including (2), in a bounded domain with homogeneous Neumann boundary conditions. More precisely, we relax the assumption that the competition terms are logistic (quadratic), and replace it with the assumption that the competition terms are given by power laws (the powers being suitably chosen). We also relax the assumption that the cross diffusion term is quadratic (proportional to $u\,v$) and replace it by the more general assumption that it writes $u\,\phi(v)$ (with ϕ a differentiable function from $\mathbb{R}_+$ to $\mathbb{R}_+$).

More precisely, we shall consider the following system:

$$\partial_t u - \Delta_x (d_u\, u + u\,\phi(v)) = u\,(r_u - r_a\, u^a - r_b\, v^b), \tag{4}$$

$$\partial_t v - d_v\, \Delta_x v = v\,(r_v - r_c\, v^c - r_d\, u^d), \tag{5}$$

$$\forall t \geq 0, x \in \partial\Omega, \qquad \nabla_x u(t,x) \cdot n(x) = \nabla_x v(t,x) \cdot n(x) = 0, \tag{6}$$

whose coefficients satisfy the

Assumption A: $d_u, d_v > 0$, $r_u, r_v, r_a, r_b, r_c, r_d > 0$, $a, b, c, d > 0$, and $\phi := \phi(v) \geq 0$ is differentiable ($W^{1,\infty}_{loc}$) with locally bounded derivative on $\mathbb{R}_+$.

We propose two theorems, corresponding to the respective cases $d < a$ and $a \leq d$. The first one writes:

Theorem 1. *We assume that Ω is a smooth bounded domain of $\mathbb{R}^N$ ($N \in \mathbb{N}^*$, and the unit normal outward vector to Ω at point $x \in \partial\Omega$ is from now on denoted by $n(x)$). We also consider Assumption A on the coefficients of system (4) – (6), and suppose moreover that $d < a$. Finally, we consider initial data $u_{in} \geq 0$, $v_{in} \geq 0$ such that $u_{in} \in L^{p_0}(\Omega)$, $v_{in} \in L^\infty(\Omega) \cap W^{2,1+p_0/d}(\Omega)$ for some $p_0 > 1$. If $1 + p_0/d \geq 3$, we also assume the compatibility condition on the boundary $\partial\Omega$ (that is, $\nabla_x v_{in} \cdot n(x) = 0$).*

Then, there exists a global weak solution (u, v) (with $u := u(t,x) \geq 0$, $v := v(t,x) \geq 0$) of system (4) – (6) with initial data (u_{in}, v_{in}) lying in $L^{p_0+a}([0,T] \times \Omega) \times L^\infty([0,T] \times \Omega)$ for all $T > 0$. More precisely, for all test functions ψ_1, $\psi_2 \in C^2_c(\mathbb{R}_+ \times \overline{\Omega})$ such that $\forall t \in \mathbb{R}_+, x \in \partial\Omega$, $\nabla_x \psi_1(t,x) \cdot n(x) = \nabla_x \psi_2(t,x) \cdot n(x) = 0$, the following identities hold:

$$-\int_0^\infty \int_\Omega \partial_t \psi_1\, u - \int_\Omega \psi_1(0,\cdot)\, u_{in} - \int_0^\infty \int_\Omega \Delta_x \psi_1\, (d_u + \phi(v))\, u = \int_0^\infty \int_\Omega \psi_1\, u\,(r_u - r_a\, u^a - r_b\, v^b), \tag{7}$$

$$-\int_0^\infty \int_\Omega \partial_t \psi_2\, v - \int_\Omega \psi_2(0,\cdot)\, v_{in} - d_v \int_0^\infty \int_\Omega \Delta_x \psi_2\, v = \int_0^\infty \int_\Omega \psi_2\, v\,(r_v - r_c\, v^c - r_d\, u^d). \tag{8}$$

If moreover ϕ is twice differentiable with locally bounded second order derivative on $\mathbb{R}_+$, and if $u_{in} \in W^{2,s_0}(\Omega)$ and $v_{in} \in W^{2,1+\frac{a}{d}(s_0-1)}(\Omega)$ (and if compatibility conditions on the boundary $\partial\Omega$ are imposed for u_{in}, (resp. v_{in}) when $s_0 \geq 3$, (resp. $1 + \frac{a}{d}(s_0 - 1) \geq 3$)) for some $s_0 > 1 + N/2$, then (u, v) is locally Hölder continuous w.r.t. variables t, x. Moreover, for all $T > 0$, $\partial_t u, \partial_{x_i x_j} u \in L^{s_0}([0,T] \times \Omega)$, $\partial_t v, \partial_{x_i x_j} v \in L^{1+\frac{a}{d}(s_0-1)}([0,T] \times \Omega)$ $(i, j = 1..N)$.

Finally if ϕ, (resp. u_{in}, v_{in}) have locally Hölder continuous second order derivatives on $\mathbb{R}_+$ (resp. $\overline{\Omega}$), and if u_{in} and v_{in} satisfy compatibility conditions on the boundary $\partial\Omega$, then u, v have locally Hölder continuous first order time derivative and second order space derivatives on $\mathbb{R}_+ \times \overline{\Omega}$ (so that system (4) – (6) is solved in the strong sense). In this last setting and provided that $b, d \geq 1$ or that the initial data are bounded below by a strictly positive *constant, if* $(u_{1,in}, v_{1,in})$ *and* $(u_{2,in}, v_{2,in})$ *are two sets of (nonnegative) initial data, then any corresponding sets of solutions* (u_1, v_1) *and* (u_2, v_2) *satisfy (for any* $T > 0$*)*

$$||u_1 - u_2||_{L^2([0,T]\times\Omega)} + ||v_1 - v_2||_{L^2([0,T]\times\Omega)} \le C_T \bigg(||u_{1,in} - u_{2,in}||_{L^2(\Omega)} + ||v_{1,in} - v_{2,in}||_{L^2(\Omega)} \bigg),$$

for some constant $C_T > 0$*. As a consequence, uniqueness holds in this last setting (among solutions having locally Hölder continuous first order time derivatives and second order space derivatives on* $\mathbb{R}_+ \times \overline{\Omega}$*).*

Then, our second theorem writes

Theorem 2. *We assume that* Ω *is a smooth bounded domain of* $\mathbb{R}^N$ *(*$N \in \mathbb{N}^*$*). We consider Assumption A on the coefficients of system (4) – (6), and suppose moreover that* $a \le d$*,* $a \le 1$*,* $d \le 2$*. Finally, we consider initial data* $u_{in} \ge 0$*,* $v_{in} \ge 0$ *such that* $u_{in} \in L^2(\Omega)$*,* $v_{in} \in L^\infty(\Omega) \cap W^{2,1+2/d}(\Omega)$ *(If* $1 + 2/d \ge 3$*, i.-e.* $d \le 1$*, we also assume the compatibility condition on the boundary* $\partial\Omega$ *(that is,* $\nabla_x v_{in} \cdot n(x) = 0$*)).*

Then, there exists a weak solution (u, v) *(with* $u := u(t, x) \ge 0$*,* $v := v(t, x) \ge 0$*) of system (4) – (6) with initial data* (u_{in}, v_{in}) *lying in* $L^2([0, T] \times \Omega) \times L^\infty([0, T] \times \Omega)$ *for all* $T > 0$*.*

More precisely, for all test functions ψ_1*,* $\psi_2 \in C^2_c(\mathbb{R}_+ \times \overline{\Omega})$ *such that* $\forall t \in \mathbb{R}_+, x \in \partial\Omega, \quad \nabla_x\psi_1(t, x)\cdot n(x) = \nabla_x\psi_2(t, x) \cdot n(x) = 0$*, identities (7), (8) hold.*

Those existence theorems are consequences of propositions showing the convergence in a singular perturbation problem. This problem, analogous to system (3) in the case of the Shigesada-Kawasaki-Teramoto model, writes:

$$\begin{cases} \partial_t u^\varepsilon_A - d_A\, \Delta_x u^\varepsilon_A = [r_u - r_a\,(u^\varepsilon_A + u^\varepsilon_B)^a - r_b\,(v^\varepsilon)^b]\, u^\varepsilon_A + \dfrac{1}{\varepsilon}\,[k(v^\varepsilon)\, u^\varepsilon_B - h(v^\varepsilon)\, u^\varepsilon_A], \\ \partial_t u^\varepsilon_B - (d_A + d_B)\, \Delta_x u^\varepsilon_B = [r_u - r_a\,(u^\varepsilon_A + u^\varepsilon_B)^a - r_b\,(v^\varepsilon)^b]\, u^\varepsilon_B - \dfrac{1}{\varepsilon}\,[k(v^\varepsilon)\, u^\varepsilon_B - h(v^\varepsilon)\, u^\varepsilon_A], \\ \partial_t v^\varepsilon - d_v\, \Delta_x v^\varepsilon = [r_v - r_c\,(v^\varepsilon)^c - r_d\,(u^\varepsilon_A + u^\varepsilon_B)^d]\, v^\varepsilon, \end{cases} \tag{9}$$

with the homogeneous Neumann boundary condition

$$\forall t \ge 0, x \in \partial\Omega, \qquad \nabla_x u^\varepsilon_A(t, x) \cdot n(x) = \nabla_x u^\varepsilon_B(t, x) \cdot n(x) = \nabla_x v^\varepsilon(t, x) \cdot n(x) = 0, \tag{10}$$

