

ENTROPY DISSIPATION ESTIMATES FOR THE LANDAU EQUATION IN THE COULOMB CASE AND APPLICATIONS

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ABSTRACT. We present in this paper an estimate which bounds from below the entropy dissipation $D(f)$ of the Landau operator with Coulomb interaction by a weighted H^1 norm of the square root of f . As a consequence, we get a weighted $L_t^1(L_v^3)$ estimate for the solutions of the spatially homogeneous Landau equation with Coulomb interaction, and the propagation of L^1 moments of any order for this equation. We also present an application of our estimate to the Landau equation with (moderately) soft potentials, providing thus a new proof of some recent results of [22].

1. INTRODUCTION AND MAIN RESULT

1.1. Description of the Landau equation. We recall the spatially homogeneous Landau equation of plasma theory (cf. [10, 23]),

$$(1) \quad \frac{\partial f}{\partial t}(t, v) = Q(f, f)(t, v), \quad v \in \mathbb{R}^N, \quad t \geq 0,$$

where $f := f(t, v)$ is a nonnegative function representing the number of particles which at time t move with velocity v , and Q is a nonlinear quadratic operator modeling the collisions between those particles. It acts on the variable v only, and writes

$$(2) \quad Q(f, f)(v) = \sum_{i=1}^N \frac{\partial}{\partial v_i} \left\{ \int_{\mathbb{R}^N} \sum_{j=1}^N a_{ij}(v - w) \left(f(w) \frac{\partial f}{\partial v_j}(v) - f(v) \frac{\partial f}{\partial v_j}(w) \right) dw \right\}.$$

We also introduce the (nonnegative) initial datum $f_{in} : \mathbb{R}^N \rightarrow \mathbb{R}_+$:

$$(3) \quad \forall v \in \mathbb{R}^N, \quad f(0, v) = f_{in}(v).$$

Here, $(a_{ij}(z))_{ij}$ ($z \in \mathbb{R}^N$) is given by

$$(4) \quad a_{ij}(z) = \Pi_{ij}(z) \psi(|z|),$$

where ψ is a nonnegative function, and

$$(5) \quad \Pi_{ij}(z) = \delta_{ij} - \frac{z_i z_j}{|z|^2}$$

is the i, j -component of the orthogonal projection Π onto $z^\perp := \{y / y \cdot z = 0\}$.

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It is customary to define the following functions:

$$(6) \quad b_i(z) = \sum_{j=1}^N \partial_j a_{ij}(z) = -(N-1) \frac{z_i}{|z|^2} \psi(|z|),$$

$$(7) \quad c(z) = \sum_{i=1}^N \sum_{j=1}^N \partial_{ij} a_{ij}(z) = -(N-1) \left((N-2) \frac{\psi(|z|)}{|z|^2} + \frac{\psi'(|z|)}{|z|} \right).$$

The computation above must be adapted when ψ has strong singularities (for example, in the important case when $N = 3$ and $\psi(|z|) = |z|^{-1}$, corresponding to the Coulomb interaction, one finds $c(z) = -8\pi \delta_0$).

Using those functions, the Landau operator can be written (at the formal level) as a (conservative or non conservative) nonlinear (quadratic) parabolic equation with nonlocal coefficients, indeed

$$(8) \quad Q(f, f) = \sum_{i=1}^N \frac{\partial}{\partial v_i} \left(\sum_{j=1}^N (a_{ij} * f) \frac{\partial f}{\partial v_j} - (b_i * f) f \right),$$

and

$$(9) \quad Q(f, f) = \sum_{i=1}^N \sum_{j=1}^N (a_{ij} * f) \frac{\partial^2 f}{\partial v_i \partial v_j} - (c * f) f.$$

We also observe that at the formal level (that is, when both f and φ are smooth functions having a reasonable behavior at infinity), the following weak version of the Landau operator can be defined:

$$(10) \quad \begin{aligned} & \int_{\mathbb{R}^N} Q(f, f)(v) \varphi(v) dv \\ &= \frac{1}{2} \sum_{i=1}^N \sum_{j=1}^N \iint_{\mathbb{R}^N \times \mathbb{R}^N} f(v) f(w) a_{ij}(v-w) \left(\partial_{ij} \varphi(v) + (\partial_{ij} \varphi)(w) \right) dv dw \\ & \quad + \sum_{i=1}^N \iint_{\mathbb{R}^N \times \mathbb{R}^N} f(v) f(w) b_i(v-w) \left(\partial_i \varphi(v) - (\partial_i \varphi)(w) \right) dv dw. \end{aligned}$$

Using the test functions $\varphi(v) = 1$, v_i (for $i = 1, \dots, N$), $\frac{|v|^2}{2}$, we see that (still at the formal level), the Landau operator conserves mass, (i -th component of the) momentum and kinetic energy, that is:

$$(11) \quad \int_{\mathbb{R}^N} Q(f, f)(v) \begin{pmatrix} 1 \\ v_i \\ |v|^2/2 \end{pmatrix} dv = 0.$$

We also get (at the formal level) the formula for the entropy dissipation $D_\psi := D_\psi(f)$ (defined on functions f from \mathbb{R}^N to \mathbb{R}_+) by considering $\varphi(v) = \ln f(v)$:

$$(12) \quad D_\psi(f) := - \int_{\mathbb{R}^N} Q(f, f)(v) \ln f(v) dv$$

$$\begin{aligned}
&= \frac{1}{2} \sum_{i=1}^N \sum_{j=1}^N \iint_{\mathbb{R}^N \times \mathbb{R}^N} f(v) f(w) \psi(|v-w|) \Pi_{ij}(v-w) \left(\frac{\partial_i f}{f}(v) - \frac{\partial_i f}{f}(w) \right) \\
&\quad \left(\frac{\partial_j f}{f}(v) - \frac{\partial_j f}{f}(w) \right) dv dw \\
&= 2 \iint_{\mathbb{R}^N \times \mathbb{R}^N} \psi(|v-w|) \left| \Pi[(\nabla_v - \nabla_w) \sqrt{f(v)f(w)}] \right|^2 dv dw \geq 0.
\end{aligned}$$

Note that this last formula (as already noticed in [29], but with ψ inside the derivatives) implies that $D_\psi(f)$ is defined (but may be infinite, in particular if the distribution $\Pi(\nabla_v - \nabla_w) \sqrt{f(v)f(w)}$ does not lie in $L^2_{loc}(\mathbb{R}^N \times \mathbb{R}^N)$) as soon as $f \in L^1_{loc}(\mathbb{R}^N)$.

As a consequence of formulas (11) and (12), the solutions of the Landau equation (1), (3) satisfy (at least formally) the conservation of mass, momentum and energy, that is

$$(13) \quad \int_{\mathbb{R}^N} f(t, v) \begin{pmatrix} 1 \\ v_i \\ |v|^2/2 \end{pmatrix} dv = \int_{\mathbb{R}^N} f_{in}(v) \begin{pmatrix} 1 \\ v_i \\ |v|^2/2 \end{pmatrix} dv,$$

and the entropy dissipation identity

$$(14) \quad \frac{d}{dt} H(f(t, \cdot)) = -D_\psi(f(t, \cdot)) \leq 0,$$

where $H := H(f)$ is the entropy functional (defined on functions from \mathbb{R}^N to \mathbb{R}_+)

$$(15) \quad H(f) := \int_{\mathbb{R}^N} f(v) \ln f(v) dv,$$

and D_ψ is the entropy dissipation functional defined in (12).

Note that identities (13) and (14) naturally furnish an *a priori* estimate: indeed integrating (14) on the time interval $[0, T]$, one can write

$$H(f(T, \cdot)) + \int_0^T D_\psi(f(t, \cdot)) dt = H(f(0, \cdot)).$$

Then,

$$\begin{aligned}
&\int_{\mathbb{R}^N} f(t, v) |\ln f(t, v)| dv = \int_{\mathbb{R}^N} f(t, v) \ln f(t, v) dv \\
&\quad + 2 \int_{\mathbb{R}^N} f(t, v) (-\ln f(t, v)) 1_{\{f(t, v) \leq 1\}} dv \\
&= H(f(t, \cdot)) + 2 \int_{\mathbb{R}^N} f(t, v) (-\ln f(t, v)) 1_{\{f(t, v) \leq e^{-1-|v|^2}\}} dv \\
&\quad + 2 \int_{\mathbb{R}^N} f(t, v) (-\ln f(t, v)) 1_{\{1 \geq f(t, v) \geq e^{-1-|v|^2}\}} dv \\
&\leq H(f(t, \cdot)) + 2 \int_{\mathbb{R}^N} e^{-1-|v|^2} (1 + |v|^2) dv + 2 \int_{\mathbb{R}^N} f(0, v) (1 + |v|^2) dv.
\end{aligned}$$

Then, for any initial datum such that the mass, energy and entropy are finite, that is

$$(16) \quad \mathcal{M}_{in} := \int_{v \in \mathbb{R}^N} f_{in}(v) \left(1 + \frac{|v|^2}{2} + |\ln(f_{in}(v))| \right) dv < +\infty,$$

we see that a solution of (1), (3) satisfies at the formal level the following *a priori* estimate:

$$(17) \quad \sup_{t \in [0, T]} \int_{v \in \mathbb{R}^N} f(t, v) \left(1 + \frac{|v|^2}{2} + |\ln f(t, v)| \right) dv \\ + \int_0^T D_\psi(f(t, \cdot)) dt \leq C(T, N, \mathcal{M}_{in}),$$

where the constant $C(T, N, \mathcal{M}_{in})$ only depends on T, N and \mathcal{M}_{in} .

The most important (that is, physically relevant) function ψ appearing in operator (1) is $\psi(z) = |z|^{-1}$ in dimension 3. It corresponds to the case of charged particles (moving according to Coulomb interaction), cf. [23], and it also naturally appears in the so-called weak coupling asymptotics of Boltzmann equation (cf. [6] and the older reference [7]). Our paper is mainly devoted to the study of this case (systematically called in the sequel the “Coulomb case”). In the Coulomb case, $b_i(z) = -2 z_i |z|^{-3}$, and $c(z) = -8\pi \delta_0(z)$.

It is however also interesting, at least from the mathematical viewpoint, to consider more general functions ψ and dimensions $N \in \mathbb{N}$ ($N \geq 2$). We refer for example to [15] to see how the Landau kernel with arbitrary ψ can be obtained from the Boltzmann kernel (with arbitrary cross section) through a scaling involving the concept of grazing collisions.

We use in this paper a terminology very close to that of [22]. When $N = 3$ (and sometimes for all $N \in \mathbb{N}$, $N \geq 2$), if ψ is given by a power law, we say that

$$(18) \quad \psi(|z|) = |z|^{\gamma+2}$$

is coming out of hard potentials when $\gamma \in]0, 1]$, Maxwellian molecules when $\gamma = 0$, (moderately) soft potentials when $\gamma \in [-2, 0]$, and very soft potentials when $\gamma \in]-4, -2[$. Note that the Coulomb case falls within the category of very soft potentials such as it is defined here, and that it does not constitute a limiting case of this category.

As we shall see, one of the main results of this paper holds as long as $\gamma > -4$; the limiting case corresponds thus to $\gamma = -4$. This provides some justification for the use of the terminology “very soft potentials” in the case when $\gamma \in]-4, -2[$, rather than for another interval whose right-hand side is -2 . Note that $\gamma = -4$ already appeared as a limiting parameter in the context of the Boltzmann equation, cf. applications in section 7 of [3].

1.2. Existing results on the spatially homogeneous Landau equation and new results in the Coulomb case. Our primary goal in this paper is to improve the existing theory on the spatially homogeneous Landau equation in the Coulomb case. We however first recall here what is known at the mathematical level in “easier” cases, and only describe at the end what is known in the specific case of the Coulomb interaction (in 3D).

The case of hard potentials, (that is, ψ given by (18) with $\gamma \in]0, 1]$), has been thoroughly treated. We refer to [16] for the existence of C^∞ solutions, and to [11], [12] and [25] for a study of smoothness beyond C^∞ . The solutions are moreover unique, and rapidly decaying (w.r.t. the v variable, and when $t > 0$) whenever

the initial datum lies in a suitably weighted L^2 space. Finally, Maxwellian lower bounds are also known to hold in this case (cf. [16]).

The special case $\gamma = 0$ in (18) (that is, Maxwellian molecules) is also quite well-known (cf. [28]): smoothness also immediately appears when the initial datum has finite mass and energy, and uniqueness of a classical solution holds for such an initial datum. Polynomial moments (with respect to the v variable) are however only propagated (not created like when $\gamma > 0$). Maxwellian lower bounds exist as in the case of hard potentials. A remarkable feature of Maxwellian molecules is that the Fisher information is decaying, so that H is not the only known Lyapunov functional in that case.

The theory is less developed in the case of (moderately) soft potentials (that is, ψ given by (18) with $\gamma \in [-2, 0[$). The best result as far as existence/smoothness is concerned can be found in the very recent paper [22] (cf. also [2] for related estimates, and [20] for measure solutions). There, propagation of L^p norms is proven, which implies existence and uniqueness of suitably defined solutions thanks to theorems proven in [19]. Note that the limiting case $\gamma = -2$, which requires a lot of care, is treated in [22]. We recover in Appendix 2 of our paper some of the results of [22], with a different proof.

In all the previous cases, the boundedness of the mass, energy and entropy of f is sufficient to give a sense to weak solutions (that is, solutions defined by using the weak form (10) of the operator). Then, extra smoothness estimates are based on the parabolic forms (8), (9) of the operator. More precisely, coercivity estimates are extracted from the term involving $a_{ij} * f$ in those formula. They take the form (cf. [2], Prop. 2.1. p.4):

$$(19) \quad \sum_{i=1}^N \sum_{j=1}^N (a_{ij} * f)(v) \xi_i \xi_j \geq C (1 + |v|^2)^{\gamma/2} |\xi|^2,$$

where C only depends on the mass, energy and entropy of f . Estimate (19) can be efficient only if the terms involving $b_i * f$ in the (conservative) parabolic equation (8) can be controlled. Such a control is rather easily obtained in the case of hard potentials or Maxwellian molecules ($\gamma \in [0, 1]$ in (18)), but becomes much more difficult to get when (moderately) soft potentials ($\gamma \in [-2, 0[$ in (18)) are concerned (cf. [22] and Appendix 2 of the present work for more details).

For very soft potentials ($\gamma \in]-4, -2[$ when ψ is given by (18)), several new difficulties appear.

First, the mere boundedness of mass, energy and entropy of f does not seem sufficient any more for defining weak solutions thanks to (10). Indeed, remembering that $a_{ij}(v-w)$ behaves like $|v-w|^{\gamma+2}$ (when $|v-w|$ is close to 0), terms appearing in the weak form (10) of the operator (written here in 3D, but the same difficulty appears in all dimensions) such as

$$(20) \quad \iint_{\mathbb{R}^3 \times \mathbb{R}^3} f(v) f(w) a_{ij}(v-w) \left(\partial_{ij} \varphi(v) + (\partial_{ij} \varphi)(w) \right) dv dw$$

are not automatically defined when f belongs to a (weighted) $L^1(\mathbb{R}^3)$ (or $L \ln L(\mathbb{R}^3)$) space. Note that they are however defined if f belongs to some $L^p(\mathbb{R}^3)$ for $p > 1$ large enough. Looking for the best (smallest) possible p which enables the definition of (20), for f which do not depend on the time variable t , gives an insight of the specific difficulties of very soft potentials. Observing that when $\gamma < -2$,

$|\cdot|^{\gamma+2} 1_{|\cdot|\leq 1}$ lies in $L^{\frac{3}{-\gamma-2}-\varepsilon}$ for all (small) $\varepsilon > 0$, we see that if $f \in L^p(\mathbb{R}^3)$, then $f * [a_{ij} 1_{|\cdot|\leq 1}] \in L^r(\mathbb{R}^3)$, with $\frac{1}{r} > \frac{1}{p} + \frac{-\gamma-2}{3} - 1$, thanks to Young's inequality for convolutions. It is then possible to define (20) if $\frac{1}{r} + \frac{1}{p} < 1$, that is $\frac{2}{p} + \frac{-\gamma-2}{3} < 2$, or $p > \frac{6}{8+\gamma}$. This computation is in fact too optimistic because of the dependence of f w.r.t. time, but it already shows that some $L^p(\mathbb{R}^3)$ (for $p > 1$) estimate is required in order to define weak solutions.

Secondly, the attempts to use the coercivity estimates such as (19) have not been able yet to provide L^p estimates with $p > 1$ in the case of very soft potentials ($\gamma \in]-4, -2[$ in (18)), because the term involving $b_i * f$ in (8) cannot be controlled. This is all the more frustrating since such L^p estimates exist when $\gamma \geq -2$ (cf. [22]), but are not needed to define weak solutions in that case (as already mentioned, an L^p estimate with $p > 1$ is needed in order to define a weak solution only in the case of very soft potentials).

Note nevertheless that (in the spatially homogeneous as well as in the spatially inhomogeneous context), it is possible (in the Coulomb case) to build a theory of local (in time) solutions, or of global solutions with small initial data, cf. [5], [18], [21] and [2].

As far as global in time solutions with general (not small) initial data are concerned, a significant step forward was achieved in [29], where it was observed that the L^p estimate (with, say, p larger than $\frac{6}{8+\gamma}$) was not really necessary to define solutions, if one allows to use the boundedness (in L^1 w.r.t. time) of the entropy dissipation $D_\psi(f)$ (and not only the mass, energy and entropy appearing in the *a priori* estimate (17)) to bound terms like (20). Solutions defined in this way are called “H-solutions” in order to distinguish them from weak solutions which need an L^p estimate (with $p > 1$ large enough) for f .

Another progress was made in [4] (see formula (70) p. 30, Lemma 13 p. 35, and Remark p. 36), when renormalized solutions of the spatially inhomogeneous Landau equation in the Coulomb case were shown to satisfy bounds of the type

$$(21) \quad \int_0^T \iint_{\mathbb{R}^{2N}} \sum_{i,j} (a_{ij} * f)(v) \partial_{v_i} \left(\frac{f}{1+f} \right) \partial_{v_j} \left(\frac{f}{1+f} \right) dv dx dt < +\infty,$$

which imply that the terms appearing in the definition of renormalized solutions (of the spatially inhomogeneous Landau equation in the Coulomb case) are well defined.

This result is related to the estimates on the non cutoff Boltzmann operator proven in [3] (cf. also the earlier works [24], [30] and [1]), which belong to the class of entropy dissipation estimates. This means that the entropy dissipation functional related to a PDE is bounded below by a functional controlling somehow the smoothness of the function (appearing in the entropy dissipation functional).

It is in fact proven in [3] that the entropy dissipation of the Boltzmann operator without cutoff controls (with the help of the mass and energy) \sqrt{f} in H_{loc}^s , where s is related to the strength of the angular singularity in the cross section of the Boltzmann equation. By suitably scaling this cross section (cf. [3] or [4]), it is possible to recover the Landau equation, thus suggesting that in the limit, the Landau entropy dissipation (with, say, ψ given by (18) and $\gamma \in]-4, 1[$), controls \sqrt{f} in H_{loc}^1 . Such a result is indeed quoted in [27] (section 5.1.3.2, p. 170), and

may be obtained either starting from the results of [3], or from those of [17]. Using then a Sobolev embedding, the *a priori* estimate (17) implies that (a solution of the spatially homogeneous Landau equation) f lies in $L^1([0, T]; L^3_{loc}(\mathbb{R}^N))$, so that the weak formulation (10) makes sense.

In this paper, we show that in fact, in the Coulomb case, it is possible to go beyond a local setting, and get a global weighted (presumably with the optimal weight) estimate in $L^1([0, T]; L^3(\mathbb{R}^3))$ for f , associated to a global weighted (presumably with the optimal weight) bound of \sqrt{f} in $L^2([0, T]; H^1(\mathbb{R}^3))$.

This estimate is a consequence of our main theorem, which is an entropy dissipation estimate enabling the control of the weighted $H^1(\mathbb{R}^3)$ norm of \sqrt{f} by the Landau entropy dissipation $D_{|\cdot|^{-1}}$ of the Landau operator in the Coulomb case.

We introduce the following definition for weighted L^p spaces: for all $l \in \mathbb{R}$, $p \in [1, +\infty]$, the weighted L^p spaces and norms are thus defined:

$$\|f\|_{L^p_l(\mathbb{R}^N)} := \|(v \mapsto (1 + |v|^2)^{l/2} f)\|_{L^p(\mathbb{R}^N)},$$

and $L^p_l(\mathbb{R}^N) = \{f : \mathbb{R}^N \rightarrow \mathbb{R}, \|f\|_{L^p_l(\mathbb{R}^N)} < +\infty\}$.

Then, our main theorem writes

Theorem 1. *Let $f := f(v) \geq 0$, belonging to $L^1_2(\mathbb{R}^3)$, be such that $\int f |\ln f| dv \leq \bar{H}$, for some $\bar{H} > 0$.*

Then, there exists a constant $C := C(\int f dv, \int f v dv, \int f |v|^2/2 dv, \bar{H}) > 0$ which (explicitly) depends only on the mass $\int f dv$, the momentum $\int f v dv$, the energy $\int f |v|^2/2 dv$ and (an upper bound on the) entropy \bar{H} , such that

$$\int_{\mathbb{R}^3} |\nabla \sqrt{f(v)}|^2 (1 + |v|^2)^{-3/2} dv \leq C (1 + D_{x \mapsto |x|^{-1}}(f)).$$

If moreover f is radially symmetric and satisfies the normalization $\int f(v) dv = 1$, $\int f(v) v dv = 0$, $\int f |v|^2 dv = 3$, then the constant can be explicitly estimated with numbers which are not too huge:

$$\begin{aligned} & \int_{\mathbb{R}^3} |\nabla \sqrt{f(v)}|^2 (1 + |v|^2)^{-3/2} dv \leq 108 \\ & \times 13^{3/2} \left(\frac{16\pi}{3}\right)^{4/3} \exp\left(\frac{16}{3} \bar{H}\right) \left(2 + \frac{128}{3} D_{x \mapsto |x|^{-1}}(f)\right). \end{aligned}$$

This theorem is proven in section 2, as a corollary of a more general theorem presented there (Theorem 3), in which the dimension is arbitrary, and the hypothesis on ψ is relaxed: we assume indeed only that ψ is bounded below by a (negative) power law at infinity. The discussion on the hypothesis of this theorem, as well as a presentation of the main arguments of the proof (and the link of this proof with other works) is postponed to section 2.

As a corollary of Theorem 1 (and as previously announced), we recover that H-solutions of the Landau equation in the Coulomb case (defined on $[0, T]$) belong to $L^1([0, T], L^3_{-3}(\mathbb{R}^3))$, and satisfy the estimate

$$\int_0^T \int_{\mathbb{R}^3} |\nabla \sqrt{f(t, v)}|^2 (1 + |v|^2)^{-3/2} dv dt < +\infty.$$

They are therefore “usual” weak solutions (of the Landau equation in the Coulomb case). This last result (the fact that H-solutions are usual weak solutions) was

in fact already suggested in [3], where a close computation was performed for the Boltzmann equation without cutoff and very soft potentials (cf. second application of section 7 in [3]). The bound suggested there (and in [27], section 5.1.3.2, p. 170) is however only local (that is, $f \in L^1([0, T], L^3_{loc}(\mathbb{R}^3))$).

More precisely, we prove the

Corollary 1.1. *Let $T > 0$, $f_{in} := f_{in}(v) \in L^1_2(\mathbb{R}^3) \cap L \ln L(\mathbb{R}^3)$ be nonnegative, and $f := f(t, v) \in L^\infty([0, T]; L^1_2(\mathbb{R}^3))$ be a (nonnegative) H -solution of the Landau equation in the Coulomb case ($\gamma = -3$ and $N = 3$) with initial datum f_{in} (such a solution exists thanks to [29]).*

Then, $f \in L^1([0, T], L^3_{-3}(\mathbb{R}^3))$, and for all $\varphi := \varphi(t, v) \in C^2_c([0, T] \times \mathbb{R}^3_v)$,

$$\begin{aligned}
 (22) \quad & - \int_{\mathbb{R}^3} f_{in}(v) \varphi(0, v) dv - \int_0^T \int_{\mathbb{R}^3} f(t, v) \partial_t \varphi(t, v) dv dt \\
 &= \frac{1}{2} \sum_{i=1}^3 \sum_{j=1}^3 \int_0^T \iint_{\mathbb{R}^3 \times \mathbb{R}^3} f(t, v) f(t, w) a_{ij}(v-w) \left(\partial_{ij} \varphi(t, v) + (\partial_{ij} \varphi)(t, w) \right) dv dw dt \\
 & \quad + \sum_{i=1}^3 \int_0^T \iint_{\mathbb{R}^3 \times \mathbb{R}^3} f(t, v) f(t, w) b_i(v-w) \left(\partial_i \varphi(t, v) - (\partial_i \varphi)(t, w) \right) dv dw dt.
 \end{aligned}$$

We underline that all the terms above (remember that $a_{ij}(z) = \Pi_{ij}(z) |z|^{-1}$ and $b_i(z) = -2 z_i |z|^{-3}$) are defined under the assumption that $f \in L^\infty([0, T]; L^1_2(\mathbb{R}^3)) \cap L^1([0, T]; L^3_{-3}(\mathbb{R}^3))$. This last statement is part of Corollary 1.1.

Note that thanks to an interpolation (see Proposition 6) between $L^\infty([0, T]; L^1_2(\mathbb{R}^3))$ and $L^1([0, T]; L^3_{-3}(\mathbb{R}^3))$, we get for $f := f(t, v)$ (weak solution of the Landau equation in the Coulomb case) an estimate in $L^{1+\delta}([0, T]; L^{3-\varepsilon}_{-3}(\mathbb{R}^3))$ for some $\varepsilon, \delta > 0$, so that the first estimate proven in [22] in the case of (moderately) soft potentials still holds for the Coulomb case, up to the weight (this restriction is removed in next paragraph). More details about the link between this work and the estimates of [22] are presented in Appendix 2, let us only add here that our estimate is uniform in time (whereas in [22] it grows polynomially or more than polynomially (when $\gamma = -2$)) in time, but it involves a negative weight in v . Removing this weight thanks to an interpolation with moments in L^1 (see paragraph below) leads, as in [22], to a polynomially growing w.r.t. time estimate.

We also indicate that our result does not allow to use the uniqueness theorems of [19] and [18], which state that for $\gamma \in [-3, -2]$, uniqueness holds (for solutions of the spatially homogeneous Landau equation) when the solution f lies in $L^\infty([0, T]; L^1_2(\mathbb{R}^3)) \cap L^1([0, T]; L^{3/(3+\gamma)}(\mathbb{R}^3))$ (with $3/(3+\gamma)$ replaced by ∞ when $\gamma = -3$, that is, in the Coulomb case). Indeed, $3/(3+\gamma) > 3$ whenever $\gamma \in [-3, -2]$.

Our main estimate (Theorem 1) can also be used in order to study the behavior of weak solutions of the Landau equation in the Coulomb case when $|v| \mapsto +\infty$. As very often when collision operators are concerned, this behavior is studied through the evolution of the (L^1) moments of the solution. As expected for (moderately or very) soft potentials, we are able to show that moments of any order are propagated

when H -solutions (or equivalently, weak solutions) of the Landau equation in the Coulomb case are considered. More precisely, we show the

Proposition 2. *Let $T > 0$, $k \in \mathbb{R}$, $k > 2$, and $f_{in} := f_{in}(v) \in L_k^1(\mathbb{R}^3)$ be nonnegative. We consider $f := f(t, v) \in L^\infty([0, T]; L_2^1(\mathbb{R}^3))$ a (nonnegative) H -solution (or equivalently, weak solution) of the Landau equation in the Coulomb case ($\gamma = -3$ in (18) and $N = 3$), with initial datum f_{in} .*

Then, $f \in L^\infty([0, T]; L_k^1(\mathbb{R}^3))$ for all $T > 0$ (and the corresponding norm can be explicitly estimated in terms of T , the initial mass, momentum, energy, (upper bound of) entropy, and initial $L_k^1(\mathbb{R}^3)$ bounds).

This proposition is the consequence of a more general proposition (Proposition 2) proven in section 4, in which the assumptions on the function ψ are relaxed, and in which all dimensions are considered. Once again, the discussion on the assumptions of the proposition, and the link with other results on propagation of moments is postponed to section 4.

We only write here this small comment: this result shows that, as expected, the behavior for large $|v|$ of the solutions of the Landau equation in the Coulomb case is similar to what is observed for (moderately) soft potentials: polynomial moments are propagated but not created (contrary to the case of hard potentials).

1.3. Structure of the paper. We start with the proof of our main theorem (Theorem 1 and its more general version Theorem 3), together with various comments on this theorem and its proof, in section 2.

Corollary 1.1 is then proven and commented in section 3.

Section 4 is devoted to the proof of the statement on moments (Proposition 2).

Finally, we add some appendices for the sake of completeness: standard interpolations are presented in Appendix 1, while some remarks on the links between the results presented here and the regularity issues on (moderately) soft potentials discussed in [22], are presented in Appendix 2.

2. THE ENTROPY DISSIPATION CONTROLS THE (WEIGHTED) FISHER INFORMATION

2.1. Description of our main result in a general setting. We first write down a theorem which is slightly more general than Theorem 1, since it concerns all dimensions of space, and general functions ψ :

Theorem 3. *Let $f := f(v) \geq 0$, belonging to $L_2^1(\mathbb{R}^N)$, be such that $\int f |\ln f| dv \leq \bar{H}$, for some $\bar{H} > 0$. Let ψ satisfy*

$$\forall z \in \mathbb{R}^N, \quad \psi(z) \geq K_3 \inf(1, |z|^{\gamma_1+2}),$$

for some $K_3 > 0$ and $\gamma_1 \leq 0$.

Then, there exists a constant $C := C(N, \int f dv, \int f v dv, \int f |v|^2/2 dv, \bar{H}, \gamma_1, K_3) > 0$ which (explicitly) depends only on the dimension, the mass, momentum, energy, (an upper bound of the) entropy and the parameters of the lower bound on ψ : γ_1 , K_3 , such that

$$\int_{\mathbb{R}^N} |\nabla \sqrt{f(v)}|^2 (1 + |v|^2)^{\inf(\gamma_1/2, -1)} dv \leq C (1 + D_\psi(f)),$$

where $D_\psi(f)$ is defined in (12).

Remarks: The assumptions on ψ in this theorem are not optimal. Indeed, one can check that as long as ψ is bounded below by any function decaying less rapidly than a Maxwellian function of v , the same result can be obtained (with a slight modification of the proof given in this paper), up to a change of weight. Our feeling is that even a Maxwellian lower bound is not compulsory (though the proof should then be changed more deeply), if one allows a weight tending to 0 very quickly at infinity.

It is clear that D_ψ behaves in a monotone way w.r.t ψ , so that one could hope to get a better estimate when ψ has a singularity (at point 0). This cannot be seen on the estimate presented here, in which we use 1 as a bound from below for ψ at finite distance. Therefore, there might be room for improvement of the estimate when ψ is given by a power law (and especially in the Coulomb case).

The weight appearing in $\int_{\mathbb{R}^N} |\nabla \sqrt{f(v)}|^2 (1 + |v|^2)^{\inf(\gamma_1/2, -1)} dv$ behaves like $|v|^{\gamma_1}$ for large $|v|$ (as soon as $\gamma_1 < -2$), whereas ψ behaves like $|v|^{\gamma_1+2}$. This loss of weight is reminiscent of the loss of weight in coercivity estimates (cf. for example [16] and [22]), which illustrates the degenerate character of the elliptic operator appearing in the parabolic formulations of the equation. Our feeling is therefore that this weight is somehow optimal.

The proof relies on computations which link the entropy dissipation (of the Landau equation) to quantities like $\frac{\nabla f}{f}$. Those computations, which use integrations against test functions of the type $f(w)$ or $w f(w)$, were used in the study of the long time behavior of the Landau equation (cf. [17]), and are themselves reminiscent of old computations linking the entropy dissipation to the distance to the space of Maxwellian states (cf. [14]), where derivations instead of integrations were used.

An important difference between the proof of Theorems 1, 3 and those previous works is that because we want to treat ψ which decay like a negative power law at infinity, and use only the moments of order less than 2 of f (because only those are *a priori* propagated by the Landau equation), one needs to change the test functions used in those previous works, that is $f(w)$ and $w f(w)$, in functions which decay fast at infinity, like $f(w) e^{-\delta |w|^2}$ and $w f(w) e^{-\delta |w|^2}$, for some $\delta > 0$. This leads to a new difficulty related to the fact that quantities like $\int w f(w) e^{-\delta |w|^2} dw$ are not as easy to manipulate as $\int w f(w) dw$ (which appear in the assumption of the theorem).

In particular, in the course of the proof, one needs (for some $\lambda > 0$) a lower bound on the quantity

$$\left| \text{Det} \left(\int_{\mathbb{R}^N} e^{-\lambda w^2} \begin{bmatrix} 1 & w_j & w_i \\ w_i & w_i w_j & w_i^2 \\ w_j & w_j^2 & w_i w_j \end{bmatrix} f(w) dw \right) \right|,$$

provided that f has a given (non zero) mass and energy, a given momentum and an upper bound on its entropy.

The same result with $\lambda = 0$ was obtained in [17], thanks to the fact that by a change of functions and variables, one can impose that the momentum is 0 (that is, $\int f(w) w dw = 0$), and that the matrix of directional temperatures $(\int_{\mathbb{R}^N} w_i w_j f(w) dw)_{i,j=1,\dots,N}$ is diagonal.

The proof of [17] seems difficult to modify in order to take into account the extra term $e^{-\lambda w^2}$ in the integrals. We therefore provide a new proof, based on the direct computation of the determinant and the fact that one can look for a parameter $\lambda > 0$ as small as desired.

However, in the special case of radially symmetric f , one has

$$\int w f(w) e^{-\delta |w|^2} dw = 0,$$

and the proof of [17] can be used after suitable modifications. Theorem 1 is then much easier to prove. We provide therefore this simplified proof in next subsection, before presenting the complete proof of Theorems 1 and 3.

The rest of this section is thus dedicated to the proof of Theorems 1 and 3. A simplified proof of Theorem 1 (that is, the Coulomb case) is first presented (subsection 2.2). It holds only when radially symmetric distribution functions f are considered. We also restrict ourselves in this first proof to the case when f has a normalized mass, momentum and energy, in order to simplify it further, and to keep track of numerical constants.