and the regularized initial data

$$u^\varepsilon_A(0, \cdot) = u^\varepsilon_{A,in} := \chi^\varepsilon\times(u_{A,in}*\rho^\varepsilon)+\varepsilon, \quad u^\varepsilon_B(0, \cdot) = u^\varepsilon_{B,in} := \chi^\varepsilon\times(u_{A,in}*\rho^\varepsilon)+\varepsilon, \quad v^\varepsilon(0, \cdot) = v^\varepsilon_{in} := v_{in}+\varepsilon, \tag{11}$$

where $u_{A,in}$ and $u_{B,in}$ are functions defined on Ω and extended (by zero) on $\mathbb{R}^N - \Omega$ (so that the convolution on $\mathbb{R}^N$ can be used), $(\rho^\varepsilon)_\varepsilon$ is a sequence of mollifiers on $\mathbb{R}^N$, and for all $\varepsilon > 0$, χ^ε is a cutoff function (given by Urysohn's lemma) lying in $C^\infty(\mathbb{R}^N)$ and satisfying

$$0 \le \chi^\varepsilon \le 1 \text{ in } \mathbb{R}^N, \qquad \chi^\varepsilon = 1 \text{ inside } \{x \in \Omega : d(x, \partial\Omega) > 2\varepsilon\}, \qquad \chi^\varepsilon = 0 \text{ outside } \{x \in \Omega : d(x, \partial\Omega) > \varepsilon\}. \tag{12}$$

We introduce the

Assumption B: $d_A, d_B, d_v > 0$, $r_u, r_v, r_a, r_b, r_c, r_d > 0$, $a, b, c, d > 0$. The functions h and k are differentiable with locally bounded derivative on $\mathbb{R}_+$ ($W^{1,\infty}_{loc}$) and are lower bounded by a strictly positive constant : $\forall v \ge 0$, $h(v), k(v) \ge h_0 > 0$.

Our results in this direction are summarized in the two following propositions:

Proposition 1. *We assume that* Ω *is a smooth bounded domain of* $\mathbb{R}^N$ *(*$N \in \mathbb{N}^*$*). We also consider Assumption B on the coefficients of system (9), and suppose moreover that* $d < a$*. Finally, we consider initial data* $u_{A,in} \ge 0$*,* $u_{B,in} \ge 0$*,* $v_{in} \ge 0$ *such that* $u_{A,in}, u_{B,in} \in L^{p_0}(\Omega)$*,* $v_{in} \in L^\infty(\Omega)\cap W^{2,1+p_0/d}(\Omega)$ *for some* $p_0 > 1$ *(If* $1+p_0/d \ge 3$*, we also assume the compatibility condition on the boundary* $\partial\Omega$ *(*$\nabla_x v_{in} \cdot n(x) = 0$ *when* $1 + p_0/d \ge 3$*)).*

Then, for any $\varepsilon > 0$*, there exists a unique (nonnegative for each component) strong (in the sense that all derivatives appearing in the equation lie in some* L^p *with* $p \in [1, \infty]$*) solution* $(u^\varepsilon_A, u^\varepsilon_B, v^\varepsilon)$ *to system (9)–(11). Moreover, when* $\varepsilon \to 0$*,* $(u^\varepsilon_A, u^\varepsilon_B, v^\varepsilon)$ *converges (up to a subsequence) for almost every* $(t, x) \in \mathbb{R}_+ \times \Omega$ *to a limit*

(u_A, u_B, v) (nonnegative for each component) lying in $L^{p_0+a}([0,T]\times\Omega)\times L^{p_0+a}([0,T]\times\Omega)\times L^\infty([0,T]\times\Omega)$ for all $T>0$. Finally, $h(v)\,u_A = k(v)\,u_B$ and $(u := u_A + u_B, v)$ is a weak solution of system (4) – (6) with $d_u + \phi(v) = d_A + d_B\,\frac{h(v)}{h(v)+k(v)}$ and initial data $u(0,\cdot) = u_{A,in} + u_{B,in}$, $v(0,\cdot) = v_{in}$. More precisely, for all test functions ψ_1, $\psi_2 \in C^2_c(\mathbb{R}_+\times\overline{\Omega})$ such that $\quad\forall t\in\mathbb{R}_+, x\in\partial\Omega,\quad \nabla_x\psi_1(t,x)\cdot n(x) = \nabla_x\psi_2(t,x)\cdot n(x) = 0$, the following identities hold:

$$-\int_0^\infty\int_\Omega \partial_t\psi_1\,u - \int_\Omega \psi_1(0,\cdot)\,(u_{A,in}+u_{B,in}) - \int_0^\infty\int_\Omega \Delta_x\psi_1\,\Big(u\,(d_A + d_B\,\frac{h(v)}{h(v)+k(v)})\Big) \tag{13}$$

$$= \int_0^\infty\int_\Omega \psi_1\,u\,(r_u - r_a\,u^a - r_b\,v^b),$$

$$-\int_0^\infty\int_\Omega \partial_t\psi_2\,v - \int_\Omega \psi_2(0,\cdot)\,v_{in} - d_v\int_0^\infty\int_\Omega \Delta_x\psi_2\,v = \int_0^\infty\int_\Omega \psi_2\,v\,(r_v - r_c\,v^c - r_d\,u^d). \tag{14}$$

Proposition 2. *We assume that Ω is a smooth bounded domain of $\mathbb{R}^N$ ($N\in\mathbb{N}^*$). We also consider Assumption B on the coefficients of system (9), and suppose moreover that $a\le d$, $a\le 1$, $d\le 2$. Finally, we consider initial data $u_{A,in}\ge 0$, $u_{B,in}\ge 0$, $v_{in}\ge 0$ such that $u_{A,in}, u_{B,in}\in L^2(\Omega)$, $v_{in}\in L^\infty(\Omega)\cap W^{2,1+2/d}(\Omega)$ (if $1+2/d\ge 3$, i.-e. $d\le 1$, we also assume the compatibility condition on the boundary $\partial\Omega$ ($\nabla_x v_{in}\cdot n(x) = 0$)).*

Then, for any $\varepsilon>0$, there exists a unique (nonnegative for each component) strong (in the sense that all derivatives appearing in the equation lie in some L^p with $p\in[1,\infty]$) solution $(u^\varepsilon_A, u^\varepsilon_B, v^\varepsilon)$ to system (9)–(11). Moreover, when $\varepsilon\to 0$, $(u^\varepsilon_A, u^\varepsilon_B, v^\varepsilon)$ converges (up to a subsequence) for almost every $(t,x)\in\mathbb{R}_+\times\Omega$ to a limit (u_A,u_B,v) (nonnegative for each component) lying in $L^2([0,T]\times\Omega)\times L^2([0,T]\times\Omega)\times L^\infty([0,T]\times\Omega)$ for all $T>0$. Finally, $h(v)\,u_A = k(v)\,u_B$ and $(u := u_A+u_B, v)$ is a weak solution to system (4) – (6) with $d_u+\phi(v) = d_A + d_B\,\frac{h(v)}{h(v)+k(v)}$ and initial data $u(0,\cdot) = u_{A,in}+u_{B,in}$, $v(0,\cdot) = v_{in}$. More precisely, for all test functions ψ_1, $\psi_2\in C^2_c(\mathbb{R}_+\times\overline{\Omega})$ such that $\quad\forall t\in\mathbb{R}_+, x\in\partial\Omega,\quad \nabla_x\psi_1(t,x)\cdot n(x) = \nabla_x\psi_2(t,x)\cdot n(x) = 0$, identities (13), (14) hold.

Remark 1. *Theorems 1 and 2 use classical parabolic ($W^{2,1}_s$ with the notations of [14]) estimates. For the sake of simplicity, we chose to use a non-optimal version, formulated below in Proposition 3. Note that the assumptions could be improved (see [14]): first, the estimates do not require a full compatibility condition on the boundary $\partial\Omega$ in the critical case $s=3$; secondly, some of the initial data assumed to belong to $W^{2,s}(\Omega)$ in our theorems and propositions can be assumed to belong only to the fractional Sobolev space $W^{2-2/s,s}(\Omega)$.*

Remark 2. *In the case of Theorem 2, the compactness of the nonlinear reaction terms u^{1+a} and u^d is obtained thanks to an L^p estimate for some $p>2$ given by a duality lemma. Notice first that this allows to consider parameters $a>1$ sufficiently close to 1 and $d>2$ sufficiently close to 2 in Theorem 2. Secondly, the duality lemma (stated in Lemma 4) for initial data in $L^2(\Omega)$ is actually valid for initial data in $L^p(\Omega)$ for $p<2$ sufficiently close to 2. This allows to replace in Theorem 2 the assumption $u_{in}\in L^2(\Omega)$ by the weaker assumption $u_{in}\in L^p(\Omega)$ for some $p<2$ sufficiently close to 2.*

Remark 3. *Since (as will be seen), v satisfies a maximum principle in the previous theorems, those theorems have an easy extension when the functions $v\mapsto r_b\,v^b$ and $v\mapsto r_c\,v^c$ are replaced by any smooth functions of v (with an arbitrary growth when $v\to\infty$). The functions $u\mapsto r_a\,u^a$ and $u\mapsto r_d\,u^d$ can also be replaced by smooth functions in the previous theorems, provided that those functions behave in the same way as $u\mapsto r_a\,u^a$ and $u\mapsto r_d\,u^d$ when $u\to\infty$.*

Remark 4. *In the last setting of Theorem 1, a minimum principle for v allows to replace the assumption that ϕ'' is locally Hölder continuous on $\mathbb{R}_+$ with the assumption that ϕ'' is locally Hölder continuous on $\mathbb{R}^*_+$, provided that the initial data for v is bounded below by a strictly positive constant.*

The model (1) was proposed by Shigesada, Teramoto and Kawasaki in [22]. For modeling issues, see also [19]. As far as mathematical analysis is concerned, two directions have been widely investigated: on the one hand, steady-states and stability, motivated by the formation of patterns (see [12] and the references inside); on the other hand, existence, smoothness and uniqueness of solutions.