The complete proof of Theorems 1 and 3 is then presented in subsection 2.3.

2.2. Proof in a simplified case. In this subsection, we first prove Theorem 1 when f is supposed to be radially symmetric and satisfies the normalization

$$(23) \quad \int_{\mathbb{R}^3} f(v) dv = 1, \quad \int_{\mathbb{R}^3} f(v) v dv = 0, \quad \int_{\mathbb{R}^3} f|v|^2 dv = 3.$$

We also assume that f has a sufficient (strictly positive) bound from below for the computations below to make sense.

Proof of Theorem 1 when f is supposed to be radially symmetric and satisfies the normalization (23):

We first observe that (for all $x, y \in \mathbb{R}^3$)

$$y^T (|x|^2 Id - x \otimes x) y = \frac{1}{2} \sum_{i,j=1,\dots,3} |x_i y_j - x_j y_i|^2.$$

Then, defining, for $i, j = 1, \dots, 3, i \neq j$,

$$q_{ij}^f(v, w) := (v_i - w_i) \left(\frac{\partial_j f(v)}{f(v)} - \frac{\partial_j f(w)}{f(w)} \right) - (v_j - w_j) \left(\frac{\partial_i f(v)}{f(v)} - \frac{\partial_i f(w)}{f(w)} \right),$$

we see that (remember (12))

$$\begin{aligned} D_{x \mapsto |x|^{-1}}(f) &= \frac{1}{2} \int \int_{\mathbb{R}^3 \times \mathbb{R}^3} f(v) f(w) |v - w|^{-1} \left(\frac{\nabla f(v)}{f(v)} - \frac{\nabla f(w)}{f(w)} \right)^T \\ &\quad \left(Id - \frac{(v - w) \otimes (v - w)}{|v - w|^2} \right) \left(\frac{\nabla f(v)}{f(v)} - \frac{\nabla f(w)}{f(w)} \right) dv dw \\ &= \frac{1}{4} \sum_{i,j=1,\dots,3; i \neq j} \int \int_{\mathbb{R}^3 \times \mathbb{R}^3} f(v) f(w) |v - w|^{-3} \left| q_{ij}^f(v, w) \right|^2 dv dw. \end{aligned}$$

Expanding q_{ij}^f , we get (for $i, j = 1, \dots, 3, i \neq j$)

$$q_{ij}^f(v, w) = \left[v_i \frac{\partial_j f(v)}{f(v)} - v_j \frac{\partial_i f(v)}{f(v)} \right] + w_j \frac{\partial_i f(v)}{f(v)} - w_i \frac{\partial_j f(v)}{f(v)}$$

$$-v_i \frac{\partial_j f(w)}{f(w)} + v_j \frac{\partial_i f(w)}{f(w)} + \left[w_i \frac{\partial_j f(w)}{f(w)} - w_j \frac{\partial_i f(w)}{f(w)} \right].$$

Choosing some $i, j = 1, \dots, 3$ with $i \neq j$, multiplying the above identity by $w_j (1 + |w|^2)^{-3/4} f(w)$, and integrating w.r.t. the variable w on \mathbb{R}^3 , we end up with (remember that f is assumed to be radially symmetric)

$$\begin{aligned} & \left(\int w_j^2 (1 + |w|^2)^{-3/4} f(w) dw \right) \frac{\partial_i f}{f}(v) \\ &= v_i \left(\int \left[(1 + |w|^2)^{-3/4} - \frac{3}{2} w_j^2 (1 + |w|^2)^{-7/4} \right] f(w) dw \right) \\ & \quad + \int w_j (1 + |w|^2)^{-3/4} q_{ij}^f(v, w) f(w) dw. \end{aligned}$$

As a consequence,

$$\begin{aligned} & \int \left| \frac{\partial_i f}{f}(v) \right|^2 f(v) (1 + |v|^2)^{-3/2} dv \\ & \leq 2 \left(\int v_i^2 (1 + |v|^2)^{-3/2} f(v) dv \right) \left(\int w_j^2 (1 + |w|^2)^{-3/4} f(w) dw \right)^{-2} \\ & \quad \times \left(\int \left[(1 + |w|^2)^{-3/4} - \frac{3}{2} w_j^2 (1 + |w|^2)^{-7/4} \right] f(w) dw \right)^2 \\ & \quad + 2 \left(\int w_j^2 (1 + |w|^2)^{-3/4} f(w) dw \right)^{-2} \\ & \quad \times \int f(v) (1 + |v|^2)^{-3/2} \left(\int |w_j| (1 + |w|^2)^{-3/4} |q_{ij}^f(v, w)| f(w) dw \right)^2 dv \\ & \leq \left(\int w_j^2 (1 + |w|^2)^{-3/4} f(w) dw \right)^{-2} \left(2 + 2 \int \int f(v) f(w) |v - w|^{-3} |q_{ij}^f(v, w)|^2 dv dw \right. \\ & \quad \times \sup_{v \in \mathbb{R}^3} \left[(1 + |v|^2)^{-3/2} \int f(w) |v - w|^3 |w_j|^2 (1 + |w|^2)^{-3/2} dw \right] \Big), \end{aligned}$$

since (thanks to the radial symmetry) $\int v_i^2 (1 + |v|^2)^{-3/2} f(v) dv \leq 1$, and

$$\begin{aligned} & \left| \int \left[(1 + |w|^2)^{-3/4} - \frac{3}{2} w_j^2 (1 + |w|^2)^{-7/4} \right] f(w) dw \right| \\ &= \left| \int \frac{1 + (\sum_{k \neq j} w_k^2) - \frac{1}{2} w_j^2}{(1 + |w|^2)^{7/4}} f(w) dw \right| \leq 1. \end{aligned}$$

Then, estimating

$$\begin{aligned} & \int f(w) |v - w|^3 |w_j|^2 (1 + |w|^2)^{-3/2} dw \\ & \leq 4 |v|^3 \int f(w) |w_j|^2 (1 + |w|^2)^{-3/2} dw + 4 \int f(w) |w|^3 |w_j|^2 (1 + |w|^2)^{-3/2} dw \\ & \leq \frac{4}{3} |v|^3 + \frac{16}{3}, \end{aligned}$$

we see that

$$\begin{aligned} & \int \left| \frac{\partial_i f}{f}(v) \right|^2 f(v) (1 + |v|^2)^{-3/2} dv \\ & \leq \left(\int w_j^2 (1 + |w|^2)^{-3/4} f(w) dw \right)^{-2} \left(2 + \frac{128}{3} D_{x \mapsto |x|^{-1}}(f) \right). \end{aligned}$$

We finally observe that (for all $0 < \delta < R$, and $K > 1$)

$$\begin{aligned}
\int w_j^2 (1 + |w|^2)^{-3/4} f(w) dw &= \frac{1}{3} \int |w|^2 (1 + |w|^2)^{-3/4} f(w) dw \\
&\geq \frac{1}{3} \int_{\delta \leq |w| \leq R} \delta^2 (1 + R^2)^{-3/4} f(w) dw \\
&\geq \frac{1}{3} \delta^2 (1 + R^2)^{-3/4} \left(1 - \int_{\delta \geq |w|} f(w) 1_{f(w) \leq K} dw \right. \\
&\quad \left. - \int_{\delta \geq |w|} f(w) \frac{|\ln f(w)|}{\ln K} dw - R^{-2} \int |w|^2 f(w) dw \right) \\
&\geq \frac{1}{3} \delta^2 (1 + R^2)^{-3/4} \left(1 - \frac{4}{3} \pi \delta^3 K - \frac{\bar{H}}{\ln K} - \frac{3}{R^2} \right).
\end{aligned}$$

Taking $R = \sqrt{12}$, $K = \exp(4\bar{H})$ and $\delta^3 = \frac{3}{16\pi} \exp(-4\bar{H})$, we end up with

$$\int w_j^2 (1 + |w|^2)^{-3/4} f(w) dw \geq \frac{(13)^{-3/4}}{12} \left(\frac{3}{16\pi} \right)^{2/3} \exp\left(-\frac{8}{3} \bar{H}\right),$$

which ends the proof of Theorem 1 in the case when f is radially symmetric and satisfies the normalization (23). \square

Note that an expression similar to $\exp(-\frac{8}{3} \bar{H})$ appears in the proof of Theorem 1 of [3], which can be seen as a local version of the same result, but for the Boltzmann equation.

2.3. Proof of our main result in the general case. We begin by establishing two lemmas, which can be seen as a quantitative version of the statement that it is impossible for f to be concentrated on an hyperplane of \mathbb{R}^N , if it has a bounded entropy.

Lemma 1. *Let $f := f(v) \geq 0$ belong to $L_2^1(\mathbb{R}^N)$ and be such that $\int_{\mathbb{R}^N} f(v) dv = 1$, $\int_{\mathbb{R}^N} f(v) v dv = 0$, and $\int_{\mathbb{R}^N} f(v) |v|^2 dv = N$.*

Then, for all $i, j \in \mathbb{N}$ such that $i \neq j$, all $S > 0$, and all $\lambda > 0$,

$$\begin{aligned}
&- \text{Det} \left(\int_{\mathbb{R}^N} e^{-\lambda y^2} \begin{bmatrix} 1 & y_j & y_i \\ y_i & y_i y_j & y_i^2 \\ y_j & y_j^2 & y_i y_j \end{bmatrix} f(y) dy \right) \\
&\geq \frac{1}{2} e^{-2N\lambda} \left(\int_{\mathbb{R}^N} e^{-\lambda y^2} f(y) y_i^2 dy \int_{\mathbb{R}^N} e^{-\lambda y^2} f(y) y_j^2 dy - \left[\int_{\mathbb{R}^N} e^{-\lambda y^2} f(y) y_i y_j dy \right]^2 \right) \\
&\quad - 4N (S (1 - e^{-\lambda S^2}) + 2N S^{-1})^2.
\end{aligned}$$

Proof of Lemma 1 : Expanding the determinant w.r.t. the first line, we get (for any $i, j \in \mathbb{N}$ such that $i \neq j$ and all $\lambda > 0$),

$$\begin{aligned}
&- \text{Det} \left(\int_{\mathbb{R}^N} e^{-\lambda y^2} \begin{bmatrix} 1 & y_j & y_i \\ y_i & y_i y_j & y_i^2 \\ y_j & y_j^2 & y_i y_j \end{bmatrix} f(y) dy \right) \\
&= \left(\int_{\mathbb{R}^N} e^{-\lambda y^2} f(y) dy \right) \left(\int_{\mathbb{R}^N} e^{-\lambda y^2} f(y) y_i^2 dy \int_{\mathbb{R}^N} e^{-\lambda y^2} f(y) y_j^2 dy \right. \\
&\quad \left. - \left[\int_{\mathbb{R}^N} e^{-\lambda y^2} f(y) y_i y_j dy \right]^2 \right)
\end{aligned}$$

$$\begin{aligned}
& + \left(\int_{\mathbb{R}^N} e^{-\lambda y^2} f(y) y_j dy \right) \left(\int_{\mathbb{R}^N} e^{-\lambda y^2} f(y) y_i dy \int_{\mathbb{R}^N} e^{-\lambda y^2} f(y) y_i y_j dy \right. \\
& \quad \left. - \int_{\mathbb{R}^N} e^{-\lambda y^2} f(y) y_j dy \int_{\mathbb{R}^N} e^{-\lambda y^2} f(y) y_i^2 dy \right) \\
& - \left(\int_{\mathbb{R}^N} e^{-\lambda y^2} f(y) y_i dy \right) \left(\int_{\mathbb{R}^N} e^{-\lambda y^2} f(y) y_i dy \int_{\mathbb{R}^N} e^{-\lambda y^2} f(y) y_j^2 dy \right. \\
& \quad \left. - \int_{\mathbb{R}^N} e^{-\lambda y^2} f(y) y_j dy \int_{\mathbb{R}^N} e^{-\lambda y^2} f(y) y_i y_j dy \right) \\
& \geq \left(\int_{\mathbb{R}^N} e^{-\lambda y^2} f(y) dy \right) \left(\int_{\mathbb{R}^N} e^{-\lambda y^2} f(y) y_i^2 dy \int_{\mathbb{R}^N} e^{-\lambda y^2} f(y) y_j^2 dy \right. \\
& \quad \left. - \left[\int_{\mathbb{R}^N} e^{-\lambda y^2} f(y) y_i y_j dy \right]^2 \right) \\
& \quad - N \left(\left| \int_{\mathbb{R}^N} e^{-\lambda y^2} f(y) y_i dy \right| + \left| \int_{\mathbb{R}^N} e^{-\lambda y^2} f(y) y_j dy \right| \right)^2.
\end{aligned}$$

Then, we observe (since $\int_{B(0, \sqrt{2N})^c} f(y) dy \leq (2N)^{-1} \int_{B(0, \sqrt{2N})^c} y^2 f(y) dy \leq 1/2$) that

$$\int_{\mathbb{R}^N} e^{-\lambda y^2} f(y) dy \geq e^{-2N\lambda} \int_{B(0, \sqrt{2N})} f(y) dy \geq \frac{1}{2} e^{-2N\lambda},$$

and (for any $S > 0$)

$$\begin{aligned}
& \left| \int_{\mathbb{R}^N} e^{-\lambda y^2} f(y) y_i dy \right| = \left| \int_{\mathbb{R}^N} (e^{-\lambda y^2} - 1) f(y) y_i dy \right| \\
& \leq \int_{B(0, S)} S(1 - e^{-\lambda S^2}) f(y) dy + 2 \int_{B(0, S)^c} |y_i| f(y) dy \\
& \leq S(1 - e^{-\lambda S^2}) + 2N S^{-1}.
\end{aligned}$$

This ends the proof of Lemma 1. \square

Lemma 2. *Let $f := f(v) \geq 0$ belonging to $L^1_2(\mathbb{R}^N) \cap L \ln L(\mathbb{R}^N)$ be such that $\int_{\mathbb{R}^N} f(v) dv = 1$, $\int_{\mathbb{R}^N} f(v) v dv = 0$, $\int_{\mathbb{R}^N} f(v) |v|^2 dv = N$, and $\int_{\mathbb{R}^N} f(v) |\ln f(v)| dv \leq \bar{H}$, where $\bar{H} > 0$.*

Then there exists a constant $\lambda_0 := \lambda_0(N, \bar{H}) > 0$ depending only on N, \bar{H} such that for all $i, j \in \mathbb{N}$, $i \neq j$ and $\lambda \in]0, \lambda_0]$,

$$\left| \text{Det} \left(\int_{\mathbb{R}^N} e^{-\lambda y^2} \begin{bmatrix} 1 & y_j & y_i \\ y_i & y_i y_j & y_i^2 \\ y_j & y_j^2 & y_i y_j \end{bmatrix} f(y) dy \right) \right| \geq C(N, \bar{H}),$$

where $C := C(N, \bar{H}) > 0$ is also a constant which only depends on N and \bar{H} .

Proof of Lemma 2 : We observe that (for all $i, j \in \mathbb{N}$, $i \neq j$, and $\lambda > 0$)

$$\begin{aligned}
& \left(\int e^{-\lambda y^2} y_i^2 f(y) dy \right) \left(\int e^{-\lambda y^2} y_j^2 f(y) dy \right) - \left(\int e^{-\lambda y^2} y_i y_j f(y) dy \right)^2 \\
& = \left(\int e^{-\lambda y^2} y_i^2 f(y) dy \right) \left(\int e^{-\lambda y^2} \left| y_j - y_i \frac{\int e^{-\lambda z^2} z_i z_j f(z) dz}{\int e^{-\lambda z^2} z_i^2 f(z) dz} \right|^2 f(y) dy \right) \\
& = \left(\int e^{-\lambda y^2} y_j^2 f(y) dy \right) \left(\int e^{-\lambda y^2} \left| y_i - y_j \frac{\int e^{-\lambda z^2} z_i z_j f(z) dz}{\int e^{-\lambda z^2} z_j^2 f(z) dz} \right|^2 f(y) dy \right).
\end{aligned}$$

Then, thanks to Cauchy-Schwarz (or Young) inequality,

$$\int e^{-\lambda z^2} z_i z_j f(z) dz \leq \sup_{k=i,j} \int e^{-\lambda z^2} z_k^2 f(z) dz,$$

so that

$$\begin{aligned} & \left(\int e^{-\lambda y^2} y_i^2 f(y) dy \right) \left(\int e^{-\lambda y^2} y_j^2 f(y) dy \right) - \left(\int e^{-\lambda y^2} y_i y_j f(y) dy \right)^2 \\ & \geq \inf_{|\theta| \leq 1} \inf_{k=i,j; l=i,j; k \neq l} \left(\int e^{-\lambda y^2} |y_k - \theta y_l|^2 f(y) dy \right)^2. \end{aligned}$$

Using this estimate and Lemma 1, we get for any $S > 0$,

$$\begin{aligned} \Gamma_{\lambda,i,j}(f) &:= -\text{Det} \left(\int_{\mathbb{R}^N} e^{-\lambda y^2} \begin{bmatrix} 1 & y_j & y_i \\ y_i & y_i y_j & y_i^2 \\ y_j & y_j^2 & y_i y_j \end{bmatrix} f(y) dy \right) \\ &\geq \frac{1}{2} e^{-2N\lambda} \left(\inf_{|\theta| \leq 1} \inf_{k=i,j; l=i,j; k \neq l} \int_{\mathbb{R}^N} e^{-\lambda y^2} f(y) |y_k - \theta y_l|^2 dy \right)^2 \\ &\quad - 4N (S(1 - e^{-\lambda S^2}) + 2N S^{-1})^2. \end{aligned}$$

Then, for any $\delta > 0$ (and for any $|\theta| \leq 1$, $k = i, j$; $l = i, j$; $k \neq l$),

$$\begin{aligned} \int_{\mathbb{R}^N} e^{-\lambda y^2} f(y) |y_k - \theta y_l|^2 dy &\geq \delta^2 \int_{|y_k - \theta y_l| \geq \delta} e^{-\lambda y^2} f(y) dy \\ &\geq \delta^2 e^{-2N\lambda} \int_{|y_k - \theta y_l| \geq \delta, |y| \leq \sqrt{2N}} f(y) dy \\ &= \delta^2 e^{-2N\lambda} \left(1 - \int_{|y_k - \theta y_l| \leq \delta, |y| \leq \sqrt{2N}} f(y) dy - \int_{|y| \geq \sqrt{2N}} f(y) dy \right) \\ &\geq \delta^2 e^{-2N\lambda} \left(\frac{1}{2} - \int_{|y_k - \theta y_l| \leq \delta, |y| \leq \sqrt{2N}} f(y) dy \right). \end{aligned}$$

As a consequence, for any $K > 1$, $S > 0$, $\delta > 0$,

$$\begin{aligned} \Gamma_{\lambda,i,j}(f) &\geq \frac{1}{2} e^{-6N\lambda} \delta^4 \left(\frac{1}{2} - \sup_{|\theta| \leq 1} \sup_{k=i,j; l=i,j; k \neq l} \int_{\left| \frac{y_k - \theta y_l}{\sqrt{1+\theta^2}} \right| \leq \frac{\delta}{\sqrt{1+\theta^2}}, |y| \leq \sqrt{2N}} f(y) dy \right)^2 \\ &\quad - 4N (S(1 - e^{-\lambda S^2}) + 2N S^{-1})^2 \\ &\geq \frac{1}{2} e^{-6N\lambda} \delta^4 \left(\frac{1}{2} - 2K (2\sqrt{2N})^{N-1} \delta - \frac{\bar{H}}{\ln K} \right)^2 - 4N (S(1 - e^{-\lambda S^2}) + 2N S^{-1})^2. \end{aligned}$$

Taking $K = e^{4\bar{H}}$, $\delta = 16^{-1} (8N)^{-\frac{N-1}{2}} e^{-4\bar{H}}$, we see that (for any $S > 0$)

$$\Gamma_{\lambda,i,j}(f) \geq \frac{1}{128} (16 (8N)^{\frac{N-1}{2}})^{-4} e^{-16\bar{H}} e^{-6N\lambda} - 4N (2S^2 (1 - e^{-\lambda S^2})^2 + 8N^2 S^{-2}).$$

Finally, taking $S > 0$ in such a way that

$$32N^3 S^{-2} = 2^{-24-6(N-1)} N^{-2(N-1)} e^{-16\bar{H}},$$

we get

$$\Gamma_{\lambda,i,j}(f) \geq 2^{-17-6N} N^{-2(N-1)} e^{-16\bar{H}} \left(e^{-6N\lambda} - \frac{1}{2} \right)$$

$$-2^{26+6N} N^{2(N+1)} e^{16\bar{H}} (1 - e^{-2^{23+6N} N^{2N+1} e^{16\bar{H}} \lambda})^2.$$

The limit of this formula when $\lambda \rightarrow 0$ is $2^{-18-6N} N^{-2(N-1)} e^{-16\bar{H}}$. This ends the proof of Lemma 2.

Note that when $N = 3$, Lemma 2 can be made numerically explicit: $\Gamma_{\lambda,i,j}(f) \geq 2^{-38} 3^{-4} \exp(-16\bar{H})$, provided that $0 < \lambda \leq \lambda_0 = 2^{-82} 3^{-13} \exp(-24\bar{H})$. \square

We now turn to the

Proof of Theorem 3: We shall assume in this proof, without loss of generality, that

$$(24) \quad \int_{\mathbb{R}^N} f(v) dv = 1, \quad \int_{\mathbb{R}^N} f(v) v dv = 0, \quad \int_{\mathbb{R}^N} f(v) |v|^2 dv = N.$$

It indeed amounts to a simple change of unknown and variables of the form

$$(25) \quad f(v) \longrightarrow a f(bv + c), \quad a, b \in \mathbb{R}, c \in \mathbb{R}^N.$$

In the computations below, we do not take care of the points where $f = 0$. We explain how to justify the computations where $\frac{1}{f}$ appears at the end of the proof of Theorem 3.

We first observe that (for all $x, y \in \mathbb{R}^N$)

$$y^T (|x|^2 Id - x \otimes x) y = \frac{1}{2} \sum_{i,j=1,\dots,N} |x_i y_j - x_j y_i|^2.$$

Then, defining, for $i, j = 1, \dots, N$, $i \neq j$,

$$q_{ij}^f(v, w) := (v_i - w_i) \left(\frac{\partial_j f(v)}{f(v)} - \frac{\partial_j f(w)}{f(w)} \right) - (v_j - w_j) \left(\frac{\partial_i f(v)}{f(v)} - \frac{\partial_i f(w)}{f(w)} \right),$$

we see that (remember (12))

$$\begin{aligned} D_\psi(f) &= \frac{1}{2} \int \int_{\mathbb{R}^N \times \mathbb{R}^N} f(v) f(w) \psi(|v - w|) \left(\frac{\nabla f(v)}{f(v)} - \frac{\nabla f(w)}{f(w)} \right)^T \\ &\quad \left(Id - \frac{(v - w) \otimes (v - w)}{|v - w|^2} \right) \left(\frac{\nabla f(v)}{f(v)} - \frac{\nabla f(w)}{f(w)} \right) dv dw \\ &= \frac{1}{4} \sum_{i,j=1,\dots,N} \int \int_{\mathbb{R}^N \times \mathbb{R}^N} f(v) f(w) \frac{\psi(|v - w|)}{|v - w|^2} |q_{ij}^f(v, w)|^2 dv dw. \end{aligned}$$

Expanding q_{ij}^f , we get

$$\begin{aligned} q_{ij}^f(v, w) &= \left[v_i \frac{\partial_j f(v)}{f(v)} - v_j \frac{\partial_i f(v)}{f(v)} \right] + w_j \frac{\partial_i f(v)}{f(v)} - w_i \frac{\partial_j f(v)}{f(v)} \\ &\quad - v_i \frac{\partial_j f(w)}{f(w)} + v_j \frac{\partial_i f(w)}{f(w)} + \left[w_i \frac{\partial_j f(w)}{f(w)} - w_j \frac{\partial_i f(w)}{f(w)} \right]. \end{aligned}$$

We now select some $i, j = 1, \dots, N$, $i \neq j$, and integrate the identity above against $e^{-\lambda w^2} f(w) dw$, for some $\lambda > 0$. We get

$$\begin{aligned} \int_{\mathbb{R}^N} q_{ij}^f(v, w) e^{-\lambda w^2} f(w) dw &= \left[v_i \frac{\partial_j f(v)}{f(v)} - v_j \frac{\partial_i f(v)}{f(v)} \right] \left(\int_{\mathbb{R}^N} e^{-\lambda w^2} f(w) dw \right) \\ &\quad + \left(\int_{\mathbb{R}^N} w_j e^{-\lambda w^2} f(w) dw \right) \frac{\partial_i f(v)}{f(v)} - \left(\int_{\mathbb{R}^N} w_i e^{-\lambda w^2} f(w) dw \right) \frac{\partial_j f(v)}{f(v)} \end{aligned}$$

$$-2\lambda v_i \left(\int_{\mathbb{R}^N} w_j e^{-\lambda w^2} f(w) dw \right) + 2\lambda v_j \left(\int_{\mathbb{R}^N} w_i e^{-\lambda w^2} f(w) dw \right).$$

We then integrate it against $w_k e^{-\lambda w^2} f(w) dw$, for some $k = 1, \dots, N$ (and some $\lambda > 0$) and get

$$\begin{aligned} \int_{\mathbb{R}^N} q_{ij}^f(v, w) w_k e^{-\lambda w^2} f(w) dw &= \left[v_i \frac{\partial_j f(v)}{f(v)} - v_j \frac{\partial_i f(v)}{f(v)} \right] \left(\int_{\mathbb{R}^N} w_k e^{-\lambda w^2} f(w) dw \right) \\ &+ \left(\int_{\mathbb{R}^N} w_j w_k e^{-\lambda w^2} f(w) dw \right) \frac{\partial_i f(v)}{f(v)} - \left(\int_{\mathbb{R}^N} w_i w_k e^{-\lambda w^2} f(w) dw \right) \frac{\partial_j f(v)}{f(v)} \\ &+ v_i \left(\int_{\mathbb{R}^N} (\delta_{jk} - 2\lambda w_j w_k) e^{-\lambda w^2} f(w) dw \right) - v_j \left(\int_{\mathbb{R}^N} (\delta_{ik} - 2\lambda w_i w_k) e^{-\lambda w^2} f(w) dw \right) \\ &+ \int_{\mathbb{R}^N} (\delta_{ik} w_j - \delta_{jk} w_i) e^{-\lambda w^2} f(w) dw. \end{aligned}$$

Using $k = i, j$, and considering the above identities as a 3×3 system for the unknowns $v_i \frac{\partial_j f(v)}{f(v)} - v_j \frac{\partial_i f(v)}{f(v)}$, $\frac{\partial_i f(v)}{f(v)}$ and $\frac{\partial_j f(v)}{f(v)}$, Cramer formulas enable the computation of $\frac{\partial_i f(v)}{f(v)}$ in terms of q_{ij}^f :

$$\frac{\partial_i f(v)}{f(v)} = \frac{\text{Det} \left(\int_{\mathbb{R}^N} e^{-\lambda w^2} f(w) \begin{bmatrix} 1 & Z_1(f)(v, w) & w_i \\ w_i & Z_2(f)(v, w) & w_i^2 \\ w_j & Z_3(f)(v, w) & w_i w_j \end{bmatrix} dw \right)}{\text{Det} \left(\int_{\mathbb{R}^N} e^{-\lambda w^2} f(w) \begin{bmatrix} 1 & w_j & w_i \\ w_i & w_j w_i & w_i^2 \\ w_j & w_j^2 & w_i w_j \end{bmatrix} dw \right)},$$

where

$$\begin{aligned} Z_1(f)(v, w) &= q_{ij}^f(v, w) + 2\lambda v_i w_j - 2\lambda v_j w_i, \\ Z_2(f)(v, w) &= q_{ij}^f(v, w) w_i - v_i (-2\lambda w_j w_i) + v_j (1 - 2\lambda w_i^2) - w_j, \\ Z_3(f)(v, w) &= q_{ij}^f(v, w) w_j - v_i (1 - 2\lambda w_j^2) + v_j (-2\lambda w_i w_j) + w_i. \end{aligned}$$

From now on, we denote by $C_1(\cdot), C_2(\cdot), \dots$ various constants depending only on the parameters indicated in the parenthesis.

Using Lemma 2 (and the notation $\Gamma_{\lambda, i, j}$ introduced in its proof), we see that, taking $\lambda = \lambda_0(N, \bar{H})$ (and denoting it by λ_0),

$$\begin{aligned} \left| \frac{\partial_i f(v)}{f(v)} \right| &\leq (\Gamma_{\lambda_0}(f))^{-1} 2N^{3/2} \int_{\mathbb{R}^N} e^{-\lambda_0 w^2} f(w) \\ &\times \left(|Z_1(f)(v, w)| + |Z_2(f)(v, w)| + |Z_3(f)(v, w)| \right) dw \\ &\leq C(N, \bar{H})^{-1} C_1(N, \bar{H}) \left([1 + |v|] + \int_{\mathbb{R}^N} q_{ij}^f(v, w) (1 + |w|^2)^{1/2} e^{-\lambda_0 w^2} f(w) dw \right). \end{aligned}$$

Then

$$\begin{aligned} &\int_{\mathbb{R}^N} f(v) \left| \frac{\partial_i f(v)}{f(v)} \right|^2 (1 + |v|^2)^{\inf(\gamma_1/2, -1)} dv \\ &\leq C_2(N, \bar{H}) \left(\int_{\mathbb{R}^N} f(v) (1 + |v|^2)^{\inf(1+\gamma_1/2, 0)} dv \right. \\ &\left. + \int_{\mathbb{R}^N} f(v) (1 + |v|^2)^{\inf(\gamma_1/2, -1)} \left| \int_{\mathbb{R}^N} q_{ij}^f(v, w) (1 + |w|^2)^{1/2} e^{-\lambda_0 w^2} f(w) dw \right|^2 dv \right) \end{aligned}$$

$$\begin{aligned}
&\leq C_2(N, \bar{H}) \left(1 + \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} f(v) f(w) \frac{\psi(|v-w|)}{|v-w|^2} \left| q_{ij}^f(v, w) \right|^2 dv dw \right. \\
&\times \sup_{v \in \mathbb{R}^N} \left[(1 + |v|^2)^{\inf(\gamma_1/2, -1)} \int_{\mathbb{R}^N} \frac{|v-w|^2}{\psi(|v-w|)} (1 + |w|^2) e^{-2\lambda_0 w^2} f(w) dw \right] \Big) \\
&\leq C_3(N, \bar{H}, K_3) \left(1 + D_\psi(f) \sup_{v \in \mathbb{R}^N} \left[(1 + |v|^2)^{\inf(\gamma_1/2, -1)} \right. \right. \\
&\times \left. \int_{\mathbb{R}^N} \sup \left(|v-w|^{-\gamma_1}, |v-w|^2 \right) (1 + |w|^2) e^{-2\lambda_0 w^2} f(w) dw \right] \Big) \\
&\leq C_4(N, \bar{H}, K_3, \gamma_1) (1 + D_\psi(f)).
\end{aligned}$$

This concludes the proof of Theorem 3, up to the treatment of the points where $f = 0$.

We briefly explain here how to treat this problem. First, one needs to start from the definition of $D_\psi(f)$ appearing in the last line of (12), which requires no extra assumption on f (other than $f \in L_{loc}^1$). Then, one uses $g = \sqrt{f}$, and the integration against $e^{-\lambda w^2} f(w) dw$, $e^{-\lambda w^2} w_i f(w) dw$ is replaced by an integration against $e^{-\lambda w^2} g(w) dw$, $e^{-\lambda w^2} w_i g(w) dw$. Though such a presentation is more rigorous, we did not adopt it in our proof in order to keep working with f (and not $g = \sqrt{f}$) and the definition of $D_\psi(f)$ appearing in the second line of (12) (and not the one appearing in the last line), since those quantities are much more familiar to specialists of kinetic theory. \square

We now turn to the

Proof of Theorem 1: The first part is a direct consequence of Theorem 3, when $N = 3$, $\gamma_1 = -3$, $K_3 = 1$. The second part (case of radially symmetric f) was already proven in subsection 2.2. \square

3. H -SOLUTIONS OF THE LANDAU EQUATION ARE WEAK SOLUTIONS

Though a large part of the material of this section is either classical or analogous to well-known results on the Boltzmann equation (cf. the remarks at the end of the section), we provide detailed proofs for the sake of completeness.

3.1. Weighted L^p estimates. We begin this section with a direct application of Sobolev inequalities in the whole space. We write a general lemma (the dimension N as well as the exponent of the weight are arbitrary) linking the weighted H^1 norm of \sqrt{f} to a weighted $L^{\frac{N}{N-2}}$ norm of f .

Lemma 3. *Let $N \in \mathbb{N}$, $N \geq 3$, and $\gamma_1 \in \mathbb{R}$. Then, there exists a constant depending only on N and γ_1 (denoted by $C := C(N, \gamma_1) > 0$) such that for all $f := f(v) \geq 0$ lying in $L_2^1(\mathbb{R}^N)$*

$$\begin{aligned}
&\left(\int_{\mathbb{R}^N} |f(v)|^{\frac{N}{N-2}} (1 + |v|^2)^{\frac{N}{N-2} \inf(\frac{\gamma_1}{2}, -1)} dv \right)^{\frac{N-2}{N}} \\
&\leq C \left[\int_{\mathbb{R}^N} f(v) (1 + |v|^2) dv + \int_{\mathbb{R}^N} |\nabla \sqrt{f(v)}|^2 (1 + |v|^2)^{\inf(\frac{\gamma_1}{2}, -1)} dv \right].
\end{aligned}$$

In the inequality above, the right-hand side is considered as equal to $+\infty$ if \sqrt{f} is not in $H_{loc}^1(\mathbb{R}^N)$.