The local (in time) existence was established by Amann: in his series of papers [1]-[3], he proved a general result of existence of local (in time) solutions for parabolic systems, including (1) and (4)-(5).

The global (in time) existence has then been proved under various assumptions. One of the difficulties arising is the use of Sobolev inequalities in the parabolic estimates, which only provides results in low dimension. Indeed, for the well studied triangular quadratic case (that is, (1) with $d_{21} = 0$), most papers allowing strong cross diffusion (that is, when no restriction is imposed on d_{12}) only deal with low dimensions: for results in dimension 1, see [16], [17] and [21]. In [27], Yagi showed the global existence in dimension 2 in the presence of self diffusion, and Lou, Ni and Wu obtained it in [15] without condition on self diffusion, together with a stability result. Choi, Lui and Yamada first got rid of the restriction on the dimension in [7] (without self diffusion in the second equation), provided that the cross diffusion coefficient d_{12} is sufficiently small. In a following paper [8], they removed the smallness assumption on the cross diffusion in the presence of self diffusion in the first equation. However, in the presence of self diffusion in the second equation, they require that the dimension is lower than 6. Finally, Phan improved this result up to dimension lower than 10 in [24], and in any dimension under the assumption that the self diffusion dominates the cross diffusion in [25]. For the quadratic system (2) without self diffusion, our Theorem 2 gives the existence of global solutions in any dimension, without restriction on the strength of the cross diffusion.

When it comes to systems with general reaction terms of the form (4)-(5), Posio and Tesei first showed the existence (in any dimension) of global solutions under some strong assumption on the reaction coefficients in [20]. This assumption was relaxed in [28] by Yamada, who obtained the existence of global strong solutions under the assumption $a > d$, which is exactly our assumption in Theorem 1. The main differences between our work (in the case $a > d$) and [28] are the following: first, our Theorem 1 allows singular initial data leading to weak solutions (and provide results very close to those of [28] when initial data are smooth). Then our method, based on simple energy estimates, presents an unifying proof for a wide range of parameters including both the quadratic case and the case $a > d$. Finally, the approximating system that we use leads to self contained proofs without reference to abstract existence theorems. Note also that (for general reaction terms) Wang got similar results in [26] in the presence of self diffusion in the first equation, under a condition (depending on the dimension) of smallness of the parameter d w. r. t. the parameter a.

Systems of reaction diffusion equations such as (3) were introduced by Iida, Mimura and Ninomiya in [12] to approximate cross diffusion systems, in particular from the point of view of stability. The convergence of the stationary problem was explored by Izuhara and Mimura in [13], both numerically and theorically. In [9], Conforto and Desvillettes showed the convergence of the solutions of (3) towards a solution of the system (2) in dimension one. Our paper generalizes their result to a wider set of admissible reaction terms and in any dimension. Note finally that Murakawa obtained similar results for a class of non triangular systems in [18].

The rest of our paper is structured as follows: Propositions 1 and 2 are proven in Section 2. Then, Section 3 is devoted to the proof of Theorems 1 and 2.

2 Proof of the validity of the singular perturbation

We begin with the

Proof of Proposition 1. We fix $T > 0$, and write (for any $q \in [1, \infty]$) $L^q = L^q([0, T] \times \Omega)$. In the proof of this proposition and of the following proposition, the constant $C_T > 0$ only depends on the parameters d_A, d_B, d_v, $r_u, r_v, r_a, r_b, r_c, r_d$, a, b, c, d, the domain Ω, the initial data $u_{A,in}, u_{B,in}, v_{in}$, the functions h and k, and the time T. In particular, all the estimates are uniform w.r.t ε.

We first observe that for a given $\varepsilon > 0$, standard theorems for reaction-diffusion equations show the existence of a (nonnegative for each component) unique strong (in the sense that all derivatives appearing in the equations are defined a.e.) solution $(u_A^\varepsilon, u_B^\varepsilon, v^\varepsilon)$ to system (9), (10), (11).
We refer to [10] for complete proofs.

We now establish three lemmas stating the (uniform w.r.t. ε) *a priori* estimates for this solution $(u_A^\varepsilon, u_B^\varepsilon, v^\varepsilon)$.

Lemma 1. *The following (uniform w.r.t ε) estimates hold:*

$$\sup_{0 \le t \le T} \int_\Omega (u_A^\varepsilon + u_B^\varepsilon)(t) \le C_T; \qquad \|u_A^\varepsilon + u_B^\varepsilon\|_{L^{1+a}} \le C_T. \tag{15}$$

Proof of Lemma 1. We notice that $u_A^\varepsilon + u_B^\varepsilon$ satisfies the equation

$$\partial_t(u_A^\varepsilon + u_B^\varepsilon) - \Delta_x[M^\varepsilon(u_A^\varepsilon + u_B^\varepsilon)] = [r_u - r_a(u_A^\varepsilon + u_B^\varepsilon)^a - r_b(v^\varepsilon)^b](u_A^\varepsilon + u_B^\varepsilon) \le C_T, \tag{16}$$

where $M^\varepsilon = \frac{d_A u_A^\varepsilon + (d_A + d_B) u_B^\varepsilon}{u_A^\varepsilon + u_B^\varepsilon}$. We integrate w.r.t. space and time to get

$$\sup_{0\le t\le T} \int_\Omega (u_A^\varepsilon + u_B^\varepsilon)(t) \le \int_\Omega (u_{A,in}^\varepsilon + u_{B,in}^\varepsilon) + C_T \le C_T, \tag{17}$$

so that

$$\sup_{0\le t\le T} \int_\Omega (u_A^\varepsilon + u_B^\varepsilon)(t) + r_a \int_0^T \int_\Omega (u_A^\varepsilon + u_B^\varepsilon)^{1+a} \le \int_\Omega (u_{A,in}^\varepsilon + u_{B,in}^\varepsilon) + r_u \int_0^T \int_\Omega (u_A^\varepsilon + u_B^\varepsilon) \le C_T. \tag{18}$$

□

Lemma 2. *For all $1 < q \le 1 + p_0/d$, the following (uniform w.r.t ε) estimates hold:*

$$\|v^\varepsilon\|_{L^\infty} \le C_T; \qquad \|\nabla_x v^\varepsilon\|_{L^{2q}}^2 \le C_T\,(1 + \|(u_A^\varepsilon + u_B^\varepsilon)^d\|_{L^q}); \qquad \|\partial_t v^\varepsilon\|_{L^{a+p/d}} \le C_T\,(1 + \|(u_A^\varepsilon + u_B^\varepsilon)^d\|_{L^q}). \tag{19}$$

Proof of Lemma 2. The first estimate is a consequence of the maximum principle for the equation satisfied by v^ε. We can then apply the maximal regularity result for the heat equation (satisfied by v when the reaction term is considered as given) in order to get the third estimate. The same bound also holds for $\partial_{x_i x_j} v^\varepsilon$, so that interpolating with the first estimate, the second estimate holds. □

We now write down an (uniform w.r.t. ε) estimate obtained thanks to the use of a Lyapounov-like (entropy) functional:

Lemma 3. *For all $p \in]1, p_0]$, the following inequalities hold:*

$$\|u_A^\varepsilon + u_B^\varepsilon\|_{L^{p+a}}^{p+a} \le C_T(1 + \|u_A^\varepsilon + u_B^\varepsilon\|_{L^{p+d}}^{p+d}), \tag{20}$$

$$\|\nabla_x (u_A^\varepsilon)^{p/2}\|_{L^2}^2 + \|\nabla_x (u_B^\varepsilon)^{p/2}\|_{L^2}^2 + \frac{1}{\varepsilon}\|(h(v^\varepsilon)u_A^\varepsilon)^{p/2} - (k(v^\varepsilon)u_B^\varepsilon)^{p/2}\|_{L^2}^2 \le C_T\,(1 + \|u_A^\varepsilon + u_B^\varepsilon\|_{L^{p+d}}^{p+d}). \tag{21}$$

Proof of Lemma 3. We define the following entropy for any $p > 0$ (with $p \ne 1$):

$$\mathscr{E}^\varepsilon(t) = \int_\Omega h(v^\varepsilon)^{p-1} \frac{(u_A^\varepsilon)^p}{p}(t) + \int_\Omega k(v^\varepsilon)^{p-1} \frac{(u_B^\varepsilon)^p}{p}(t) \qquad (=: \mathscr{E}_A^\varepsilon(t) + \mathscr{E}_B^\varepsilon(t)). \tag{22}$$