In particular, for $N = 3$ and $\gamma_1 = -3$ (that is, the Coulomb case),

$$\begin{aligned} & \left(\int_{\mathbb{R}^3} |f(v)|^3 (1 + |v|^2)^{-\frac{3}{2}} dv \right)^{\frac{1}{3}} \\ & \leq \frac{6}{\sqrt{\pi}} \int_{\mathbb{R}^3} f(v) dv + \frac{8}{3\sqrt{\pi}} \int_{\mathbb{R}^3} |\nabla \sqrt{f(v)}|^2 (1 + |v|^2)^{\frac{3}{2}} dv. \end{aligned}$$

If $N = 2$, and $\gamma_1 \in \mathbb{R}$, then for any $q \geq 1$, there exists a constant depending only on q and γ_1 (denoted by $C := C(q, \gamma_1) > 0$) such that for all $f \geq 0$ lying in $L^1_2(\mathbb{R}^2)$,

$$\begin{aligned} & \left(\int_{\mathbb{R}^2} |f(v)|^q (1 + |v|^2)^{q \inf(\frac{\gamma_1}{2}, -1)} dv \right)^{\frac{1}{q}} \\ & \leq C \left[\int_{\mathbb{R}^2} f(v) dv + \int_{\mathbb{R}^2} |\nabla \sqrt{f(v)}|^2 (1 + |v|^2)^{\inf(\frac{\gamma_1}{2}, -1)} dv \right]. \end{aligned}$$

Proof of Lemma 3 : We denote by $C(N)$ (resp. $C(N, \gamma_1)$) any constant depending only on N (resp. N, γ_1). We recall the standard Sobolev inequality for functions of $H^1(\mathbb{R}^N)$ (for $N \geq 3$):

$$\int_{\mathbb{R}^N} |h|^{\frac{2N}{N-2}} \leq C(N) \left(\int_{\mathbb{R}^N} |\nabla h|^2 \right)^{\frac{N}{N-2}}.$$

Applying it to $h(v) = g(v) (1 + |v|^2)^{\frac{1}{2} \inf(\gamma_1/2, -1)}$, we see that

$$\begin{aligned} & \int_{\mathbb{R}^N} |g(v)|^{\frac{2N}{N-2}} (1 + |v|^2)^{\frac{N}{N-2} \inf(\frac{\gamma_1}{2}, -1)} dv \leq C(N) \left(\int_{\mathbb{R}^N} |\nabla g(v)|^2 (1 + |v|^2)^{\inf(\frac{\gamma_1}{2}, -1)} dv \right)^{\frac{N}{N-2}} \\ & + C(N, \gamma_1) \left(\int_{\mathbb{R}^N} |g(v)|^2 dv \right)^{\frac{N}{N-2}}. \end{aligned}$$

Then, taking $f = g^2$, we end up with

$$\begin{aligned} & \int_{\mathbb{R}^N} |f(v)|^{\frac{N}{N-2}} (1 + |v|^2)^{\frac{N}{N-2} \inf(\frac{\gamma_1}{2}, -1)} dv \leq C(N) \left(\int_{\mathbb{R}^N} |\nabla \sqrt{f(v)}|^2 (1 + |v|^2)^{\inf(\frac{\gamma_1}{2}, -1)} dv \right)^{\frac{N}{N-2}} \\ & + C(N, \gamma_1) \left(\int_{\mathbb{R}^N} f(v) dv \right)^{\frac{N}{N-2}}. \end{aligned}$$

The Coulomb case is treated by following the constants in the above computations, and by using the best constant in the Sobolev embedding.

In the case when $N = 2$, we proceed as in the case when $N \geq 3$, but starting from the Sobolev inequality (which holds for all $q \in [1, +\infty[$, and with a constant $C(q)$ which depends only on q):

$$\int_{\mathbb{R}^2} |h|^{2q} \leq C(q) \left(\int_{\mathbb{R}^2} |\nabla h|^2 \right)^q.$$

□

3.2. Weak solutions. We also start with a lemma which holds for all dimensions, and any function ψ bounded above in the vicinity of 0 by a power corresponding to very soft potentials. It is a simple consequence of Young's inequality for convolutions.

Lemma 4. *Let $N \in \mathbb{N}$, $N \geq 1$. We assume that ψ is a function satisfying*

$$\forall z \geq 0, \quad 0 \leq \psi(|z|) \leq K_1 |z|^2 + K_2 |z|^{\gamma_2+2},$$

for some $K_1, K_2 > 0$ and $\gamma_2 \in]-4, -2[$.

Then for any $R > 0$, and $r < N/(-\gamma_2 - 2)$, there exists a constant $C := C(r, N, K_1, K_2, \gamma_2) > 0$ depending only on the parameters of ψ , N and r , such that for any $f \in L_2^1(\mathbb{R}^N) \cap L_{loc}^{r'}(\mathbb{R}^N)$ (with $\frac{1}{r} + \frac{1}{r'} = 1$),

$$\begin{aligned} & \int_{\mathbb{R}^N} \int_{B(0,R)} f(v) f(w) \psi(|v-w|) dv dw \\ & \leq C \|f\|_{L^1(\mathbb{R}^N)} \left(\|f\|_{L_2^1(\mathbb{R}^N)} + \|f 1_{|\cdot| \leq R}\|_{L^{r'}(\mathbb{R}^N)} \right). \end{aligned}$$

In particular, if $N \geq 3$, there exists a constant $C := C(N, K_1, K_2, \gamma_2) > 0$ depending only on the parameters of ψ and N , such that for any $f \in L_2^1(\mathbb{R}^N) \cap L_{loc}^{\frac{N}{N-2}}(\mathbb{R}^N)$,

$$\begin{aligned} & \int_{\mathbb{R}^N} \int_{B(0,R)} f(v) f(w) \psi(|v-w|) dv dw \\ & \leq C \|f\|_{L^1(\mathbb{R}^N)} \left(\|f\|_{L_2^1(\mathbb{R}^N)} + \|f 1_{|\cdot| \leq R}\|_{L^{\frac{N}{N-2}}(\mathbb{R}^N)} \right). \end{aligned}$$

Proof of Lemma 4 :

$$\begin{aligned} & \int_{\mathbb{R}^N} \int_{B(0,R)} f(v) f(w) \psi(|v-w|) dv dw \\ & \leq \int_{\mathbb{R}^N} \int_{B(0,R)} f(v) f(w) (K_1 + K_2 |v-w|^{\gamma_2+2}) 1_{|v-w| \leq 1} dv dw \\ & \quad + \int_{\mathbb{R}^N} \int_{B(0,R)} f(v) f(w) (K_1 |v-w|^2 + K_2) 1_{|v-w| \geq 1} dv dw \\ & \leq K_1 \|f\|_{L^1(\mathbb{R}^N)}^2 + K_2 \int_{\mathbb{R}^N} \int_{B(0,R)} f(v) f(w) |v-w|^{\gamma_2+2} 1_{|v-w| \leq 1} dv dw \\ & \quad + 4 K_1 \|f\|_{L^1(\mathbb{R}^N)} \|f\|_{L_2^1(\mathbb{R}^N)} + K_2 \|f\|_{L^1(\mathbb{R}^N)}^2 \\ & \leq \left(4 K_1 \|f\|_{L_2^1(\mathbb{R}^N)} + (K_1 + K_2) \|f\|_{L^1(\mathbb{R}^N)} \right) \|f\|_{L^1(\mathbb{R}^N)} \\ & \quad + K_2 \|f 1_{|\cdot| \leq R}\|_{L^{r'}(\mathbb{R}^N)} \|f\|_{L^1(\mathbb{R}^N)} \|x \mapsto |x|^{\gamma_2+2} 1_{|\cdot| \leq 1}\|_{L^r(\mathbb{R}^N)}, \end{aligned}$$

thanks to Young's inequality for convolutions.

Note that the last term is finite when $r < N/(-\gamma_2 - 2)$, which is equivalent to $r' > N/(N + \gamma_2 + 2)$. Observing that (when $N \geq 3$) $N/(N + \gamma_2 + 2) < N/(N - 2)$ since $\gamma_2 > -4$, we see that one can take $r' = N/(N - 2)$, so that the particular case (at the end of the statement of Lemma 4) also holds. \square

We are now in a position to write down the

Proof of Corollary 1.1: We know (cf. [29]) that H -solutions $f := f(t, v)$ of the Landau equation (in the Coulomb case) satisfy the *a priori* estimate (17). As

a consequence, they lie in $L^\infty([0, T]; L^1_2(\mathbb{R}^3))$ and satisfy the boundedness of the entropy dissipation, that is

$$\int_0^T D_{x \mapsto |x|^{-1}}(f(t, \cdot)) dt < +\infty.$$

Using the main result of this paper (that is, Theorem 1), we see that

$$\int_0^T \int_{\mathbb{R}^3} |\nabla \sqrt{f(t, v)}|^2 (1 + |v|^2)^{-3/2} dv dt < +\infty.$$

Then, using Lemma 3, we end up with

$$\int_0^T \left(\int_{\mathbb{R}^3} |f(t, v)|^3 (1 + |v|^2)^{-9/2} dv \right)^{1/3} dt < +\infty,$$

so that $f \in L^1([0, T]; L^3_{-3}(\mathbb{R}^3))$, which is the first statement of Corollary 1.1

Then, we consider $\varphi \in C_c^2([0, T] \times \mathbb{R}^3)$, and $R > 0$ such that

$$\cup_{t \in [0, T[} \text{Supp } f(t, \cdot) \subset B(0, R).$$

Then (since v and w play the same role), for any $i, j = 1, \dots, 3$,

$$\begin{aligned} & \int_0^T \iint_{\mathbb{R}^3 \times \mathbb{R}^3} f(t, v) f(t, w) a_{ij}(v - w) \left| \partial_{ij} \varphi(t, v) + \partial_{ij} \varphi(t, w) \right| dv dw dt \\ & \leq 2 \|\partial_{ij} \varphi\|_{L^\infty([0, T] \times \mathbb{R}^3)} \int_0^T \iint_{\mathbb{R}^3 \times B(0, R)} f(t, v) f(t, w) |v - w|^{-1} dv dw dt \\ & \leq C(\varphi) \|f\|_{L^\infty([0, T]; L^1(\mathbb{R}^3))} \left(T \|f\|_{L^\infty([0, T]; L^1_2(\mathbb{R}^3))} + \|(t, v) \mapsto f(t, v) 1_{|v| \leq R}\|_{L^1([0, T]; L^3(\mathbb{R}^3))} \right), \end{aligned}$$

thanks to Lemma 4 (particular case, with $N = 3$), and with $C(\varphi)$ depending only on $\|\nabla^2 \varphi\|_{L^\infty(\mathbb{R}^3)}$. This last quantity is finite since $f \in L^1([0, T]; L^3_{-3}(\mathbb{R}^3))$.

We recall that in the Coulomb case, for $i = 1, \dots, 3$, $|b_i(z)| \leq 2|z|^{-2}$. Then (since v and w play the same role), taking φ as above, and $i = 1, \dots, 3$,

$$\begin{aligned} & \int_0^T \iint_{\mathbb{R}^3 \times \mathbb{R}^3} f(t, v) f(t, w) |b_i(v - w)| \left| (\partial_i \varphi)(t, v) - (\partial_i \varphi)(t, w) \right| dv dw dt \\ & \leq 4 \int_0^T \iint_{\mathbb{R}^3 \times B(0, R)} f(t, v) f(t, w) |v - w|^{-2} \left| (\partial_i \varphi)(t, v) - (\partial_i \varphi)(t, w) \right| dv dw dt \\ & \leq 4 \sup_{j=1, \dots, 3} \|\partial_{ij} \varphi\|_{L^\infty([0, T] \times \mathbb{R}^3)} \int_0^T \iint_{\mathbb{R}^3 \times B(0, R)} f(t, v) f(t, w) |v - w|^{-1} dv dw dt, \end{aligned}$$

which, once again, is finite thanks to Lemma 4 and the estimate $f \in L^\infty([0, T]; L^1_2(\mathbb{R}^3)) \cap L^1([0, T]; L^3_{-3}(\mathbb{R}^3))$.

The last statement of Corollary 1.1 is thus proven. \square

Remark: As stated earlier, the computations of this section are either classical or direct extensions and variants of classical computations. They are close, for example, to computations of [3] (section 7, second application) or [19] (proof of Proposition 3.3, step 1). Note that in the proof, we only need the estimate $f 1_{|v| \leq R} \in L^1([0, T]; L^3(\mathbb{R}^3))$, and not $f \in L^1([0, T]; L^3_{-3}(\mathbb{R}^3))$, so that the “local in v ” approach of [3] is well adapted to get this result.

Observing the assumptions of Theorem 1 and Lemma 4, we see that if

$$\forall z \geq 0, \quad K_3 \inf(1, |z|^{\gamma_1+2}) \leq \psi(|z|) \leq K_1 |z|^2 + K_2 |z|^{\gamma_2+2},$$

for some $K_1, K_2, K_3 > 0$ and $\gamma_1 < 0$, $\gamma_2 \in]-4, -2[$, then reasonable solutions f of the Landau equation with initial data having a finite mass, energy and entropy, naturally satisfy (for all $T > 0$) the estimate $\int_0^T \int_{\mathbb{R}^N} |\nabla \sqrt{f(t, v)}|^2 (1 + |v|^2)^{\inf(\gamma_1/2, -1)} dv dt < +\infty$, and thus belong to $L^1([0, T]; L^{\frac{N}{N-2}}_{\inf(\gamma_1, -2)}(\mathbb{R}^N))$ when $N \geq 3$ (and $L^1([0, T]; L^q_{\inf(\gamma_1, -2)}(\mathbb{R}^2))$ for all $q \in [1, +\infty[$ when $N = 2$). This is sufficient to define weak solutions according to Lemma 4, and the proof of Corollary 1.1.

We also wish to emphasize the appearance of the limiting case $\gamma = -4$ in Lemma 4. This was already observed in [3], and, from our point of view, justifies the terminology of “very soft potentials” for the case when $\gamma \in]-4, -2[$, rather than $\gamma \in]-3, -2[$ or $\gamma \in [-3, -2[$. Note also that the value $\gamma = -4$ suprisingly appears for all dimensions N .

4. ESTIMATES FOR MOMENTS

We present here an estimate of propagation of moments which holds for any weak solutions of the Landau equation, in arbitrary dimension, when the function ψ is bounded above around 0 by a power law of either very soft or (moderately) soft type. We also impose that ψ is bounded below at infinity by an arbitrary power law, and that ψ is bounded above at infinity by a (moderately) soft potential.

This estimate implies Proposition 2 in the Coulomb case.

Our result of propagation of moments writes

Proposition 4. *Let ψ satisfy*

$$\forall z \geq 0, \quad K_3 \inf(1, |z|^{\gamma_1+2}) \leq \psi(z) \leq K_1 |z|^{2-\delta} + K_2 |z|^{\gamma_2+2},$$

for some $K_1, K_2, K_3 > 0$ and $\delta \in]0, 2]$, $\gamma_1 \leq 0$, $\gamma_2 \in]-4, -\delta]$.

Let $N \in \mathbb{N}$, $N \geq 2$, $T > 0$, and $f := f(t, v) \geq 0$ be a weak solution of the Landau equation (1) on $[0, T] \times \mathbb{R}^N$ associated to ψ and an initial datum $f_{in} \in L^1_2 \cap L \ln L(\mathbb{R}^N)$, in the following sense: We assume that $f \in C([0, T]; \mathcal{D}'(\mathbb{R}^N)) \cap L^\infty([0, T]; L^1_2 \cap L \ln L(\mathbb{R}^N))$, satisfies the conservation of mass, momentum, energy,

$$\forall t \in [0, T], \quad \int_{\mathbb{R}^N} f(t, v) \begin{pmatrix} 1 \\ v_i \\ |v|^2/2 \end{pmatrix} dv = \int_{\mathbb{R}^N} f_{in}(v) \begin{pmatrix} 1 \\ v_i \\ |v|^2/2 \end{pmatrix} dv,$$

the (uniform in time) boundedness of the entropy

$$\sup_{t \in [0, T]} \int_{\mathbb{R}^N} f(t, v) |\ln f(t, v)| dv < +\infty,$$

and has a finite (integrated in time) $L^{\frac{N}{N-2}}_{\inf(\gamma_1, -2)}$ norm, that is

$$(26) \quad \mathcal{Q}_{T, N, \gamma_1}(f) := \int_0^T \left(\int_{\mathbb{R}^N} |f(v)|^{\frac{N}{N-2}} (1 + |v|^2)^{\frac{N}{N-2} \inf(\frac{\gamma_1}{2}, -1)} dv \right)^{\frac{N-2}{N}} dt < +\infty,$$

if $N \geq 3$ (in this assumption, $N/(N-2)$ is replaced by all $q \in [1, +\infty[$ if $N = 2$).

Moreover, we assume that $f(0, \cdot) = f_{in}$ and f satisfies the (strong w.r.t. t) weak form of the Landau equation, that is, for all $\varphi := \varphi(v) \in C_c^2(\mathbb{R}^N)$, and $t_1 \leq t_2 \in [0, T]$,

$$(27) \quad \begin{aligned} & \int_{\mathbb{R}^N} f(t_2, v) \varphi(v) dv - \int_{\mathbb{R}^N} f(t_1, v) \varphi(v) dv \\ &= \frac{1}{2} \sum_{i=1}^3 \sum_{j=1}^3 \int_{t_1}^{t_2} \iint_{\mathbb{R}^N \times \mathbb{R}^N} f(s, v) f(s, w) a_{ij}(v-w) \left(\partial_{ij} \varphi(v) + (\partial_{ij} \varphi)(w) \right) dv dw ds \\ & \quad + \sum_{i=1}^3 \int_{t_1}^{t_2} \iint_{\mathbb{R}^N \times \mathbb{R}^N} f(s, v) f(s, w) b_i(v-w) \left(\partial_i \varphi(v) - (\partial_i \varphi)(w) \right) dv dw ds. \end{aligned}$$

We recall that (for $i, j = 1, \dots, 3$) a_{ij} and b_i are defined by (4) and (6), and that thanks to estimate (26), Lemma 4, and a direct extension of the proof of Corollary 1.1, each term in (27) is well defined.

Finally, we assume that $(\gamma_2 + 2)(1 - \inf(\gamma_1/2, -1)) > -4$.

Then, for all $k \in \mathbb{R}$ such that $\int_{\mathbb{R}^N} f_{in}(v) (1 + |v|^2)^k dv < +\infty$, the moment of order $2k$ of f is locally (in time) bounded: for all $T > 0$,

$$(28) \quad \sup_{t \in [0, T]} \int_{\mathbb{R}^N} f(t, v) (1 + |v|^2)^k dv < +\infty.$$

More precisely, this last quantity only depends on T, N , the parameters of ψ , the initial mass, momentum, energy and (an upper bound of the) entropy, the quantity $\mathcal{Q}_{T, N, \gamma_1}(f)$, and the initial moment $\int_{\mathbb{R}^N} f_{in}(v) (1 + |v|^2)^k dv$.

Note that when $\gamma_1 = \gamma_2$, the condition $(\gamma_2 + 2)(1 - \inf(\gamma_1/2, -1)) > -4$ is equivalent to $\gamma_1 > -2\sqrt{3}$, so that the Coulomb case, which corresponds to $\gamma_1 = \gamma_2 = -3$, falls within the range of application of this proposition.

Remarks: Note first that it is possible to take $\delta = 0$ in this proposition, but the proof has then to be somewhat modified (and approximate solutions must be introduced), so we discarded this case (which would be very far from any ψ coming out physics anyway). For $\delta < 0$, the behavior of moments can become completely different since ψ can in a such a case look like a hard potential, we do not therefore investigate such functions ψ .

As stated in the remarks of the previous section, all reasonable solutions of the Landau equation (for ψ satisfying the estimates of Proposition 4) with initial data having a finite mass, energy and entropy, will satisfy the estimate (for all $T > 0$) $\int_0^T \int_{\mathbb{R}^N} |\nabla \sqrt{f(t, v)}|^2 (1 + |v|^2)^{\inf(\gamma_1/2, -1)} dv dt < +\infty$, thanks to Theorem 3, and therefore estimate (26), thanks to Lemma 3. The assumptions of Proposition 4 are therefore quite reasonable.

Surprisingly, the critical parameter $\gamma = -2\sqrt{3}$ appears in the issue of propagation of moments. We are not sure that this critical parameter is really significant since our proof might be far from optimal (our main purpose was to treat the Coulomb case).

One can observe from the proof that the dependence w.r.t. T in estimate (28) is of polynomial type, since no Gronwall-type argument is used.

Note that the best result (in terms of dependence of the constants w.r.t. the order of the moment) of propagation of moments in (moderately) soft potentials

is to be found in [26]. We insist that the result of propagation of moments which is presented here is typical of soft potentials: as for (moderately) soft potentials, no appearance of moments is expected in such a case. Propagation of (reasonable) higher than polynomial moments can probably be obtained thanks to the summation of polynomial moments.

We end up this series of remarks by the following (rather vague) statement: in some sense, the propagation of moments of any order shows that the main problems in the theory of the Landau equation in the Coulomb case are rather related to smoothness (or integrability) issues than to issues related to large $|v|$.

The proof that we propose follows the usual lines of proofs of propagation of moments. The main modification w.r.t. (moderately) soft potentials consists in using the weighted $L_t^1(L_v^p)$ ($p > 1$) estimate on f (coming out of the weighted $L_t^2(H_v^1)$ norm of \sqrt{f}) in order to treat the singularity which is typical of very soft potentials. It is important here that the weights in the $L_t^1(L_v^p)$ estimate be polynomial, and not arbitrary. This was not the case in the previous section, where the fact that H -solutions are standard weak solutions only requires the information that some $L_t^1(L_{v,loc}^p)$ bound holds.

We choose to present the proof as an induction in order to avoid an interpolation leading to a differential inequality that would necessitate to work on an approximate solution of the Landau equation rather than on the (weak) solution itself. As a consequence, the proof of Proposition 4 is obtained by applying inductively the following lemma:

Lemma 5. *Let N, T, ψ (together with $K_1, K_2, K_3, \delta, \gamma_1, \gamma_2$), and f be as in Proposition 4.*

We define for all $l \in \mathbb{R}$,

$$(29) \quad M_l(f; T) := \|f\|_{L^\infty([0, T]; L_{2l}^1(\mathbb{R}^N))} = \sup_{t \in [0, T]} \int_{\mathbb{R}^N} f(t, v) (1 + |v|^2)^l dv.$$

Let $k > 1$ be such that $M_k(f; 0) < +\infty$, $M_{k-\delta/2}(f; T) < +\infty$. If $\gamma_2 < -2$, we assume moreover that

$$M_{\frac{2(k-1)+(\gamma_2+2-\varepsilon)\inf(\gamma_1/2, -1)}{6+\gamma_2-\varepsilon}}(f, T) < +\infty,$$

for some $\varepsilon \in]0, \gamma_2 + 4[$.

Then,

$$M_k(f; T) < +\infty.$$

More precisely, $M_k(f; T)$ is bounded by a constant only depending on N, T, k , the parameters of ψ , the bound $\mathcal{Q}_{T, N, \gamma_1}(f)$, the initial moment $M_k(f; 0)$, the moment $M_{k-\delta/2}(f; T)$, and, when $\gamma_2 \in]-4, -2[$, the moment $M_{\frac{2(k-1)+(\gamma_2+2-\varepsilon)\inf(\gamma_1/2, -1)}{6+\gamma_2-\varepsilon}}(f, T)$ and $\varepsilon \in]0, \gamma_2 + 4[$.

Proof of Lemma 5: We consider $k > 1$, $\chi \in D(\mathbb{R})$ such that $\chi|_{[0, 1]} = 1$, $\chi|_{[0, 2]^c} = 0$, $0 \leq \chi \leq 1$, $\eta \in]0, 1[$, and we use the test function $v \mapsto \phi(v) := (1 + |v|^2)^k \chi(\eta(1 + |v|^2)^{1/2})$.

Then (for $i, j = 1, \dots, N$),

$$\partial_i \phi(v) = 2k v_i (1 + |v|^2)^{k-1} \chi(\eta(1 + |v|^2)^{1/2})$$

$$+ \eta v_i (1 + |v|^2)^{k-1/2} \chi'(\eta(1 + |v|^2)^{1/2}),$$

and

$$\begin{aligned} \partial_{ij} \phi(v) &= 2k [(1 + |v|^2) \delta_{ij} + 2(k-1) v_i v_j] (1 + |v|^2)^{k-2} \chi(\eta(1 + |v|^2)^{1/2}) \\ &\quad + [(4k-1) v_i \eta v_j + \eta \delta_{ij} (1 + |v|^2)] (1 + |v|^2)^{k-3/2} \chi'(\eta(1 + |v|^2)^{1/2}) \\ &\quad + (\eta v_i) (\eta v_j) (1 + |v|^2)^{k-1} \chi''(\eta(1 + |v|^2)^{1/2}), \end{aligned}$$

so that

$$|\partial_{ij} \phi(v)| \leq C(k) (1 + |v|^2)^{k-1},$$

where $C(k)$ is a constant depending only on k (and a bound on $\|\chi'\|_\infty, \|\chi''\|_\infty$).

Using definitions (4) and (6), we see that (for $i, j = 1, \dots, N$)

$$|a_{ij}(z)| \leq \psi(|z|), \quad |b_i(z)| \leq (N-1) |z|^{-1} \psi(|z|),$$

so that

$$|a_{ij}(v-w) (\partial_{ij} \phi(v) + \partial_{ij} \phi(w))| \leq C(k) \psi(|v-w|) (1 + |v|^2 + |w|^2)^{k-1},$$

$$|b_i(v-w) (\partial_i \phi(v) - \partial_i \phi(w))| \leq C(N, k) \psi(|v-w|) (1 + |v|^2 + |w|^2)^{k-1},$$

where $C(k)$ (resp. $C(N, k)$) is a constant depending only on k (resp. on N, k).

Then, we get

$$\begin{aligned} &\int f(T, v) (1 + |v|^2)^k \chi(\eta(1 + |v|^2)^{1/2}) dv - \int f(0, v) (1 + |v|^2)^k \chi(\eta(1 + |v|^2)^{1/2}) dv \\ &\leq C(N, k) \int_0^T \int \int f(t, v) f(t, w) \psi(|v-w|) (1 + |v|^2 + |w|^2)^{k-1} dv dw dt \\ &\leq C(N, k) \int_0^T \int \int_{|v-w| \leq 1, |v| \geq 2, |w| \geq 1} f(t, v) f(t, w) (K_1 + K_2 |v-w|^{\gamma_2+2}) \\ &\quad \times (1 + |v|^2 + |w|^2)^{k-1} dv dw dt \\ &\quad + C(N, k) \int_0^T \int \int_{|v-w| \leq 1, |v| \leq 2, |w| \leq 3} f(t, v) f(t, w) (K_1 + K_2 |v-w|^{\gamma_2+2}) \\ &\quad \times (1 + |v|^2 + |w|^2)^{k-1} dv dw dt \\ &\quad + C(N, k) \int_0^T \int \int_{|v-w| \geq 1} f(t, v) f(t, w) (K_1 + K_2) |v-w|^{2-\delta} \\ &\quad \times (1 + |v|^2 + |w|^2)^{k-1} dv dw dt. \end{aligned}$$

If $|v-w| \leq 1$ and $|v| \geq 2$, then $\frac{1}{2}|v| \leq |w| \leq \frac{3}{2}|v|$, so that (still when $|v-w| \leq 1$ and $|v| \geq 2$), we get the estimates

$$1 + |v|^2 \leq 1 + |v|^2 + |w|^2 \leq \frac{13}{4} (1 + |v|^2),$$

and

$$1 + |w|^2 \leq 1 + |v|^2 + |w|^2 \leq 5 (1 + |w|^2).$$

As a consequence, for any $q \in \mathbb{R}$,

$$\int f(T, v) (1 + |v|^2)^k \chi(\eta(1 + |v|^2)^{1/2}) dv - \int f(0, v) (1 + |v|^2)^k \chi(\eta(1 + |v|^2)^{1/2}) dv$$

$$\begin{aligned}
&\leq C(N, k) K_1 T M_0(f; T) M_{k-1}(f; T) + C(N, k) K_2 \sup \left(1, \left(\frac{13}{4} \right)^{q/2} \right) \sup(1, 5^{k-1-q/2}) \\
&\times \int_0^T \int \int_{|v-w| \leq 1, |v| \geq 2, |w| \geq 1} f(t, v) f(t, w) |v-w|^{\gamma_2+2} (1+|w|^2)^{k-1-q/2} (1+|v|^2)^{q/2} dv dw dt \\
&\quad + C(N, k) K_1 T M_0(f; T)^2 \\
&\quad + C(N, k) K_2 \int_0^T \int \int_{|v-w| \leq 1, |v| \leq 2, |w| \leq 3} f(t, v) f(t, w) |v-w|^{\gamma_2+2} dv dw dt \\
&\quad + C(N, k) (K_1 + K_2) T M_{k-\delta/2}(f; T)^2.
\end{aligned}$$

If $\gamma_2 \in [-2, -\delta]$, we take $q = 0$ in the estimate above, and obtain that

$$\int f(T, v) (1 + |v|^2)^k \chi(\eta(1 + |v|^2)^{1/2}) dv \leq C$$

where

$$C := C(T, K_1, K_2, \gamma_2, M_1(f; 0), M_k(f; 0), M_{k-\delta/2}(f; T), N, k)$$

only depends on N, T , the parameters of ψ , the initial mass and energy, the initial moment $M_k(f; 0)$, and the moment $M_{k-\delta/2}(f; T)$.

We now suppose that $\gamma_2 \in]-4, -2[$. Thanks to Lemma 4,

$$\begin{aligned}
&\int_0^T \int \int_{|v| \leq 2, |w| \leq 3} f(t, v) f(t, w) |v-w|^{\gamma_2+2} dv dw dt \\
&\leq \int_0^T C(N, \gamma_2) \|f(t, \cdot)\|_{L^1(\mathbb{R}^N)} \left(\|f(t, \cdot)\|_{L^1_2(\mathbb{R}^N)} + \|f(t, \cdot) 1_{|\cdot| \leq 2}\|_{L^{\frac{N}{N-2}}(\mathbb{R}^N)} \right) dt \\
&\leq C(N, \gamma_2) M_0(f; T) \left(M_1(f; T) T + 5^{-in f(\gamma_1/2, -1)} \mathcal{Q}_{T, N, \gamma_1}(f) \right),
\end{aligned}$$

where $C(N, \gamma_2)$ is a constant only depending on N and γ_2 . If $N = 2$, one must replace $N/(N-2)$ by some $q \geq 1$ (depending on γ_2).

As a consequence, for all $r \in [1, N/(-\gamma_2 - 2)[$ (note that $-\gamma_2 - 2 < 2 \leq N$) and all $q \in \mathbb{R}$ (and defining r' by the identity $\frac{1}{r} + \frac{1}{r'} = 1$), using Young's inequality for convolutions, and denoting in the sequel $C(a, b, \dots)$ a constant depending only on a, b, \dots ,

$$\begin{aligned}
&\int f(T, v) (1 + |v|^2)^k \chi(\eta(1 + |v|^2)^{1/2}) dv \\
&\leq C(N, T, K_1, K_2, \gamma_2, \gamma_1, M_k(f; 0), M_{k-\delta/2}(f; T), \mathcal{Q}_{T, N, \gamma_1}(f), q, k) \\
&\quad \times \left(1 + \int_0^T \|(1 + |\cdot|^2)^{q/2} f\|_{L^{r'}(\mathbb{R}^N)} \right. \\
&\quad \left. \times \|(1 + |\cdot|^2)^{k-1-q/2} f\|_{L^1(\mathbb{R}^N)} \|z \mapsto |z|^{\gamma_2+2} 1_{|z| \leq 1}\|_{L^r(\mathbb{R}^N)} dt \right) \\
&\leq C(N, T, K_1, K_2, \gamma_2, \gamma_1, M_k(f; 0), M_{k-\delta/2}(f; T), \mathcal{Q}_{T, N, \gamma_1}(f), q, k, r) \\
&\quad \times \left(1 + \int_0^T \|(1 + |\cdot|^2)^{q/2} f\|_{L^{r'}(\mathbb{R}^N)} \|(1 + |\cdot|^2)^{k-1-q/2} f\|_{L^1(\mathbb{R}^N)} dt \right).
\end{aligned}$$

Taking (for some $\varepsilon \in]0, \gamma_2 + 4[$) $r' = \frac{N}{N + \gamma_2 + 2 - \varepsilon}$, $\beta = \frac{-\gamma_2 - 2 + \varepsilon}{2} \in]0, 1[$, and $q = \frac{\beta \inf(\gamma_1, -2) + 2(k-1)(1-\beta)}{2-\beta}$, we see that $r \in [1, N/(-\gamma_2 - 2)[$, and

$$\frac{1}{r'} = \beta \frac{N-2}{N} + 1 - \beta, \quad q = \beta \inf(\gamma_1, -2) + (1-\beta)(2(k-1) - q),$$

so that thanks to the interpolation inequality of Proposition 5 in Appendix 1,

$$\|(1+|\cdot|^2)^{q/2} f\|_{L^{r'}(\mathbb{R}^N)} \leq \|(1+|\cdot|^2)^{\inf(\gamma_1/2, -1)} f\|_{L^{\frac{N}{N-2}}(\mathbb{R}^N)}^\beta \|(1+|\cdot|^2)^{k-1-q/2} f\|_{L^1(\mathbb{R}^N)}^{1-\beta}.$$

We end up with

$$\begin{aligned} & \int f(T, v) (1 + |v|^2)^k \chi(\eta(1 + |v|^2)^{1/2}) dv \\ & \leq C(N, T, K_1, K_2, \gamma_2, M_k(f; 0), M_{k-\delta/2}(f; T), \mathcal{Q}_{T, N, \gamma_1}(f), \varepsilon, k, \gamma_1) \\ & \times \left(1 + \int_0^T \|(1 + |\cdot|^2)^{\inf(\gamma_1/2, -1)} f\|_{L^{\frac{N}{N-2}}}^\beta \|(1 + |\cdot|^2)^{k-1-q/2} f\|_{L^1}^{2-\beta} dt \right) \\ & \leq C \left(N, T, K_1, K_2, \gamma_2, M_k(f; 0), M_{k-\delta/2}(f; T), \right. \\ & \quad \left. M_{\frac{2(k-1) + (\gamma_2 + 2 - \varepsilon) \inf(\gamma_1/2, -1)}{6 + \gamma_2 - \varepsilon}}(f, T), \mathcal{Q}_{T, N, \gamma_1}(f), \varepsilon, k, \gamma_1 \right), \end{aligned}$$

since $k - 1 - q/2 = \frac{2(k-1) + (\gamma_2 + 2 - \varepsilon) \inf(\gamma_1/2, -1)}{6 + \gamma_2 - \varepsilon}$.