We compute the derivative

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\mathscr{E}_A^\varepsilon(t) &= \int_\Omega \partial_t\{h(v^\varepsilon)^{p-1}\frac{(u_A^\varepsilon)^p}{p}\}(t) \\
&= \frac{p-1}{p}\int_\Omega \partial_t v^\varepsilon h'(v^\varepsilon) h(v^\varepsilon)^{p-2}(u_A^\varepsilon)^p + \int_\Omega \partial_t u_A^\varepsilon (u_A^\varepsilon)^{p-1} h(v^\varepsilon)^{p-1} \\
&= \frac{p-1}{p}\int_\Omega \partial_t v^\varepsilon h'(v^\varepsilon) h(v^\varepsilon)^{p-2}(u_A^\varepsilon)^p + \int_\Omega [r_u - r_a(u_A^\varepsilon + u_B^\varepsilon)^a - r_b(v^\varepsilon)^b](u_A^\varepsilon)^p h(v^\varepsilon)^{p-1} \\
&\quad + \frac{1}{\varepsilon}\int_\Omega [k(v^\varepsilon)u_B^\varepsilon - h(v^\varepsilon)u_A^\varepsilon](u_A^\varepsilon)^{p-1} h(v^\varepsilon)^{p-1} + d_A \int_\Omega \Delta_x u_A^\varepsilon (u_A^\varepsilon)^{p-1} h(v^\varepsilon)^{p-1},
\end{aligned} \tag{23}$$

where the last term is estimated by integrating by part (and using the inequality $2|ab| \le a^2 + b^2$) in the case when $p > 1$:

$$\begin{aligned}
& d_A \int_\Omega \Delta_x u_A^\varepsilon (u_A^\varepsilon)^{p-1} h(v^\varepsilon)^{p-1} \\
&= -\, d_A\,(p-1)\int_\Omega |\nabla_x u_A^\varepsilon|^2 (u_A^\varepsilon)^{p-2} h(v^\varepsilon)^{p-1} - d_A\,(p-1)\int_\Omega \nabla_x u_A^\varepsilon \cdot \nabla_x h(v^\varepsilon)(u_A^\varepsilon)^{p-1} h(v^\varepsilon)^{p-2} \\
&\le -\,\frac{d_A}{2}\,(p-1)\int_\Omega |\nabla_x u_A^\varepsilon|^2 (u_A^\varepsilon)^{p-2} h(v^\varepsilon)^{p-1} + \frac{d_A}{2}\,(p-1)\int_\Omega |\nabla_x h(v^\varepsilon)|^2 (u_A^\varepsilon)^p h(v^\varepsilon)^{p-3} \\
&= -\,2\,d_A\,\frac{(p-1)}{p^2}\int_\Omega |\nabla_x (u_A^\varepsilon)^{p/2}|^2 h(v^\varepsilon)^{p-1} + \frac{(p-1)}{2}\,d_A \int_\Omega |\nabla_x v^\varepsilon|^2 (u_A^\varepsilon)^p (h'(v^\varepsilon))^2 h(v^\varepsilon)^{p-3}.
\end{aligned} \tag{24}$$

Similarly, we get for u_B^ε,

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\mathscr{E}_B^\varepsilon(t) \le \frac{p-1}{p}\int_\Omega \partial_t v^\varepsilon k'(v^\varepsilon)k(v^\varepsilon)^{p-2}(u_B^\varepsilon)^p \\ + \int_\Omega [r_u - r_a(u_B^\varepsilon + u_B^\varepsilon)^a - r_b(v^\varepsilon)^b](u_B^\varepsilon)^p k(v^\varepsilon)^{p-1} - \frac{1}{\varepsilon}\int_\Omega [k(v^\varepsilon)u_B^\varepsilon - h(v^\varepsilon)u_A^\varepsilon](u_B^\varepsilon)^{p-1}k(v^\varepsilon)^{p-1} \\ -2(d_A+d_B)\frac{p-1}{p^2}\int_\Omega |\nabla_x (u_B^\varepsilon)^{p/2}|^2 k(v^\varepsilon)^{p-1} + \frac{p-1}{2}(d_A+d_B)\int_\Omega |\nabla_x v^\varepsilon|^2 (u_B^\varepsilon)^p (k'(v^\varepsilon))^2 k(v^\varepsilon)^{p-3}.\end{aligned} \tag{25}$$

We sum the two estimates and integrate w.r.t time to get (still for any $p > 1$)

$$\begin{aligned}\int_\Omega h(v^\varepsilon)^{p-1}\frac{(u_A^\varepsilon)^p}{p}(T) + k(v^\varepsilon)^{p-1}\frac{(u_B^\varepsilon)^p}{p}(T) \\ +2\, d_A\,\frac{p-1}{p^2}\int_0^T\int_\Omega |\nabla_x (u_A^\varepsilon)^{p/2}|^2 h(v^\varepsilon)^{p-1} + 2\,(d_A+d_B)\,\frac{p-1}{p^2}\int_0^T\int_\Omega |\nabla_x (u_B^\varepsilon)^{p/2}|^2 k(v^\varepsilon)^{p-1} \\ +\frac{1}{\varepsilon}\int_0^T\int_\Omega [k(v^\varepsilon)u_B^\varepsilon - h(v^\varepsilon)u_A^\varepsilon][(u_B^\varepsilon)^{p-1}k(v^\varepsilon)^{p-1} - (u_A^\varepsilon)^{p-1}h(v^\varepsilon)^{p-1}] \\ + r_a\int_0^T\int_\Omega (u_A^\varepsilon + u_B^\varepsilon)^a[(u_A^\varepsilon)^p h(v^\varepsilon)^{p-1} + (u_B^\varepsilon)^p k(v^\varepsilon)^{p-1}] \\ \le \int_\Omega h(v_{in}^\varepsilon)^{p-1}\frac{(u_{A,in}^\varepsilon)^p}{p} + k(v_{in}^\varepsilon)^{p-1}\frac{(u_{A,in}^\varepsilon)^p}{p} \\ +\frac{p-1}{p}\int_0^T\int_\Omega \partial_t v^\varepsilon[h'(v^\varepsilon)h(v^\varepsilon)^{p-2}(u_A^\varepsilon)^p + k'(v^\varepsilon)k(v^\varepsilon)^{p-2}(u_B^\varepsilon)^p] \\ + r_u\int_0^T\int_\Omega (u_A^\varepsilon)^p h(v^\varepsilon)^{p-1} + (u_B^\varepsilon)^p k(v^\varepsilon)^{p-1} \\ +\frac{p-1}{2}\int_0^T\int_\Omega [d_A(u_A^\varepsilon)^p(h'(v^\varepsilon))^2 h(v^\varepsilon)^{p-3} + (d_A+d_B)(u_B^\varepsilon)^p(k'(v^\varepsilon))^2 k(v^\varepsilon)^{p-3}]|\nabla_x v^\varepsilon|^2.\end{aligned} \tag{26}$$

Let us estimate the right-hand side of inequality (26): the first term is finite since $p \le p_0$. Thanks to the maximum principle for the density v^ε (obtained in Lemma 2) and the regularity of the functions h and k in Assumption B, the terms $h(v^\varepsilon)$, $h'(v^\varepsilon)$ and $k(v^\varepsilon)$, $k'(v^\varepsilon)$ are uniformly bounded in L^∞. We then can estimate the third term with Jensen's inequality. The second and the last terms are estimated thanks to Hölder's inequality and bounds given by Lemma 2. More precisely, for the second term, we get

$$\begin{aligned}\left|\frac{p-1}{p}\int_0^T\int_\Omega \partial_t v^\varepsilon[h'(v^\varepsilon)h(v^\varepsilon)^{p-2}(u_A^\varepsilon)^p + k'(v^\varepsilon)k(v^\varepsilon)^{p-2}(u_B^\varepsilon)^p]\right| \\ \le C_p\|h'(v^\varepsilon)h(v^\varepsilon)^{p-2} + k'(v^\varepsilon)k(v^\varepsilon)^{p-2}\|_{L^\infty}\|\partial_t v^\varepsilon\|_{L^{1+p/d}}\|(u_A^\varepsilon + u_B^\varepsilon)^p\|_{L^{1+d/p}} \le C_T\,(1 + \|u_A^\varepsilon + u_B^\varepsilon\|_{L^{p+d}}^{p+d}),\end{aligned} \tag{27}$$

and for the last term, we get

$$\begin{aligned}\left|\frac{p-1}{2}\int_0^T\int_\Omega [d_A(u_A^\varepsilon)^p(h'(v^\varepsilon))^2 h(v^\varepsilon)^{p-3} + (d_A+d_B)(u_B^\varepsilon)^p(k'(v^\varepsilon))^2 k(v^\varepsilon)^{p-3}]|\nabla_x v^\varepsilon|^2\right| \\ \le C_p\|h'(v^\varepsilon)^2 h(v^\varepsilon)^{p-3} + k'(v^\varepsilon)^2 k(v^\varepsilon)^{p-3}\|_{L^\infty}\||\nabla_x v^\varepsilon|^2\|_{L^{1+p/d}}\|(u_A^\varepsilon + u_B^\varepsilon)^p\|_{L^{1+d/p}} \le C_T\,(1 + \|u_A^\varepsilon + u_B^\varepsilon\|_{L^{p+d}}^{p+d}).\end{aligned} \tag{28}$$

The terms of the left-hand side of (26) being all nonnegative, they are dominated by the quantity $(1 + \|u_A^\varepsilon + u_B^\varepsilon\|_{L^{p+d}}^{p+d})$. We then conclude the estimates on space derivatives by using the lower bound of h and k (remember Assumption B) and the following elementary inequality for all positive x, y : $(x-y)\,(x^{p-1} - y^{p-1}) \ge C_p\,|x^{p/2} - y^{p/2}|^2$. ☐

As a first consequence, we can improve the Lebesgue space in which we get a uniform estimate for $u_A^\varepsilon + u_B^\varepsilon$. Taking $p = p_0$ and using Hölder's inequality, we see that $\|u_A^\varepsilon + u_B^\varepsilon\|_{L^{p_0+a}}$ is uniformly (w.r.t. ε) bounded.