We conclude the proof of Lemma 5 by letting η tend to 0 and by using Fatou's lemma. \square

We now turn to the

Proof of Proposition 4: Since we assume that $(\gamma_2 + 2)(1 - \inf(\gamma_1/2, -1)) > -4$, we can find $\varepsilon \in]0, \gamma_2 + 4[$ such that $(\gamma_2 + 2 - \varepsilon)(1 - \inf(\gamma_1/2, -1)) > -4$. Then we define by induction (for $n \in \mathbb{N}$) the sequence

$$k_0 = k, \quad k_{n+1} = \sup \left(k_n - \delta/2, \frac{2}{6 + \gamma_2 - \varepsilon} k_n + \frac{\inf(\gamma_1/2, -1)(\gamma_2 + 2 - \varepsilon) - 2}{6 + \gamma_2 - \varepsilon} \right),$$

when $\gamma_2 \in]-4, -2[$, and

$$k_0 = k, \quad k_{n+1} = k_n - \delta/2,$$

when $\gamma_2 \in [-2, -\delta]$.

We observe that the fixed point of $k \mapsto \frac{2}{6 + \gamma_2 - \varepsilon} k + \frac{\inf(\gamma_1/2, -1)(\gamma_2 + 2 - \varepsilon) - 2}{6 + \gamma_2 - \varepsilon}$ is $\bar{k} = \frac{\inf(\gamma_1/2, -1)(\gamma_2 + 2 - \varepsilon) - 2}{4 + \gamma_2 - \varepsilon}$, so that $\bar{k} < 1$ is equivalent to $(\gamma_2 + 2 - \varepsilon)(1 - \inf(\gamma_1/2, -1)) > -4$, which holds thanks to our assumption and our choice of ε . Then, the sequence $(k_n)_{n \in \mathbb{N}}$ decreases, and we can pick up $n_0 \in \mathbb{N}$ such that $k_{n_0} > 1$, and $k_{n_0+1} \leq 1$. By (finite) induction, starting from k_{n_0} and using Lemma 5 repeatedly, we get that $M_{k_n}(f, T) < +\infty$ for all $n = n_0, n_0 - 1, \dots, 0$, so that Proposition 4 holds. \square

Proof of Proposition 2: We first observe that $\psi(z) = |z|^{-3}$ satisfies the assumption on ψ in Proposition 4, with $\gamma_1 = \gamma_2 = -3$, so that $(\gamma_2 + 2)(1 - \inf(\gamma_1/2, -1)) = -5/2 > -4$. Then, the H-solutions of the Landau equation in the Coulomb case satisfy the conservation of mass, momentum and energy, and have an upper-bounded entropy. According to Corollary 1.1, they also lie in $L^\infty([0, T]; L^3_{-3}(\mathbb{R}^3))$ for all $T > 0$, and (cf. [29]) satisfy the (strong w.r.t. t)

weak form of the Landau equation. Proposition 2 is then a direct consequence of Proposition 4. \square .

5. CONCLUSIONS AND PERSPECTIVES

In this paper, we have shown that the existing solutions f of the Landau equation in the Coulomb case (that is, the H -solutions) are such that \sqrt{f} lies in a weighted $L^2(H^1)$ space, so that f lies in a weighted $L^1(L^3)$ space. This is sufficient to show that those solutions can be defined in the standard weak framework, and that all polynomial L^1 moments are propagated.

According to [22], such an estimate enables the propagation of L^p norms (for $p > 1$) when (moderately) soft potentials are concerned (we also refer to Appendix 2 of this work). However, we weren't able to extend this result in the Coulomb case (or for any $\gamma < -2$), so that the smoothness issues for the (spatially homogeneous) Landau equation in the Coulomb case are still completely open (together with the related issue of uniqueness).

We believe that the results presented for the Landau equation in the Coulomb case can be extended to the Boltzmann equation with very soft potentials and angular cutoff. Such an extension would be quite close to what is obtained in [3]. The main difference would be the treatment of large velocities, which is not investigated in [3].

Appendix 1: Interpolations

We recall two classical interpolation inequalities, which can be proven by an application of Hölder's inequality.

The first one deals with weighted L^p spaces.

Proposition 5. *Assume that for some $q_1, q_2 \in [1, \infty]$, $a_1, a_2 \in \mathbb{R}$, $x \mapsto (1 + |x|^2)^{a_1} f(x) \in L^{q_1}(\mathbb{R}^N)$, $x \mapsto (1 + |x|^2)^{a_2} f(x) \in L^{q_2}(\mathbb{R}^N)$.*

Then, for and any $\beta \in [0, 1]$, $x \mapsto (1 + |x|^2)^a f \in L^q(\mathbb{R}^N)$, where

$$\frac{1}{q} = \frac{\beta}{q_1} + \frac{1-\beta}{q_2}, \quad a = \beta a_1 + (1-\beta) a_2.$$

Moreover,

$$(30) \quad \begin{aligned} \|x \mapsto (1 + |x|^2)^a f(x)\|_{L^q(\mathbb{R}^N)} &\leq \|x \mapsto (1 + |x|^2)^{a_1} f(x)\|_{L^{q_1}(\mathbb{R}^N)}^\beta \\ &\quad \times \|x \mapsto (1 + |x|^2)^{a_2} f(x)\|_{L^{q_2}(\mathbb{R}^N)}^{1-\beta}, \end{aligned}$$

or, in abridged form,

$$(31) \quad \|f\|_{L_a^q(\mathbb{R}^N)} \leq \|f\|_{L_{a_1}^{q_1}(\mathbb{R}^N)}^\beta \|f\|_{L_{a_2}^{q_2}(\mathbb{R}^N)}^{1-\beta}.$$

Then, the second one deals with weighted $L^p(L^q)$ spaces, the weight concerning the second variable.

Proposition 6. *Assume that for some $p_1, q_1, p_2, q_2 \in [1, \infty]$, $a_1, a_2 \in \mathbb{R}$, $(t, x) \mapsto (1+|x|^2)^{a_1} f(t, x) \in L^{p_1}([0, T]; L^{q_1}(\mathbb{R}^N))$ on one hand, and $(t, x) \mapsto (1+|x|^2)^{a_2} f(t, x) \in L^{p_2}([0, T]; L^{q_2}(\mathbb{R}^N))$ on the other hand.*

Then, for any $\beta \in [0, 1]$,

$$(t, x) \mapsto (1+|x|^2)^a f(t, x) \in L^p([0, T]; L^q(\mathbb{R}^N)),$$

where

$$\frac{1}{p} = \frac{\beta}{p_1} + \frac{1-\beta}{p_2}, \quad \frac{1}{q} = \frac{\beta}{q_1} + \frac{1-\beta}{q_2}, \quad a = \beta a_1 + (1-\beta) a_2.$$

Moreover,

$$\begin{aligned} (32) \quad & \| (t, x) \mapsto (1+|x|^2)^a f(t, x) \|_{L^p([0, T]; L^q(\mathbb{R}^N))} \\ & \leq \| (t, x) \mapsto (1+|x|^2)^{a_1} f(t, x) \|_{L^{p_1}([0, T]; L^{q_1}(\mathbb{R}^N))}^\beta \\ & \quad \times \| (t, x) \mapsto (1+|x|^2)^{a_2} f(t, x) \|_{L^{p_2}([0, T]; L^{q_2}(\mathbb{R}^N))}^{1-\beta}, \end{aligned}$$

or, in abridged notation,

$$(33) \quad \|f\|_{L^p([0, T]; L_a^q(\mathbb{R}^N))} \leq \|f\|_{L^{p_1}([0, T]; L_{a_1}^{q_1}(\mathbb{R}^N))}^\beta \|f\|_{L^{p_2}([0, T]; L_{a_2}^{q_2}(\mathbb{R}^N))}^{1-\beta}.$$

Appendix 2: Further regularity estimates in the case of (moderately) soft potentials

This second appendix is devoted to the application of the main estimate of the paper (that is, Theorem 3), to the case of moderately soft potentials. Since a well-established theory exists in this case (cf. in particular [22] and [2]), we merely present alternative proofs of already existing results, or proofs of variants of those results.

We consider functions ψ in the Landau kernel which are bounded above around point 0 by power laws of (moderately) soft potentials type. We also choose to impose a bound from below involving an arbitrary power at infinity, and a bound above at infinity which behaves like a (moderately) soft potential, but with an exponent which needn't be the same as the exponent appearing around point 0.

In order to keep the proofs of our results as simple as possible, we consider smooth solutions of the Landau equation rapidly decaying at infinity (w.r.t. v) such that $\ln f$ is slowly growing at infinity (w.r.t. v), in the presentation of the results. In other words, we only describe the *a priori* estimates.

Moreover, for the sake of readability of the results, we present estimates which depend on an arbitrary number of moments (in L^1) on the initial data. Those results still (partially) hold when only a finite number of moments initially exist.

The first part of the estimates of [22], that is the bound in a weighted $L^1([0, T]; L^3(\mathbb{R}^3))$ (or, in dimension $N \geq 2$, in a weighted $L^1([0, T]; L^{N/(N-2)}(\mathbb{R}^N))$), and by interpolation with $L^\infty([0, T]; L^1(\mathbb{R}^3))$, in a weighted $L^{1+\delta}([0, T]; L^{3-\varepsilon}(\mathbb{R}^3))$ for some $\delta, \varepsilon > 0$, is a direct consequence of the main estimate of this paper and Sobolev inequalities (cf. Theorem 3, and Lemma 3).

The second part of the estimates of [22] are related to the propagation of L^p norms. The bounds that we present in this appendix also concern the propagation of L^p norms. The main differences with the results of [22] are the following:

- We use the spaces $\cup_{p < q} L^q$ whereas the spaces L^p are used in [22];
- We assume here that all moments in L^1 of the solution are (initially) finite, whereas only a finite number of them are assumed to be finite in [22];
- Our estimates are polynomially growing w.r.t. time, whereas they grow more rapidly in [22];
- More importantly from the structural point of view, in the proof of the bounds, the coercivity estimate $\sum_{i,j} (a_{ij} * f) \xi_i \xi_j \geq C(1 + |v|^2)^{\gamma/2} |\xi|^2$ proven in [22], [2], is replaced by our main theorem (Theorem 3);
- We do not treat here the limiting case $\gamma = -2$, whereas this case is treated in [22];
- Finally, we authorize here dimensions different from 3 and functions ψ which are not necessarily power laws.

We start with the most technical case, which appears when $\psi(z) \leq K_2 |z|^{\gamma_2+2}$ with $\gamma_2 \in]-2, -1[$ (for $|z|$ close to 0), and one looks for the propagation of L^q norms with $q < 1 + \frac{1}{N+1+\gamma_2}$. This corresponds to a soft potential in some intermediate regime (still moderately soft, but already far from Maxwell molecules), and a parameter q close to 1.

Our proposition writes:

Proposition 7. *Let ψ be a function satisfying*

$$\forall z \geq 0, \quad K_3 \inf(1, |z|^{\gamma_1+2}) \leq \psi(z) \leq K_1 |z|^{2-\delta} + K_2 |z|^{\gamma_2+2},$$

for some $K_1, K_2, K_3 > 0$ and $\delta \in]0, 1]$, $\gamma_1 \leq 0$, $\gamma_2 \in]-2, -1[$.

Let $T > 0$, and $f := f(t, v) \geq 0$ be a strong solution of the Landau equation (1) on $[0, T] \times \mathbb{R}^N$ (more precisely, we assume that f is smooth, rapidly decaying when $|v| \mapsto +\infty$. We also assume that $\ln f$ is slowly growing when $|v| \mapsto +\infty$).

We consider $q \in]1, 1 + \frac{1}{N+1+\gamma_2}]$. Then, for all $m \in \mathbb{R}$, $\varepsilon_0 \in]0, q-1]$, there exists C depending only on $N, K_1, K_2, K_3, \gamma_1, \gamma_2, \delta, \varepsilon_0, m, q$, on the initial mass, momentum, energy and (upper part of) the entropy, on the initial moments (in L^1) ($\|f(0, \cdot)\|_{L^1_\mu(\mathbb{R}^N)}\big)_{\mu \geq 1}$, and initial $L^{q-\varepsilon}$ norm ($\|f(0, \cdot)\|_{L^{q-\varepsilon}(\mathbb{R}^N)}\big)_{\varepsilon \in]0, q-1]}$, such that

$$\sup_{t \in [0, T]} \|f(t, \cdot)\|_{L^{q-\varepsilon_0}_m(\mathbb{R}^N)} \leq C.$$

This proposition is obtained by using inductively the following lemma:

Lemma 6. *Let ψ and f satisfy the assumptions of Proposition 7.*

We consider $A \in]1, \frac{N}{N+1+\gamma_2}]$.

Then for all $\varepsilon_0 \in]0, A-1]$ and $m_0 \in \mathbb{R}$,

$$\sup_{t \in [0, T]} \|f(t, \cdot)\|_{L^{A(1+(\gamma_2+2)/N)-\varepsilon_0}_{m_0}(\mathbb{R}^N)}$$

is bounded by a constant depending on the same parameters as in Proposition 7 (except that q is replaced by A , and the initial $L^{q-\varepsilon}$ norm by the initial $L^{A-\varepsilon}$ norm), and on the quantities

$$\sup_{t \in [0, T]} \|f(t, \cdot)\|_{L^{A-\varepsilon}_m(\mathbb{R}^N)},$$

for all $\varepsilon \in]0, A-1]$, and $m \in \mathbb{R}$.

Proof of Lemma 6: We observe that multiplying Landau's equation by f^k for some $k > 0$ and integrating on $[0, T] \times \mathbb{R}^N$, we end up with

$$\begin{aligned} & \int \frac{f^{k+1}}{k+1} dv(T) + k \int_0^T \int \sum_{i=1}^N \sum_{j=1}^N f^{k-1} (a_{ij} * f) \partial_i f \partial_j f dv dt \\ &= \int \frac{f^{k+1}}{k+1} dv(0) + k \int_0^T \int \sum_{i=1}^N (b_i * f) \partial_i f f^k dv \\ &\leq \int \frac{f^{k+1}}{k+1} dv(0) + k \mathcal{F}_{T,N,\gamma_1}(f)^{1/2} \left(\int_0^T \int \left| \sum_{i=1}^N b_i * f \right|^2 f^{2k+1} (1+|v|^2)^{\sup(-\gamma_1/2,1)} dv dt \right)^{1/2}, \end{aligned}$$

with, thanks to Theorem 3,

$$\mathcal{F}_{T,N,\gamma_1}(f) := \int_0^T \int_{\mathbb{R}^N} \frac{|\nabla_v f|^2}{f} (1+|v|^2)^{\inf(\gamma_1/2,-1)} dv dt \leq C,$$

where C depends only on N, K_3, γ_1 , and the initial mass, momentum, energy and (an upper bound of the) initial entropy. In the rest of the proof of this lemma, we denote by $C(a, b, \dots)$ a constant only depending on a, b, \dots

Then, for all $r \in [1, \infty]$,

$$\begin{aligned} & \int_0^T \int \left| \sum_{i=1}^N b_i * f \right|^2 f^{2k+1} (1+|v|^2)^{\sup(-\gamma_1/2,1)} dv dt \\ &\leq C(N) \int_0^T \int f^{2k+1}(t, v) \left| \int_{|v-w| \geq 1} (K_1 |v-w| + K_2) f(t, w) dw \right|^2 (1+|v|^2)^{\sup(-\gamma_1/2,1)} dv dt \\ &+ C(N) \int_0^T \int f^{2k+1}(t, v) \left| \int_{|v-w| \leq 1} (K_1 + K_2 |v-w|^{\gamma_2+1}) f(t, w) dw \right|^2 (1+|v|^2)^{\sup(-\gamma_1/2,1)} dv dt \\ &\leq C(N, K_1, K_2, \|f(0, \cdot)\|_{L^1_2(\mathbb{R}^N)}) \int_0^T \int f^{2k+1}(t, v) (1+|v|^2)^{\sup(1-\gamma_1/2,2)} dv dt \\ &+ C(N, K_2, \|f(0, \cdot)\|_{L^1(\mathbb{R}^N)}) \int_0^T \int f^{2k+1}(t, v) \left| |\cdot|^{\gamma_2+1} 1_{|\cdot| \leq 1} * f(t, \cdot) \right|^2 (v) (1+|v|^2)^{\sup(-\gamma_1/2,1)} dv dt \\ &\leq C(N, K_1, K_2, \|f(0, \cdot)\|_{L^1_2(\mathbb{R}^N)}) \|f\|_{L^{2k+1}^{2k+1}([0,T]; L^{\frac{2k+1}{\sup(2-\gamma_1,4)}}(\mathbb{R}^N))}^{2k+1} \\ &+ C(N, K_2, \|f(0, \cdot)\|_{L^1(\mathbb{R}^N)}) \int_0^T \left\| (1+|\cdot|^2)^{\sup(-\gamma_1/2,1)} f^{2k+1}(t, \cdot) \right\|_{L^{r'}(\mathbb{R}^N)} \left\| |\cdot|^{\gamma_2+1} 1_{|\cdot| \leq 1} * f(t, \cdot) \right\|_{L^{2r}(\mathbb{R}^N)}^2 dt, \end{aligned}$$

for all $r \in [1, +\infty]$ and r' defined by $\frac{1}{r} + \frac{1}{r'} = 1$.