Let us combine the previous estimate and Lemma 2 with $q = 1 + p_0/d > 1$:

$$\|\nabla_x v^\varepsilon\|^2_{L^{2(1+p_0/d)}} + \|\partial_t v^\varepsilon\|_{L^{1+p_0/d}} \le C_T\,(1 + \|(u_A^\varepsilon + u_B^\varepsilon)^d\|_{L^{1+p_0/d}}) \le C_T. \tag{29}$$

Then, thanks to Jensen's inequality, the sequence $(v^\varepsilon)_\varepsilon$ is included in a bounded subset of $\{v \in L^2([0,T], H^1(\Omega)) : \partial_t v \in L^1([0,T], L^1(\Omega))\}$. From Aubin's lemma (see Theorem 5 in [23]) we then can extract a subsequence - still called $(v^\varepsilon)_\varepsilon$ - which converges towards a limit v (lying in L^∞) a.e.:

$$v^\varepsilon(t,x) \to v(t,x) \qquad \text{almost everywhere on } [0,T]\times\Omega. \tag{30}$$

Recall eq. (16) for $u_A^\varepsilon + u_B^\varepsilon$. Notice that the reaction term in (16) is uniformly bounded in L^λ with $\lambda = \frac{p_0+a}{1+a} > 1$. As a consequence, $\partial_t(u_A^\varepsilon + u_B^\varepsilon)$ in (16) is uniformly bounded in $L^\lambda([0,T], W^{-2,\lambda})$. Furthermore, let us choose some p in the interval $]1, \min\{p_0, 2\}[$. Then for $C = A$ or B,

$$\begin{aligned}\|\nabla_x u_C^\varepsilon\|_{L^1} \le \|(u_C^\varepsilon)^{p/2-1}\nabla_x u_C^\varepsilon\|_{L^2}\,\|(u_C^\varepsilon)^{1-p/2}\|_{L^2} &= \frac{2}{p}\,\|\nabla_x (u_C^\varepsilon)^{p/2}\|_{L^2}\,\|(u_C^\varepsilon)^{2-p}\|^{1/2}_{L^1}\\ &\le C_T\,(1 + \|u_A^\varepsilon + u_B^\varepsilon\|^{\frac{p+d}{2}}_{L^{p+d}})\,\|(u_C^\varepsilon)^{2-p}\|^{1/2}_{L^1} \le C_T,\end{aligned} \tag{31}$$

thanks to Lemma 3. We therefore can apply Aubin's lemma to extract a subsequence - still called $(u_A^\varepsilon + u_B^\varepsilon)_\varepsilon$ - which converges towards a limit u (lying in L^{a+p_0}) a.e.:

$$u_A^\varepsilon(t,x) + u_B^\varepsilon(t,x) \to u(t,x) \qquad \text{almost everywhere on } [0,T]\times\Omega. \tag{32}$$

Let us take p in the interval $]1, \min\{p_0, 2\}[$. We use the elementary inequality (for all $x > 0$) $|x-1| \le C_p\,|x^{p/2} - 1| \times |x^{1-p/2} + 1|$ to get

$$\begin{aligned}&\int_0^T\!\!\int_\Omega |k(v^\varepsilon)u_B^\varepsilon - h(v^\varepsilon)u_A^\varepsilon| \le C_p \int_0^T\!\!\int_\Omega |(u_B^\varepsilon k(v^\varepsilon))^{p/2} - (u_A^\varepsilon h(v^\varepsilon))^{p/2}| \times [(u_B^\varepsilon k(v^\varepsilon))^{1-p/2} + (u_A^\varepsilon h(v^\varepsilon))^{1-p/2}]\\ &\le C_p \left(\int_0^T\!\!\int_\Omega |(u_B^\varepsilon k(v^\varepsilon))^{p/2} - (u_A^\varepsilon h(v^\varepsilon))^{p/2}|^2\right)^{1/2} \times \left(\int_0^T\!\!\int_\Omega [(u_B^\varepsilon k(v^\varepsilon))^{1-p/2} + (u_A^\varepsilon h(v^\varepsilon))^{1-p/2}]^2\right)^{1/2} \le \sqrt{\varepsilon}\,C_T,\end{aligned} \tag{33}$$

thanks to Lemma 3. Then, up to a subsequence,

$$h(v^\varepsilon(t,x))\,u_A^\varepsilon(t,x) - k(v^\varepsilon(t,x))\,u_B^\varepsilon(t,x) \to 0 \qquad \text{almost everywhere on } [0,T]\times\Omega. \tag{34}$$

Now thanks to the convergences (30), (32) and (34), we can compute

$$u_A^\varepsilon(t,x) = \frac{k(v^\varepsilon)\,(u_A^\varepsilon + u_B^\varepsilon) + [h(v^\varepsilon)\,u_A^\varepsilon - k(v^\varepsilon)\,u_B^\varepsilon]}{h(v^\varepsilon) + k(v^\varepsilon)} \to \frac{k(v)\,u}{h(v)+k(v)} =: u_A(t,x) \text{ almost everywhere on } [0,T]\times\Omega, \tag{35}$$

and similarly

$$u_B^\varepsilon(t,x) = \frac{h(v^\varepsilon)\,(u_A^\varepsilon + u_B^\varepsilon) - [h(v^\varepsilon)\,u_A^\varepsilon - k(v^\varepsilon)\,u_B^\varepsilon]}{h(v^\varepsilon) + k(v^\varepsilon)} \to \frac{h(v)\,u}{h(v)+k(v)} =: u_B(t,x) \text{ almost everywhere on } [0,T]\times\Omega. \tag{36}$$

We now can pass to the limit in equation (16) for $u_A^\varepsilon + u_B^\varepsilon$ and equation (9) for v^ε. We use Assumption B and the previous estimates to avoid L^1 concentration in $(u_A^\varepsilon + u_B^\varepsilon)^{1+a}$ and $(u_A^\varepsilon + u_B^\varepsilon)^d$. The limit of M^ε in (16) is computed with the definitions of u_A and u_B in (35) and (36). This concludes the proof of Proposition 1. ☐

We now turn to the

Proof of Proposition 2. As in Proposition 1, we consider a strong solution $(u_A^\varepsilon, u_B^\varepsilon, v^\varepsilon)$ of eq. (9), (10).

Note first that the estimates of Lemmas 1 and 2 still hold under the assumptions of Proposition 2 (with $p_0 = 2$ in the case of Lemma 2).

We now introduce a duality lemma in the spirit of the one used in [5]:

Lemma 4. *We consider a function $M := M(t,x)$ satisfying*

$$0 < m_0 \le M(t,x) \le m_1 \qquad \textit{for } t \ge 0 \textit{ and } x \in \Omega, \tag{37}$$

for some constants $m_0, m_1 > 0$. Then one can find $p^* > 2$ such that for all $p \in [2, p^*[$, for all $T \ge 0$, there exists a constant $C_T > 0$ depending only on Ω, N, T, and the constants m_0, m_1, p, such that for any initial data u_{in} in $L^2(\Omega)$ and any $K > 0$, all nonnegative solutions u of the system

$$\begin{cases} \partial_t u - \Delta_x(Mu) \le K \textit{ in } [0,T]\times\Omega, \\ u(0,x) = u_{in}(x) \textit{ in } \Omega, \\ \nabla_x(Mu)(t,x)\cdot n(x) = 0 \textit{ on } [0,T]\times\partial\Omega, \end{cases} \tag{38}$$

satisfy

$$\|u\|_{L^p([0,T]\times\Omega)} \le C_T\left(\|u_{in}\|_{L^2(\Omega)} + K\right). \tag{39}$$

Proof of Lemma 4. It relies on the study of the dual problem

$$\begin{cases} \partial_t v + M\Delta_x v = -f \text{ in } [0,T]\times\Omega, \\ v(T,x) = 0 \text{ in } \Omega, \\ \nabla_x v(t,x)\cdot n(x) = 0 \text{ on } [0,T]\times\partial\Omega, \end{cases} \tag{40}$$

for f a nonnegative function in $L^{p'}([0,T]\times\Omega)$.

Using the notations of [5], we define the constant $C_{m,q} > 0$ for $m > 0$, $q \in]1,2]$ as the best constant in the parabolic estimate

$$\|\Delta_x w\|_{L^q([0,T]\times\Omega)} \le C_{m,q}\,\|g\|_{L^q([0,T]\times\Omega)}, \tag{41}$$

where g is any function in $L^q([0,T]\times\Omega)$ and w is the solution of the backward heat equation

$$\begin{cases} \partial_t w + m\,\Delta_x w = g \text{ in } [0,T]\times\Omega, \\ w(T,x) = 0 \text{ in } \Omega, \\ \nabla_x w(t,x)\cdot n(x) = 0 \text{ on } [0,T]\times\partial\Omega. \end{cases} \tag{42}$$

Let $p \ge 2$, $q = p' \le 2$ and let f be any nonnegative function in L^q. We consider the solution v of system (40). Notice that from a minimum principle, v is nonnegative. Then, from Lemma 2.2 and Remark 2.3 in [5], there exists a constant C_T depending only on Ω, N, T and m_0, m_1, q such that v satisfies

$$\|\Delta_x v\|_{L^q} \le C_T\,\|f\|_{L^q}, \tag{43}$$

and

$$\|v(0,\cdot)\|_{L^2(\Omega)} \le C_T\,\|f\|_{L^q}, \tag{44}$$

provided that $q > 2\frac{N+2}{N+4}$ and

$$C_{\frac{m_0+m_1}{2},q}\,\frac{m_1-m_0}{2} < 1. \tag{45}$$

Let us first assume that condition (45) holds for some fixed $q \in]2\frac{N+2}{N+4}, 2]$. Then we compute

$$\frac{d}{dt}\int_\Omega u(t)v(t) \le K\int_\Omega v(t) - \int_\Omega u(t)f(t), \tag{46}$$

so that integrating w.r.t. time, and using the condition $v(T,\cdot) = 0$,

$$\int_0^T\int_\Omega uf \le K\int_0^T\int_\Omega v + \int_\Omega u_{in}\,v(0,\cdot). \tag{47}$$

The first term is estimated with (43)

$$\int_0^T\int_\Omega v = -\int_0^T\int_\Omega\int_t^T \partial_t v = \int_0^T\int_\Omega\int_t^T (f + M\Delta_x v) \tag{48}$$

$$\leq T \left(\int_0^T \int_\Omega f + m_1 |\Delta_x v| \right) \leq T^{1+1/p} \, |\Omega|^{1/p} \, (1 + m_1 C_T) \, \|f\|_q,$$

and the second term with (44)

$$\int_\Omega u_{in} \, v(0,\cdot) \leq \|u_{in}\|_{L^2(\Omega)} \, \|v(0,\cdot)\|_{L^2(\Omega)} \leq C_T \, \|f\|_{L^q} \, \|u_{in}\|_{L^2(\Omega)}. \tag{49}$$

Recombining,

$$\int_0^T \int_\Omega u \, f \leq C_T \left(K + \|u_{in}\|_{L^2(\Omega)} \right) \|f\|_{L^q}, \tag{50}$$

which by duality gives estimate (39).

It remains to check that there exists an interval $[2, p^*[$ in which any p satisfies condition (45) with $q = p'$. This is done in [5]. $\square$

We apply Lemma 4 to eq. (16) and see that for some $p^* > 2$,

$$||u_A^\varepsilon + u_B^\varepsilon||_{L^{p^*}} \leq C_T. \tag{51}$$

Recalling definition (22) and computation (23) in the case when $p \in]0,1[$, we use the inequality

$$-d_A \int_\Omega \Delta_x u_A^\varepsilon (u_A^\varepsilon)^{p-1} h(v^\varepsilon)^{p-1} \leq -2 \, d_A \, \frac{(1-p)}{p^2} \int_\Omega |\nabla_x (u_A^\varepsilon)^{p/2}|^2 h(v^\varepsilon)^{p-1}$$
$$+ \frac{(1-p)}{2} \, d_A \int_\Omega |\nabla_x v^\varepsilon|^2 (u_A^\varepsilon)^p (h'(v^\varepsilon))^2 h(v^\varepsilon)^{p-3},$$

and get the estimate

$$\begin{aligned}
&\int_\Omega h(v_{in}^\varepsilon)^{p-1} \frac{(u_{A,in}^\varepsilon)^p}{p} + k(v_{in}^\varepsilon)^{p-1} \frac{(u_{B,in}^\varepsilon)^p}{p} \\
+ 2 \, d_A \frac{1-p}{p^2} \int_0^T \int_\Omega |\nabla_x (u_A^\varepsilon)^{p/2}|^2 h(v^\varepsilon)^{p-1} &+ 2 \, (d_A + d_B) \frac{1-p}{p^2} \int_0^T \int_\Omega |\nabla_x (u_B^\varepsilon)^{p/2}|^2 k(v^\varepsilon)^{p-1} \\
&- \frac{1}{\varepsilon} \int_\Omega [k(v^\varepsilon) u_B^\varepsilon - h(v^\varepsilon) u_A^\varepsilon][(u_B^\varepsilon)^{p-1} k(v^\varepsilon)^{p-1} - (u_A^\varepsilon)^{p-1} h(v^\varepsilon)^{p-1}] \\
&\leq \int_\Omega h(v^\varepsilon)^{p-1} \frac{(u_A^\varepsilon)^p}{p}(T) + k(v^\varepsilon)^{p-1} \frac{(u_B^\varepsilon)^p}{p}(T) \\
&+ \frac{1-p}{p} \int_0^T \int_\Omega \partial_t v^\varepsilon [h'(v^\varepsilon) h(v^\varepsilon)^{p-2} (u_A^\varepsilon)^p + k'(v^\varepsilon) k(v^\varepsilon)^{p-2} (u_B^\varepsilon)^p] \\
&- \int_0^T \int_\Omega [r_u - r_a (u_A^\varepsilon + u_B^\varepsilon)^a - r_b (v^\varepsilon)^b][(u_A^\varepsilon)^p h(v^\varepsilon)^{p-1} + (u_B^\varepsilon)^p k(v^\varepsilon)^{p-1}] \\
+ \frac{1-p}{2} \int_0^T \int_\Omega [d_A (u_A^\varepsilon)^p (h'(v^\varepsilon))^2 h(v^\varepsilon)^{p-3} &+ (d_A + d_B)(u_B^\varepsilon)^p (k'(v^\varepsilon))^2 k(v^\varepsilon)^{p-3}] |\nabla_x v^\varepsilon|^2.
\end{aligned} \tag{52}$$

Note that in estimate (52), the first and third term of the r.h.s. are clearly bounded (w.r.t. ε) thanks to Lemma 1 and estimate (51). The second and fourth terms are estimated thanks to inequalities (27) and (28) and lead to estimate (21).

Using (51) and the elementary inequality

$$(x - y) \, (x^{p-1} - y^{p-1}) \geq C_p \, |x^{p/2} - y^{p/2}|^2$$

for $p \in]0, \min(1, p^* - d)[$, we see that

$$||\nabla_x (u_A^\varepsilon)^{p/2}||_{L^2} \leq C_T, \qquad ||\nabla_x (u_B^\varepsilon)^{p/2}||_{L^2} \leq C_T,$$

$$||(h(v^\varepsilon)\, u_A^\varepsilon)^{p/2} - (k(v^\varepsilon)\, u_B^\varepsilon)^{p/2}||_{L^2} \le C_T\, \sqrt{\varepsilon}.$$

Using Lemma 2 with $p_0 = 2$, we see that we then can extract a subsequence - still called $(v^\varepsilon)_\varepsilon$ - which converges towards a limit v (lying in L^∞) for a.e. $(t,x) \in [0,T] \times \Omega$. Moreover, thanks to (51), $\partial_t(u_A^\varepsilon + u_B^\varepsilon)$ is bounded in H^{-2}. Finally, thanks to estimate (31) (still with $p \in]0, \min(1, p^* - d)[$), we see that we can extract a subsequence - also called $(u_A^\varepsilon + u_B^\varepsilon)_\varepsilon$ - which converges towards a limit u (lying in L^2) for a.e. $(t,x) \in [0,T] \times \Omega$. We also can use inequality (33) (still with $p \in]0, \min(1, p^* - d)[$), and end the proof of Proposition 2 as that of Proposition 1. $\square$

3 Proof of existence, regularity and stability

Proof of Theorem 1. First step: existence

We denote by $v_1 = v_1(v_{in}, \Omega, N, d_v, r_v, r_c, c) > 0$ a positive constant given by the maximum principle such that any nonnegative function v with initial datum $v(0,\cdot) = v_{in}$, satisfying Neumann boundary conditions and the inequality

$$\partial_t v - d_v \Delta_x v \le [r_v - r_c v^c] v, \tag{53}$$

satisfies $0 \le v(t,x) \le v_1$ for all $t \ge 0$, $x \in \Omega$. Thanks to a smooth cutoff function $\chi(v)$ ($\chi(v) = 1$ for $0 \le v \le v_1$, $\chi(v) = 0$ for $v \ge 2v_1$ and $0 \le \chi(v) \le 1$ for all $v \ge 0$), we define $\phi_B(v) = \chi(v)\phi(v)$ for all $v \ge 0$. Since ϕ_B is a continuous function with compact support, it is uniformly bounded by some positive constant ϕ_1. Then Proposition 1 gives that the solution of system (9)–(11) with parameters $d_A := d_u/2 > 0$, $d_B := d_u + \phi_1 > 0$, $h(v) := d_u/2 + \phi_B(v) \ge d_u/2 > 0$, $k(v) := d_u/2 + \phi_1 - \phi_B(v) \ge d_u/2 > 0$ and initial data $u_{A,in} := \frac{k(v_{in})}{h(v_{in}) + k(v_{in})} u_{in} \ge 0$, $u_{B,in} := \frac{h(v_{in})}{h(v_{in}) + k(v_{in})} u_{in} \ge 0$ and $v_{in} \ge 0$ converge a. e. on $\mathbb{R}_+ \times \Omega$ when ε tends to zero, and its limit (u_A, u_B, v) provides a solution $(u := u_A + u_B, v) \in L^{p_0 + a}(\Omega) \times L^\infty(\Omega)$ of system (4)–(6) with $d_u + \phi(v)$ replaced by $d_A + d_B \frac{h(v)}{h(v)+k(v)} = d_u + \phi_B(v)$. Then, as a solution of eq. (6), $v(t,x)$ satisfies inequality (53), and by the maximum principle, the bound $0 \le v(t,x) \le v_1$ is valid. By definition of ϕ_B, we then have $\phi_B(v(t,x)) = \phi(v(t,x))$ for all $t \ge 0$, $x \in \Omega$, so that (u,v) is actually a solution of (4)–(6).