Note that taking $\beta = \frac{N}{N+2(A-\varepsilon_0)}$ (with $\varepsilon_0 \in]0, A-1[$), one has $\beta \in]0, 1[$ and (with a slight abuse of notation)

$$\frac{N}{N+2(A-\varepsilon_0)} = \frac{1-\beta}{\infty} + \frac{\beta}{1} = \frac{1-\beta}{A-\varepsilon_0} + \beta \frac{N-2}{N},$$

when $N \geq 3$, so that, thanks to the interpolation described in Proposition 6,

$$\|f\|_{L^{\frac{N+2(A-\varepsilon_0)}{N}}([0,T]; L_{m_0}^{\frac{N+2(A-\varepsilon_0)}{N}}(\mathbb{R}^N))}$$

is bounded by a constant which depends on $\|f\|_{L^\infty([0,T]; L_m^{A-\varepsilon_0}(\mathbb{R}^N))}$ with some $m \in \mathbb{R}$ (this quantity appears in the assumption of the lemma) and $\|f\|_{L^1([0,T]; L_{\inf(\gamma_1, -2)}^{\frac{N}{N-2}}(\mathbb{R}^N))}$, which is itself bounded by a constant depending on $\mathcal{F}_{T,N,\gamma_1}(f)$, thanks to Lemma 3. Then, $\|f\|_{L^{2k+1}([0,T]; L^{\frac{2k+1}{\sup(2-\gamma_1, 4)}}(\mathbb{R}^N))}$ is bounded (by a constant depending on the parameters appearing in the assumptions of the lemma) for all $k < A/N$. The case $N = 2$ leads to the same conclusion if one replaces $\frac{N}{N-2}$ by any $q \in [1, +\infty[$.

We now take r in such a way that

$$\frac{1}{2r} = \frac{1}{A-\varepsilon_0} - \frac{\gamma_2+1}{N} - 1 = \frac{1}{A-\varepsilon_0/2} - \frac{\gamma_2+1}{N} \left[1 - \frac{N}{\gamma_2+1} \frac{\varepsilon_0/2}{(A-\varepsilon_0)(A-\varepsilon_0/2)} \right] - 1,$$

(with $\varepsilon_0 \in]0, A-1[$ small enough). Note that $-(\gamma_2+1) < N/2$ since $N \geq 2$. Moreover $\frac{1}{A-\varepsilon_0} - \frac{\gamma_2+1}{N} - 1 > 0$ since $A \leq \frac{N}{N+\gamma_2+1}$. Then, $r > 1$, and thanks to Young's inequality for convolutions, since $|\cdot|^{\gamma_2+1} 1_{|\cdot| \leq 1} \in L^{\frac{N}{-\gamma_2-1}-\delta}$ for all $\delta > 0$ small enough,

$$\sup_{t \in [0,T]} \| |\cdot|^{\gamma_2+1} 1_{|\cdot| \leq 1} * f(t, \cdot) \|_{L^{2r}(\mathbb{R}^N)}$$

is bounded by a constant depending only on $\|f\|_{L^\infty([0,T]; L^{A-\varepsilon_0/2}(\mathbb{R}^N))}$.

It remains to bound

$$\int_0^T \|v \mapsto (1+|v|^2)^{\sup(-\gamma_1/2, 1)} f^{2k+1}(t, v)\|_{L^{r'}(\mathbb{R}^N)} dt.$$

Note that taking $\beta = \frac{1}{2(A-\varepsilon_0)(1+\frac{2+\gamma_2}{N})-1}$, one has $\beta \in]0, 1[$ (remember that $A-\varepsilon_0 > 1$), and (with a slight abuse of notation)

$$\begin{aligned} \frac{1}{2(A-\varepsilon_0)(1+\frac{2+\gamma_2}{N})-1} &= \frac{1-\beta}{\infty} + \frac{\beta}{1}, \\ \frac{1}{2(A-\varepsilon_0)(1+\frac{2+\gamma_2}{N})-1} \frac{1}{r'} &= \frac{1-\beta}{A-\varepsilon_0} + \beta \frac{N-2}{N}, \end{aligned}$$

so that

$$\|f\|_{L^{2(A-\varepsilon_0)(1+\frac{2+\gamma_2}{N})-1}([0,T]; L^{\frac{2(A-\varepsilon_0)(1+\frac{2+\gamma_2}{N})-1}{\sup(-\gamma_1, 2)}}_{r'}(\mathbb{R}^N))}$$

is bounded by a constant depending only on $\|f\|_{L^\infty([0,T]; L_m^{A-\varepsilon_0}(\mathbb{R}^N))}$ for all $m \in \mathbb{R}$ (which is controlled thanks to the assumptions of the lemma) and $\|f\|_{L^1([0,T]; L_{\inf(\gamma_1, -2)}^{\frac{N}{N-2}}(\mathbb{R}^N))}$,

which is itself bounded by a constant depending on $\mathcal{F}_{T,N,\gamma_1}(f)$, thanks to Lemma 3. Then, for $k < A(1+\frac{2+\gamma_2}{N})-1$, we see that $2k+1 \leq 2(A-\varepsilon_0)(1+\frac{2+\gamma_2}{N})-1$ for some $\varepsilon_0 \in]0, A-1[$ small enough), so that for such a value of the parameter k , $\int_0^T \|v \mapsto (1+|v|^2)^{\sup(-\gamma_1/2, 1)} f^{2k+1}(t, v)\|_{L^{r'}(\mathbb{R}^N)} dt$ can be bounded by the quantities appearing in the statement of the lemma. Finally, we observe that thanks to

Proposition 4 (and Lemma 3), f has bounds in $L^\infty([0,T]; L_M^1(\mathbb{R}^N))$ for any $M \in \mathbb{R}$, depending only on quantities appearing in the statement of the lemma. Thanks to a last interpolation, we conclude the proof of Lemma 6. \square

We now turn to the

Proof of Proposition 7: We fix $M \in \mathbb{N}$ as the smallest integer such that $q(1 + \frac{\gamma_2+2}{N})^{-M} \leq 1$. Since $q \leq 1 + \frac{1}{N+1+\gamma_2}$, we see that $q(1 + \frac{\gamma_2+2}{N})^{-1} \leq \frac{N}{N+1+\gamma_2}$. Therefore we can apply Lemma 6 by induction for $A = 1$ [in this first step, one uses a slight variant of the lemma, which holds in theory only for $A > 1$], $A = q(1 + \frac{\gamma_2+2}{N})^{-M+1}$, .., $A = q(1 + \frac{\gamma_2+2}{N})^{-1}$. We end up with the conclusion of Proposition 7. \square

We now consider a less technical case, which appears either when $\psi(z) \leq K_2 |z|^{\gamma_2+2}$ with $\gamma_2 \in]-2, -1[$ (for $|z|$ close to 0), and when one looks for the propagation of L^q norms with $q \in [1 + \frac{1}{N+1+\gamma_2}, \frac{N}{N-1}[$, or when $q \in]1, \frac{N}{N-1}[$ and the soft potential is close to Maxwellian molecules, that is $\gamma_2 \in [-1, -0[$. The specificity of this case is that in the proof of the corresponding proposition (Proposition 8), $b_i * f$ lies in $L_{loc}^\infty([0, T] \times \mathbb{R}^N)$, which leads to great simplifications in the proof (with respect to the proof of Proposition 7).

Our proposition writes

Proposition 8. *Let ψ satisfying*

$$\forall z \geq 0, \quad K_3 \inf(1, |z|^{\gamma_1+2}) \leq \psi(z) \leq K_1 |z|^{2-\delta} + K_2 |z|^{\gamma_2+2},$$

for some $K_1, K_2, K_3 > 0$ and $\delta \in]0, 2[$, $\gamma_1 \leq 0$, $\gamma_2 \in]-2, -\delta[$.

Let $T > 0$, and $f := f(t, v) \geq 0$ be a strong solution of the Landau equation (1) on $[0, T] \times \mathbb{R}^N$ (more precisely, we assume that f is smooth, rapidly decaying when $|v| \mapsto +\infty$. We also assume that $\ln f$ is slowly growing when $|v| \mapsto +\infty$).

We consider $q \in [1 + \frac{1}{N+1+\gamma_2}, \frac{N}{N-1}[$ if $\gamma_2 \in]-2, -1[$, and $q \in]1, \frac{N}{N-1}[$ if $\gamma_2 \in [-1, -0[$.

Then, for all $m \in \mathbb{R}$, $\varepsilon_0 \in]0, q-1[$, there exists C depending only on $N, K_1, K_2, K_3, \gamma_1, \gamma_2, \delta, \varepsilon_0, m, q$, on the initial mass, momentum, energy and (upper part of) the entropy, on the initial moments (in L^1) ($\|f(0, \cdot)\|_{L_\mu^1(\mathbb{R}^N)}_{\mu>1}$, and initial $L^{q-\varepsilon}$ norm ($\|f(0, \cdot)\|_{L^{q-\varepsilon}(\mathbb{R}^N)}_{\varepsilon \in]0, q-1[}$, such that

$$\sup_{t \in [0, T]} \|f(t, \cdot)\|_{L_m^{q-\varepsilon_0}(\mathbb{R}^N)} \leq C.$$

Once again, the proposition is obtained thanks to the inductive use of a technical lemma:

Lemma 7. *Let ψ and f satisfy the assumptions of Proposition 8.*

We consider $A \in]\frac{N}{N+1+\gamma_2}, \frac{N}{N-1}[$ if $\gamma_2 \in]-2, -1[$ and $A \in [1, \frac{N}{N-1}[$ if $\gamma_2 \in [-1, -0[$.

Then for all $\varepsilon_0 \in]0, A-1[$ and $m_0 \in \mathbb{R}$,

$$\sup_{t \in [0, T]} \|f(t, \cdot)\|_{L_{m_0}^{1+(A-\varepsilon_0)/N}(\mathbb{R}^N)}$$

is bounded by a constant depending on the same parameters as in Proposition 7 (except that q is replaced by A , and the initial $L^{q-\varepsilon}$ norm by the initial $L^{A-\varepsilon}$ norm), and on the quantities

$$\sup_{t \in [0, T]} \|f(t, \cdot)\|_{L_m^{A-\varepsilon}(\mathbb{R}^N)},$$

for all $\varepsilon \in]0, A-1[$, and $m \in \mathbb{R}$.

Proof of Lemma 7: As in Lemma 6, we have (for any $k > 0$)

$$\begin{aligned}
 (34) \quad & \int \frac{f^{k+1}}{k+1} dv(T) + k \int_0^T \int f^{k-1} \sum_{i=1}^N \sum_{j=1}^N (a_{ij} * f) \partial_i f \partial_j f dv dt \\
 & \leq \int \frac{f^{k+1}}{k+1} dv(0) \\
 & + C(k, T, N, \gamma_1) \left(\int_0^T \int \left| \sum_{i=1}^N b_i * f \right|^2 f^{2k+1} (1 + |v|^2)^{\sup(-\gamma_1/2, 1)} dv dt \right)^{1/2},
 \end{aligned}$$

where $C(a, b, \dots)$ is (here and in the rest of the lemma) a constant depending only on a, b, \dots

We compute (for $i = 1, \dots, 3$)

$$\begin{aligned}
 |b_i * f(t, \cdot)|(v) & \leq C(N, K_1, K_2) \left(|| \cdot |^{1-\delta} * f(t, \cdot)|(v) + || \cdot |^{1+\gamma_2} * f(t, \cdot)|(v) \right) \\
 & \leq C(N, K_1, K_2, \|f(t, \cdot)\|_{L^1_1(\mathbb{R}^N)}) (1 + |v| + || \cdot |^{1+\gamma_2} 1_{|\cdot| \leq 1} * f(t, \cdot)|(v)).
 \end{aligned}$$

Then, we use Young's inequality for convolutions. If $\gamma_2 \in [-1, 0]$, we directly obtain that $\sup_{t \in [0, T]} \|b_i * f(t, \cdot)\|_{L^\infty_1(\mathbb{R}^N)}$ is bounded by a constant only depending on the parameters appearing in the statement of the lemma. If $\gamma_2 \in]-2, -1[$, we use the bound $\sup_{t \in [0, T]} \|f(t, \cdot)\|_{L^{A-\varepsilon}(\mathbb{R}^N)}$, where $A > \frac{N}{N+1+\gamma_2}$, and $\varepsilon > 0$ is sufficiently small. We also get that $\sup_{t \in [0, T]} \|b_i * f(t, \cdot)\|_{L^\infty_1(\mathbb{R}^N)}$ is bounded by a constant only depending on the parameters appearing in the statement of the lemma.

Then, it remains to bound $\int_0^T \int f^{2k+1} (1 + |v|^2)^{\sup(1-\gamma_1/2, 2)} dv dt$, with $k = \frac{A-\varepsilon_0}{N}$.

We consider $\beta = \frac{1}{2k+1} = \frac{N}{N+2(A-\varepsilon_0)}$, so that $\beta \in]0, 1[$ and (with a slight abuse of notation)

$$\frac{1}{2k+1} = \frac{1-\beta}{\infty} + \frac{\beta}{1} = \frac{1-\beta}{A-\varepsilon_0} + \beta \frac{N-2}{N}.$$

Using the bound assumed in the lemma (more precisely, $\|f\|_{L^\infty([0, T]; L^{A-\varepsilon_0}_m(\mathbb{R}^N))}$ for all $m \in \mathbb{R}$), and remembering that $\|f\|_{L^1([0, T]; L^{\frac{N}{N-2}}_{\inf(\gamma_1, -2)}(\mathbb{R}^N))}$ is bounded thanks

to Theorem 3 and Lemma 3 (in dimension 2, the quantity $N/(N-2)$ is replaced by all $q \geq 1$), we see that the interpolation described in Proposition 6 implies that $\|f\|_{L^{2k+1}([0, T]; L^{\frac{2k+1}{\sup(2-\gamma_1, 4)}}(\mathbb{R}^N))}$ is bounded by a constant only depending on the parameters in the statement of the lemma.

A last interpolation, like at the end of Lemma 6, leads to the conclusion of the proof of Lemma 7. \square

We now turn to the

Proof of Proposition 8: We build the sequence $q_0 := q$, and $q_{n+1} := (q_n - 1)N$, for all $n \in \mathbb{N}$. We fix $M \in \mathbb{N}$ as the smallest integer such that $q_M \leq 1$ if $\gamma_2 \in [-1, 0]$, and $q_M \leq 1 + \frac{1}{N+1+\gamma_2}$ if $\gamma_2 \in]-2, -1[$. Since $q < \frac{N}{N-1}$, we see that such an integer exists.

If $\gamma_2 \in [-1, 0]$, we can apply Lemma 7 by induction for $A = 1$ (for this step, one uses a slight variant of the lemma), $A = q_{M-1}, \dots, A = q_1$, and get the conclusion of Proposition 8.

If $\gamma_2 \in]-2, -1[$, then $\frac{N}{N+1+\gamma_2} < q_M \leq 1 + \frac{1}{N+1+\gamma_2}$. Using the result of Proposition 7, we can apply Lemma 7 by induction for $A = q_M$, $A = q_{M-1}$, .. , $A = q_1$, and also get the conclusion of Proposition 8. \square

Gathering Propositions 7 and 8, we see that propagation holds for L^p norms (more precisely, it holds in the spaces $\cup_{q < p, m \in \mathbb{R}} L_m^q$) as soon as $p < N/(N-1)$, under the assumptions on ψ in those propositions, and for all dimensions N .

We think that propagation in L^p for larger p is difficult to treat if one does not want to use the coercivity estimate $\sum_{i,j} (a_{ij} * f) \xi_i \xi_j \geq C(1 + |v|^2)^{\gamma/2} |\xi|^2$.

Using this coercivity estimate leads to proofs which are much closer to those of [22] than those of Propositions 7 and 8. We briefly describe how they work: first, in estimate (34), we use the coercivity estimate and Cauchy-Schwarz inequality in order to replace f^{2k+1} by f^{k+1} . Then we can take $\beta = 1/(1+k)$, so that a bound in $L^\infty(\cup_{\varepsilon > 0, m \in \mathbb{R}} L_m^{A-\varepsilon})$ implies a bound in $L^\infty(\cup_{\varepsilon > 0, m \in \mathbb{R}} L_m^{1+2(A-\varepsilon)/N})$, which improves the bound of Lemma 7. In this way, we can get the propagation of L^p norms (more precisely, we consider the spaces $\cup_{q < p, m \in \mathbb{R}} L_m^q$) as soon as $p < N/(N-2)$.

This is sufficient to show that $c * f$ is bounded in $L_{loc}^\infty([0, T] \times \mathbb{R}^N)$, so that using the parabolic form (9) of the equation, it is possible (for $N = 3$) to show that the bound of f in $L^\infty([0, T]; H^1(\mathbb{R}^3)) \cap L^2([0, T]; H^2(\mathbb{R}^3))$ only depends on the initial norm $\|f_{in}\|_{H^1(\mathbb{R}^3)}$, the initial moments (of any order) and the other parameters appearing in Proposition 7.

Further smoothness estimates are obtained by differentiating the equation. Note that in all the steps briefly described above, the dependence with respect to T of the constant is always polynomial, since no Gronwall-type argument is used.

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