Second step: regularity, first part

Using the maximum principle, and the maximal regularity for the heat equation, we see that v lies in L^∞ and (for $q_0 > s_0$ defined by $a\,(s_0 - 1) = d\,(q_0 - 1)$),

$$\|\partial_t v\|_{L^{q_0}} \le C_T\, (1 + \|u^d\|_{L^{q_0}}), \qquad \|\nabla_x^2 v\|_{L^{q_0}} \le C_T\, (1 + \|u^d\|_{L^{q_0}}). \tag{54}$$

We interpolate the second estimate with the L^∞ bound to get $\|\, |\nabla_x v|^2 \|_{L^{q_0}} \le C_T\, (1 + \|u^d\|_{L^{q_0}})$. We then multiply by u^{p-1} eq. (4) and integrate w.r.t. space and time. We get

$$\int_\Omega \frac{u^p}{p}(T) + (p-1)\frac{4}{p^2} \int_0^T \int_\Omega (d_u + \phi(v))\, |\nabla_x u^{p/2}|^2$$
$$= \int_\Omega \frac{u_{in}^p}{p} + \int_0^T \int_\Omega u^p\, (r_u - r_a u^a - r_b v^b) - \frac{2}{p}(p-1) \int_0^T \int_\Omega u^{p/2} \nabla_x u^{p/2} \nabla_x(\phi(v)).$$

Selecting $p = a\,(s_0 - 1) = d\,(q_0 - 1)$, we estimate the last term thanks to the L^∞ bound for v and the L^{q_0} estimate for $|\nabla_x v|^2$:

$$\left| \int_0^T \int_\Omega u^{p/2} \nabla_x u^{p/2} \nabla_x(\phi(v)) \right| \le \int_0^T \int_\Omega \frac{d_u}{p} |\nabla_x u^{p/2}|^2 + \frac{p}{4\, d_u} \int_0^T \int_\Omega u^p\, |\nabla_x(\phi(v))|^2$$
$$\le \int_0^T \int_\Omega \frac{d_u}{p} |\nabla_x u^{p/2}|^2 + C_T \left(\int_0^T \int_\Omega u^{p+d} \right)^{p/(p+d)} \times \left(\int_0^T \int_\Omega |\nabla_x v|^{2(p+d)/d} \right)^{d/p+d} \tag{55}$$
$$\le \int_0^T \int_\Omega \frac{d_u}{p} |\nabla_x u^{p/2}|^2 + C_T \left(1 + \int_0^T \int_\Omega u^{p+d} \right).$$

We get

$$\int_\Omega \frac{u^p}{p}(T) + (p-1)\frac{2}{p^2}\int_0^T \int_\Omega (d_u + 2\phi(v))|\nabla_x u^{p/2}|^2 + r_a \int_0^T \int_\Omega u^{p+a} \leq \int_\Omega \frac{u_{in}^p}{p} + C_T \left(1 + \int_0^T \int_\Omega u^{p+d}\right).$$

Remember our assumption that u_{in} lies in $W^{2,s_0}(\Omega)$ with $s_0 > 1 + N/2$. Then by Sobolev embeddings, $\int_\Omega u_{in}^p$ is clearly finite. In the case $p \geq 1$, since $a > d$, we can conclude that $\int_0^T \int_\Omega u^{p+a}$ is finite, so that u^a lies in L^{s_0}. In the case $p < 1$, a direct integration of eq. (4) w.r.t. space and time gives that u^{1+a} is in L^1, which implies that u^a is in L^{s_0} (since in this case $a\, s_0 = p + a < 1 + a$). Then in both cases u^a is in L^{s_0}, and estimate (54) ensures that $\partial_t v$ and $\nabla_x^2 v$ lie in L^{q_0}.

Using embedding results (see for example Lemma 3.3 in Chapter II of [14]) and the fact that $q_0 > 1 + N/2$, we see that v is Hölder continuous on $[0,T] \times \overline{\Omega}$. Similarly, $\partial_t \phi(v) = \phi'(v)\, \partial_t v$ and $\nabla_x^2 \phi(v) = \phi''(v)(\nabla_x v)^t(\nabla_x v) + \phi'(v)\nabla_x^2 v$ lie in L^{q_0}, so that $\phi(v)$ is also Hölder continuous on $[0,T] \times \overline{\Omega}$. We then rewrite the equation satisfied by u as

$$\partial_t u - A(t,x)\, \Delta_x u + B(t,x) \cdot \nabla_x u + C(t,x)\, u = 0, \tag{56}$$

where $A(t,x) = d_u + \phi(v)$ is Hölder continuous on $[0,T] \times \overline{\Omega}$, $B(t,x) = -2\, \nabla_x \phi(v)$ lies in L^{2q_0}, and $C(t,x) = -\Delta_x \phi(v) - r_u + r_a u^a + r_b v^b$ lies in L^{s_0}.

We now apply the following classical theorem issued from the theory of linear parabolic equations (see for example Theorem 9.1 and its corollary in Chapter IV of [14]):

Proposition 3. *Let $s > (N+2)/2$ and $T > 0$. Suppose that u is solution of the equation*

$$\partial_t u - A(t,x)\, \Delta_x u + B(t,x) \cdot \nabla_x u + C(t,x)\, u = 0, \tag{57}$$

$$\nabla_x u(t,x) \cdot n(x) = 0 \qquad \text{for} \qquad (t,x) \in \mathbb{R}_+ \times \partial\Omega,$$

where the coefficients satisfy: $A := A(t,x)$ is continuous on $[0,T] \times \overline{\Omega}$, $B := B(t,x)$ lies in L^r for some $r > \max(s, N+2)$, and $C := C(t,x)$ lies in L^s. Suppose also that $u(0,\cdot) \in W^{2,s}(\Omega)$ (and, if $s \geq 3$, that the compatibility condition on $\partial\Omega$ holds).

Then, $\partial_t u$ and $\partial^2_{x_i x_j} u$ lie in L^s (for $i,j = 1..N$), and u is Hölder continuous on $[0,T] \times \overline{\Omega}$ (w.r.t. t and x).

This concludes the first step of the study of the regularity.

Third step: regularity, second part

We now assume that ϕ, (resp. u_{in}, v_{in}) have locally Hölder continuous second order derivatives on $\mathbb{R}_+$ (resp. $\overline{\Omega}$).

We already know that (u,v) are Hölder continuous on $[0,T] \times \overline{\Omega}$. Thanks to a Sobolev embedding, the space gradient $\nabla_x v$ is also Hölder continuous on $[0,T] \times \overline{\Omega}$. It is then clear that in eq. (5), the reaction term is Hölder continuous on $[0,T] \times \overline{\Omega}$. Thanks to standard results in the theory of linear parabolic equations (see for example Theorem 5.3 in Chapter IV of [14]), $\partial_t v$ and $\nabla_x^2 v$ are also Hölder continuous on $[0,T] \times \overline{\Omega}$. Writing eq. (4) in its form (56), we see that the coefficients A, B and C are Hölder continuous on $[0,T] \times \overline{\Omega}$ (note that we use here the local Hölder continuity of ϕ''). The same result for linear parabolic equations implies that $\partial_t u$ and $\nabla_x^2 u$ are Hölder continuous on $[0,T] \times \overline{\Omega}$.

This concludes the second step of the study of the regularity.

Fourth step: stability and uniqueness

Let (u_1, v_1) and (u_2, v_2) be two solutions of (4)-(6). We recall that (under our assumptions of regularity on the initial data), these solutions (u_1, v_1) and (u_2, v_2) are continuous (and even Hölder continuous) functions on $[0,T] \times \overline{\Omega}$, and so are the space gradients $\nabla_x v_1$ and $\nabla_x v_2$. For any function $(u,v) \mapsto F(u,v)$, we write $\overline{F(u,v)} = \frac{F(u_1,v_1) + F(u_2,v_2)}{2}$.

We substract the equations satisfied by (u_2, v_2) to the equations satisfied by (u_1, v_1):

$$
\begin{aligned}
&\partial_t(u_1-u_2) - \Delta_x[(d_A + \overline{\phi(v)})\,(u_1-u_2)] - \Delta_x[(g(v_1)-g(v_2))\,\overline{u}] \\
&\qquad = [r_v - r_a\overline{u^a} - r_b\overline{v^b}]\,(u_1-u_2) - [r_a\,(u_1^a-u_2^a) + r_b\,(v_1^b-v_2^b)]\,\overline{u}, \\
&\partial_t(v_1-v_2) - d_v\,\Delta_x(v_1-v_2) \\
&\qquad = [r_v - r_c\overline{v^c} - r_d\overline{u^d}]\,(v_1-v_2) - [r_c\,(v_1^c-v_2^c) + r_d\,(u_1^d-u_2^d)]\,\overline{v}.
\end{aligned}
\tag{58}
$$

We multiply the first equation by the difference u_1-u_2 and integrate w.r.t. space and time. We get the identity

$$
\begin{aligned}
&\frac{1}{2}\int_\Omega (u_1-u_2)^2(T) + \int_0^T\int_\Omega (d_A+\overline{\phi(v)})\,|\nabla_x(u_1-u_2)|^2 + \int_0^T\int_\Omega (u_1-u_2)\nabla_x(u_1-u_2)\cdot\nabla_x(\overline{\phi(v)}) \\
&\qquad + \int_0^T\int_\Omega (\phi(v_1)-\phi(v_2))\nabla_x(u_1-u_2)\cdot\nabla_x\overline{u} + \int_0^T\int_\Omega \overline{u}\,\nabla_x(u_1-u_2)\cdot\nabla_x[(\phi(v_1)-\phi(v_2))] \\
&= \frac{1}{2}\int_\Omega (u_1-u_2)^2(0) + \int_0^T\int_\Omega [r_v - r_a\overline{u^a} - r_b\overline{v^b}]\,(u_1-u_2)^2 - \int_0^T\int_\Omega (u_1-u_2)\,[r_a\,(u_1^a-u_2^a)+r_b\,(v_1^b-v_2^b)]\,\overline{u}.
\end{aligned}
\tag{59}
$$

In the left-hand side of this identity, the two first terms are nonnegative. The other terms are controlled thanks to the smoothness of the functions $(\overline{u},\overline{v})$ and their space gradients (and the elementary inequality $2ab \le a^2+b^2$). We detail below their treatment: the third term of (59) is controlled by

$$
\begin{aligned}
\left|\int_0^T\int_\Omega (u_1-u_2)\,\nabla_x(u_1-u_2)\cdot\nabla_x(\overline{\phi(v)})\right| &\le C_T\int_0^T\int_\Omega |u_1-u_2|\,|\nabla_x(u_1-u_2)| \\
&\le \frac{d_A}{4}\int_0^T\int_\Omega |\nabla_x(u_1-u_2)|^2 + C_T\int_0^T\int_\Omega |u_1-u_2|^2,
\end{aligned}
\tag{60}
$$

the fourth term of (59) is controlled by

$$
\left|\int_0^T\int_\Omega (\phi(v_1)-\phi(v_2))\,\nabla_x(u_1-u_2)\cdot\nabla_x\overline{u}\right| \le \frac{d_A}{4}\int_0^T\int_\Omega |\nabla_x(u_1-u_2)|^2 + C_T\int_0^T\int_\Omega |\phi(v_1)-\phi(v_2)|^2, \tag{61}
$$

and the fifth term of (59) is controlled by

$$
\left|\int_0^T\int_\Omega \overline{u}\,\nabla_x(u_1-u_2)\cdot\nabla_x[(\phi(v_1)-\phi(v_2))]\right| \le \frac{d_A}{4}\int_0^T\int_\Omega |\nabla_x(u_1-u_2)|^2 + C_T\int_0^T\int_\Omega |\nabla_x[\phi(v_1)-\phi(v_2)]|^2, \tag{62}
$$

where moreover

$$
\begin{aligned}
\int_0^T\int_\Omega |\nabla_x[\phi(v_1)-\phi(v_2)]|^2 &= \int_0^T\int_\Omega |\overline{\phi'(v)}\,\nabla_x(v_1-v_2) + (\phi'(v_1)-\phi'(v_2))\,\nabla_x\overline{v}|^2 \\
&\le C_T\int_0^T\int_\Omega |\nabla_x(v_1-v_2)|^2 + C_T\int_0^T\int_\Omega |\phi'(v_1)-\phi'(v_2)|^2.
\end{aligned}
\tag{63}
$$

It remains to control the last term of the right-hand side :

$$
\begin{aligned}
-\int_0^T\int_\Omega (u_1-u_2)\,[r_a\,(u_1^a-u_2^a)+r_b\,(v_1^b-v_2^b)]\,\overline{u} &\le r_b\int_0^T\int_\Omega |u_1-u_2|\,|v_1^b-v_2^b|\,\overline{u} \\
&\le C_T\int_0^T\int_\Omega |u_1-u_2|^2 + C_T\int_0^T\int_\Omega |v_1^b-v_2^b|^2.
\end{aligned}
\tag{64}
$$

Thanks to those estimates, the identity (59) becomes

$$
\begin{aligned}
\int_\Omega (u_1-u_2)^2(T) \le \int_\Omega (u_1-u_2)^2(0) + C_T\Bigg(&\int_0^T\int_\Omega (u_1-u_2)^2 + \int_0^T\int_\Omega |\phi(v_1)-\phi(v_2)|^2 \\
&+ \int_0^T\int_\Omega |g'(v_1)-g'(v_2)|^2 + \int_0^T\int_\Omega |\nabla_x(v_1-v_2)|^2 + \int_0^T\int_\Omega |v_1^b-v_2^b|^2\Bigg).
\end{aligned}
\tag{65}
$$

We now multiply the second equation of (58) by the difference $v_1 - v_2$ and integrate w.r.t. space and time. We get

$$\begin{aligned} &\frac{1}{2}\int_\Omega (v_1-v_2)^2(T) + d_v \int_0^T \int_\Omega |\nabla_x (v_1-v_2)|^2 \\ &= \frac{1}{2}\int_\Omega (v_1-v_2)^2(0) \\ &+ \int_0^T\int_\Omega [r_v - r_c\,\overline{v^c} - r_d\,\overline{u^d}]\,(v_1-v_2)^2 - \int_0^T\int_\Omega (v_1-v_2)\,[r_c\,(v_1^c - v_2^c) + r_d\,(u_1^d-u_2^d)]\,\overline{v} \\ &\le \frac{1}{2}\int_\Omega (v_1-v_2)^2(0) + C_T\int_0^T\int_\Omega (v_1-v_2)^2 + C_T\int_0^T\int_\Omega |u_1^d - u_2^d|^2. \end{aligned} \tag{66}$$

We combine the two energy estimates (65) and (66):

$$\begin{aligned} \int_\Omega (u_1-u_2)^2(T) + \int_\Omega (v_1-v_2)^2(T) &\le \int_\Omega (u_1-u_2)^2(0) + \int_\Omega (v_1-v_2)^2(0) \\ &+ C_T\Bigg(\int_0^T\int_\Omega (u_1-u_2)^2 + \int_0^T\int_\Omega (v_1-v_2)^2 \end{aligned} \tag{67}$$
$$+ \int_0^T\int_\Omega |u_1^d-u_2^d|^2 + \int_0^T\int_\Omega |\phi(v_1)-\phi(v_2)|^2 + \int_0^T\int_\Omega |\phi'(v_1)-\phi'(v_2)|^2 + \int_0^T\int_\Omega |v_1^b - v_2^b|^2\Bigg).$$

Since ϕ'' is continuous on $\mathbb{R}_+$, the applications ϕ and ϕ' are locally Lipschitz. The assumption $b \ge 1, d \ge 1$ ensures that the applications $v \mapsto v^b$ and $u \mapsto u^d$ are also locally Lipschitz on $\mathbb{R}_+$. Therefore

$$\begin{aligned} \int_\Omega (u_1-u_2)^2(T) + \int_\Omega (v_1-v_2)^2(T) &\le \int_\Omega (u_1-u_2)^2(0) + \int_\Omega (v_1-v_2)^2(0) \\ &+ C_T\left(\int_0^T\int_\Omega (u_1-u_2)^2 + \int_0^T\int_\Omega (v_1-v_2)^2\right) \end{aligned} \tag{68}$$

and we can conclude thanks to Gronwall's lemma.

Note that thanks to the minimum principle, the assumption $b \ge 1, d \ge 1$ can be relaxed if the initial data are bounded below by a strictly positive constant.

This concludes the study of stability (and uniqueness), and ends the proof of Theorem 1.

□

Proof of Theorem 2. We define v_1, $\phi_B(v)$ and ϕ_1 as in the proof of Theorem 1. Then Proposition 1 gives that the solution of system (9)–(11) with parameters $d_A := d_u/2 > 0$, $d_B := d_u + \phi_1 > 0$, $h(v) := d_u/2 + \phi_B(v) \ge d_u/2 > 0$, $k(v) := d_u/2 + \phi_1 - \phi_B(v) \ge d_u/2 > 0$ and initial data $u_{A,in} := \frac{k(v_{in})}{h(v_{in})+k(v_{in})}\,u_{in} \ge 0$, $u_{B,in} := \frac{h(v_{in})}{h(v_{in})+k(v_{in})}\,u_{in} \ge 0$ and $v_{in} \ge 0$ converge a. e. on $\mathbb{R}_+ \times \Omega$ when ε tends to zero, and its limit (u_A, u_B, v) provides a solution $(u := u_A + u_B, v) \in L^2(\Omega) \times L^\infty(\Omega)$ of system (4)–(6) with $d_u + \phi(v)$ replaced by $d_A + d_B \frac{h(v)}{h(v)+k(v)} = d_u + \phi_B(v)$. We conclude as in the proof of Theorem 1.

□

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